

Pacific Journal of Mathematics

ON COMPARABLE MEANS

OVED SHISHA AND GERALD THOMAS CARGO

ON COMPARABLE MEANS

O. SHISHA AND G. T. CARGO

1. Let $-\infty < a < b < \infty$, and let Φ denote the set of all functions, continuous and strictly monotone in $[a, b]$. For every $\varphi \in \Phi$, every positive integer n , every x_1, x_2, \dots, x_n of $[a, b]$, and every positive q_1, q_2, \dots, q_n with $\sum_{\nu=1}^n q_\nu = 1$, we consider the mean

$$M_\varphi(x_1, x_2, \dots, x_n | q_1, q_2, \dots, q_n) = \varphi^{-1}(\sum_{\nu=1}^n q_\nu \varphi(x_\nu)).$$

Let ψ and χ be elements of Φ . We write

$$(1) \quad M_\psi \leq M_\chi$$

if and only if the inequality $M_\psi(x_1, x_2, \dots, x_n | q_1, q_2, \dots, q_n) \leq M_\chi(x_1, x_2, \dots, x_n | q_1, q_2, \dots, q_n)$ holds for every $n \geq 1$, every x_1, x_2, \dots, x_n of $[a, b]$, and every positive q_1, q_2, \dots, q_n with $\sum_{\nu=1}^n q_\nu = 1$.

A well-known necessary and sufficient condition for (1) to hold is that $\chi(\psi^{-1}(x))$ be convex in $[\psi(a), \psi(b)]$ (or $[\psi(b), \psi(a)]$) if χ is increasing, and that $\chi(\psi^{-1}(x))$ be concave there if χ is decreasing.

It is not difficult to see that (1) holds if and only if $M_\psi(x_1, x_2 | q_1, q_2) \leq M_\chi(x_1, x_2 | q_1, q_2)$ for every x_1, x_2 of $[a, b]$ and every positive q_1, q_2 with $q_1 + q_2 = 1$, which in turn holds if and only if $M_\psi(x_1, x_2 | 1/2, 1/2) \leq M_\chi(x_1, x_2 | 1/2, 1/2)$ for every x_1, x_2 of $[a, b]$.

Similarly, we write

$$(2) \quad M_\psi < M_\chi$$

if and only if the inequality

$$M_\psi(x_1, x_2, \dots, x_n | q_1, q_2, \dots, q_n) < M_\chi(x_1, x_2, \dots, x_n | q_1, q_2, \dots, q_n)$$

holds for every $n \geq 2$, every x_1, x_2, \dots, x_n (not all equal) of $[a, b]$, and every positive q_1, q_2, \dots, q_n with $\sum_{\nu=1}^n q_\nu = 1$. A necessary and sufficient condition for (2) to hold is that $\chi(\psi^{-1}(x))$ be strictly convex in $[\psi(a), \psi(b)]$ (or $[\psi(b), \psi(a)]$) if χ is increasing, and that $\chi(\psi^{-1}(x))$ be strictly concave there if χ is decreasing. Also, (2) holds if and only if $M_\psi(x_1, x_2 | q_1, q_2) < M_\chi(x_1, x_2 | q_1, q_2)$ for every x_1, x_2 ($\neq x_1$) of $[a, b]$ and every positive q_1, q_2 with $q_1 + q_2 = 1$, which in turn holds if and only if $M_\psi(x_1, x_2 | 1/2, 1/2) < M_\chi(x_1, x_2 | 1/2, 1/2)$ for every x_1 and x_2 ($\neq x_1$) of $[a, b]$.

2. In this paper we give simple criteria for the validity of (1)

and of (2), and then we give a few applications.

THEOREM 1. *Let ψ and χ be elements of Φ differentiable in (a, b) , and let $\psi' \neq 0$ there. A necessary and sufficient condition for (1) to hold is that χ'/ψ' be nondecreasing in (a, b) if ψ and χ are monotone in the same sense, and that χ'/ψ' be nonincreasing there if ψ and χ are monotone in opposite senses.*

Proof. Consider the function $u(x) \equiv \chi(\psi^{-1}(x))$. Let J denote the open interval joining $\psi(a)$ to $\psi(b)$, and let \bar{J} be the closure of J . For every $\xi \in J$, we have

$$(3) \quad u'(\xi) = \chi'(\psi^{-1}(\xi))/\psi'(\psi^{-1}(\xi)).$$

Suppose that ψ and χ are monotone in the same sense. Then (1) holds if and only if $u(x)$ is convex in \bar{J} in case χ increases, and if and only if $u(x)$ is concave there in case χ decreases. So (1) holds if and only if $u'(x)$ is nondecreasing in J in case ψ increases, and if and only if $u'(x)$ is nonincreasing there in case ψ decreases. From this, with the aid of (3), one easily infers that (1) is equivalent to χ'/ψ' being nondecreasing in (a, b) . Similarly one shows that (1) is equivalent to χ'/ψ' being nonincreasing in (a, b) , if ψ and χ are monotone in opposite senses.

One can modify Theorem 1 by replacing in it (1) by (2), "nondecreasing" by "strictly increasing," and "nonincreasing" by "strictly decreasing."

3. Given a function ψ , one may construct by means of Riemann-Stieltjes integrals functions χ such that $M_\psi \leq M_\chi$. In fact, we have the following

THEOREM 2. *Let ψ be a real function, continuous in $[a, b]$ and differentiable in (a, b) . Let $m(x)$ be nondecreasing or nonincreasing in $[a, b]$, continuous in (a, b) , and suppose $m(x)\psi'(x) \neq 0$ throughout (a, b) . Let C be a real constant, and for every $x \in [a, b]$ let*

$$\chi(x) = C + \int_a^x m(t)d\psi(t).$$

Then ψ and χ belong to Φ . If $m(x)$ is positive in (a, b) and nondecreasing in $[a, b]$, or negative in (a, b) and nonincreasing in $[a, b]$, then $M_\psi \leq M_\chi$. Otherwise, $M_\chi \leq M_\psi$.

Proof. Since $\psi' \neq 0$ in (a, b) , by a well known property of the derivative, ψ' is either positive throughout (a, b) , or negative through-

out (a, b) . Thus ψ is strictly monotone in $[a, b]$. Also, by well-known properties of the Riemann-Stieltjes integral, χ is continuous in $[a, b]$, and $\chi'(x) = m(x)\psi'(x)$ throughout (a, b) (and so χ is strictly monotone in $[a, b]$). If $m(x)$ is positive in (a, b) and nondecreasing in $[a, b]$, then ψ and χ are monotone in the same sense in $[a, b]$, χ'/ψ' is nondecreasing in (a, b) , and hence by Theorem 1, $M_\psi \leq M_\chi$. Similarly the rest of Theorem 2 follows.

Theorem 2 can be modified by replacing in it "nondecreasing" by "strictly increasing," "nonincreasing" by "strictly decreasing," " $M_\psi \leq M_\chi$ " by " $M_\psi < M_\chi$," and " $M_\chi \leq M_\psi$ " by " $M_\chi < M_\psi$."

4. A converse of Theorem 2 is the following

THEOREM 3. *Let ψ and χ be elements of Φ differentiable in (a, b) , and suppose $\psi' \neq 0$ there. Suppose, furthermore, that $M_\psi \leq M_\chi$. Then there exists a function $m(x)$, nondecreasing in (a, b) if ψ and χ are monotone in the same sense, and nonincreasing there if ψ and χ are monotone in opposite senses, such that throughout $[a, b]$*

$$(4) \quad \chi(x) = \chi(a) + \int_a^x m(t)\psi'(t)dt \quad (\text{a Lebesgue integral}).$$

Proof. For every $x \in (a, b)$, let $m(x) = \chi'(x)/\psi'(x)$. By Theorem 1, $m(x)$ has the monotonicity property stated in Theorem 3. Now for every $x \in [a, b]$

$$\chi(x) - \chi(a) = \int_a^x \chi'(t)dt = \int_a^x m(t)\psi'(t)dt$$

(cf. [5], Theorems 269 (p. 188) and 264 (p. 183)).

REMARK. Observe that the integral in (4) can be written, under appropriate conditions, as a Riemann-Stieltjes integral: $\int_a^x m(t)d\psi(t)$. [Cf. loc. cit, Theorem 322.1 (p. 254), and 322 (p. 253)].

Theorem 3 remains valid if we replace in it " $M_\psi \leq M_\chi$ " by " $M_\psi < M_\chi$," "nondecreasing" by "strictly increasing," and "nonincreasing" by "strictly decreasing."

5. It is known that if the end-point a is positive and $r < s$, $rs \neq 0$, then $M_{x^r} < M_{x^s}$, and $M_{x^{-|r|}} < M_{\log x} < M_{x^{|r|}}$. Consequently, if $a > 0$ then for every real $r (\neq 0, 1)$, $M_{(x^r)'} < M_{x^r}$, and $M_{(\log x)'} < M_{\log x}$. The question thus arises: Under what conditions on a function φ does one have $M_{\varphi'} < M_\varphi$ (or $M_{\varphi'} \leq M_\varphi$)?

THEOREM 4. *A necessary and sufficient condition for a real*

function φ to fulfill the conditions (α)–(γ) below is that $\varphi(x)$ should be (throughout $[a, b]$) of one of the forms $A + \int_a^x \exp C(t)dt$, $A - \int_a^x \exp C(t)dt$, $A + \int_a^x \exp \{-C(t)\}dt$, $A - \int_a^x \exp \{-C(t)\}dt$, where A is a real number, and $C(t)$ is a function, continuous and convex in $[a, b]$, differentiable in (a, b) , and satisfying there $C'(x) < 0$.

(α) φ is twice differentiable in (a, b) , $\varphi'(a)$ and $\varphi'(b)$ exist as right and left hand derivatives, respectively, $\varphi'(a)\varphi'(b) \neq 0$, and φ' is continuous in $[a, b]$.

(β) $\varphi'\varphi'' \neq 0$ throughout (a, b) (and hence φ and φ' are strictly monotone in $[a, b]$).

(γ) $M_{\varphi'} \leq M_{\varphi}$.

Proof.

Necessity. By Theorem 1, φ'/φ'' is either positive and nondecreasing in (a, b) , or negative and nonincreasing there. Thus, φ''/φ' is either positive and nonincreasing in (a, b) , or negative and nondecreasing there. In the first case we set $C(x) = -\log |\varphi'(x)|$ (in $[a, b]$). Then $C(x)$ is continuous in $[a, b]$ and $C'(x) < 0$ in (a, b) . Also $C'(x)$ is nondecreasing in (a, b) , and, therefore, $C(x)$ is convex in $[a, b]$. Either for every $x \in [a, b]$, $\varphi(x) = \varphi(a) + \int_a^x \exp \{-C(t)\}dt$, or for every $x \in [a, b]$, $\varphi(x) = \varphi(a) - \int_a^x \exp \{-C(t)\}dt$. In the second case, we set $C(x) = \log |\varphi'(x)|$ (in $[a, b]$). Then $C(x)$ is continuous in $[a, b]$, $C'(x) < 0$ in (a, b) , and, again, $C(x)$ is convex in $[a, b]$. Either for every $x \in [a, b]$, $\varphi(x) = \varphi(a) + \int_a^x \exp C(t)dt$, or for every $x \in [a, b]$, $\varphi(x) = \varphi(a) - \int_a^x \exp C(t)dt$.

Sufficiency. (α) and (β) clearly hold. Also, by the convexity of $C(t)$, $C'(t)$ is nondecreasing in (a, b) . Now, either throughout (a, b) , $\varphi'/\varphi'' = \{C'(t)\}^{-1}$, or throughout (a, b) , $\varphi'/\varphi'' = -\{C'(t)\}^{-1}$. In the first case, φ' and φ are monotone in opposite senses, and φ'/φ'' is nonincreasing in (a, b) . In the second case, φ' and φ are monotone in the same sense, and φ'/φ'' is nondecreasing in (a, b) . In either case, by Theorem 1, $M_{\varphi'} \leq M_{\varphi}$.

Theorem 4 can be modified by replacing in it “convex” by “strictly convex,” and “ $M_{\varphi'} \leq M_{\varphi}$ ” by “ $M_{\varphi'} < M_{\varphi}$.”

THEOREM 5. *Let φ be strictly monotone in $[a, b]$ and three-times differentiable in (a, b) . Let φ' be continuous in $[a, b]$ (where $\varphi'(a)$*

and $\varphi'(b)$ are right and left hand derivatives, respectively). Let $\varphi'' \neq 0$ throughout (a, b) . A necessary and sufficient condition for $M_{\varphi} \leq M_{\varphi'}$ to hold is that $\varphi''^2 \geq \varphi'\varphi'''$ throughout (a, b) if φ' and φ are monotone in the same sense, and that $\varphi''^2 \leq \varphi'\varphi'''$ throughout (a, b) if φ' and φ are monotone in opposite senses.

Theorem 5 follows easily from Theorem 1 by considering the derivative of φ'/φ'' .

Similarly, under the hypotheses of Theorem 5, $M_{\varphi'} < M_{\varphi}$ holds, if $\varphi''^2 > \varphi'\varphi'''$ throughout (a, b) and φ and φ' are monotone in the same sense, and also if $\varphi''^2 < \varphi'\varphi'''$ throughout (a, b) and φ and φ' are monotone in opposite senses.

As an example, let $a = 0, b = \pi/2, \varphi(x) \equiv \cos x$. φ and φ' are monotone in the same sense in $[0, \pi/2]$, and $\varphi''^2 = \cos^2 x > -\sin^2 x = \varphi'\varphi'''$ throughout $(0, \pi/2)$. Therefore, $M_{-\sin x} < M_{\cos x}$, i.e., $M_{\sin x} < M_{\cos x}$.

6. In a previous paper [3] the authors studied, for given positive q_1, q_2, \dots, q_n (with $\sum_{v=1}^n q_v = 1$), the ratio

$$(5) \quad \begin{cases} F(x_1, x_2, \dots, x_n) \\ = M_{\chi}(x_1, x_2, \dots, x_n \mid q_1, q_2, \dots, q_n) / M_{\psi}(x_1, x_2, \dots, x_n \mid q_1, q_2, \dots, q_n) \end{cases}$$

where $0 < a, \psi(x) \equiv x^r, \chi(x) \equiv x^s$ ($r < s, rs \neq 0$).

Their purpose was to find an upper bound for F in

$$I = \{(x_1, x_2, \dots, x_n) : a \leq x_k \leq b, k = 1, 2, \dots, n\}.$$

A crucial step was to show that if X^* is a point of I such that $F(X^*) = \max \{F(X) : X \in I\}$, then X^* is necessarily a vertex of I . In particular, X^* cannot be an interior point of I . This last property holds under quite general conditions:

THEOREM 6. *Let ψ and χ be elements of Φ , differentiable in (a, b) , and satisfying $\psi'\chi' \neq 0$ there. Assume $0 \notin [a, b], M_{\psi} < M_{\chi}$. Let q_1, \dots, q_n ($n > 1$) be given positive numbers with $\sum_{v=1}^n q_v = 1$, and let I be as in the last paragraph. Let F of (5) attain its maximum in I at a point $X^* = (x_1^*, \dots, x_n^*)$ of I . Then X^* is not an interior point of I .*

Proof. Suppose that some x_j^* satisfies $a < x_j^* < b$. Then $(\partial F / \partial x_j)_{x_v = x_v^*} = 0$, i.e.,

$$\begin{aligned} & \left[\psi^{-1} \left(\sum_{v=1}^n q_v \psi(x_v^*) \right) \right]^{-2} \left[q_j \chi'(x_j^*) \psi^{-1} \left(\sum_{v=1}^n q_v \psi(x_v^*) \right) \right] / \chi' \left(\chi^{-1} \left(\sum_{v=1}^n q_v \chi(x_v^*) \right) \right) \\ & - q_j \psi'(x_j^*) \chi^{-1} \left(\sum_{v=1}^n q_v \chi(x_v^*) \right) / \psi' \left(\psi^{-1} \left(\sum_{v=1}^n q_v \psi(x_v^*) \right) \right) = 0. \end{aligned}$$

Thus

$$\begin{aligned} \chi'(x_j^*)/\psi'(x_j^*) &= \left[\chi^{-1} \left(\sum_{v=1}^n q_v \chi(x_v^*) \right) \chi' \left(\chi^{-1} \left(\sum_{v=1}^n q_v \chi(x_v^*) \right) \right) \right] \\ &\quad \times \left/ \left[\psi^{-1} \left(\sum_{v=1}^n q_v \psi(x_v^*) \right) \psi' \left(\psi^{-1} \left(\sum_{v=1}^n q_v \psi(x_v^*) \right) \right) \right] \right]. \end{aligned}$$

Let C denote the right hand side of the last equality. If both x_j^* and x_k^* are interior points of $[a, b]$, then $\chi'(x_j^*)/\psi'(x_j^*) = C = \chi'(x_k^*)/\psi'(x_k^*)$, and hence, by the strict monotonicity of χ'/ψ' [see the end of § 2], $x_j^* = x_k^*$. Thus, if X^* were an interior point of I , we would have $x_1^* = x_2^* = \dots = x_n^*$, and therefore

$$1 = F(x_1^*, x_2^*, \dots, x_n^*) = \max \{F(X) : X \in I\} > 1.$$

REFERENCES

1. E. F. Beckenbach, *On the inequality of Kantorovich (abstract)*, Notices, Amer. Math. Society, **10**, No. 5 (1963), 440.
2. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag (1961).
3. G. T. Cargo and O. Shisha, *Bounds on Ratios of Means*, Journal of Research of the National Bureau of Standards, Vol. 66B, No. 4, pp. 169-170 (1962).
4. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (second edition), Cambridge University Press (1952).
5. H. Kestelman, *Modern theories of integration* (second revised edition), Dover Publications, 1960.

AEROSPACE RESEARCH LABORATORIES, WRIGHT-PATTERSON AFB, OHIO
AND
SYRACUSE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

ROBERT OSSERMAN
Stanford University
Stanford, California

J. DUGUNDI
University of Southern California
Los Angeles 7, California

M. G. ARSOVE
University of Washington
Seattle 5, Washington

LOWELL J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Erik Balslev and Theodore William Gamelin, <i>The essential spectrum of a class of ordinary differential operators</i>	755
James Henry Bramble and Lawrence Edward Payne, <i>Bounds for derivatives in elliptic boundary value problems</i>	777
Hugh D. Brunk, <i>Integral inequalities for functions with nondecreasing increments</i>	783
William Edward Christilles, <i>A result concerning integral binary quadratic forms</i>	795
Peter Crawley and Bjarni Jónsson, <i>Refinements for infinite direct decompositions of algebraic systems</i>	797
Don Deckard and Carl Mark Percy, <i>On continuous matrix-valued functions on a Stonian space</i>	857
Raymond Frank Dickman, Leonard Rubin and P. M. Swingle, <i>Another characterization of the n-sphere and related results</i>	871
Edgar Earle Enochs, <i>A note on reflexive modules</i>	879
Vladimir Filippenko, <i>On the reflection of harmonic functions and of solutions of the wave equation</i>	883
Derek Joseph Haggard Fuller, <i>Mappings of bounded characteristic into arbitrary Riemann surfaces</i>	895
Curtis M. Fulton, <i>Clifford vectors</i>	917
Irving Leonard Glicksberg, <i>Maximal algebras and a theorem of Radó</i>	919
Kyong Taik Hahn, <i>Minimum problems of Plateau type in the Bergman metric space</i>	943
A. Hayes, <i>A representation theory for a class of partially ordered rings</i>	957
J. M. C. Joshi, <i>On a generalized Stieltjes transform</i>	969
J. M. C. Joshi, <i>Inversion and representation theorems for a generalized Laplace transform</i>	977
Eugene Kay McLachlan, <i>Extremal elements of the convex cone B_n of functions</i>	987
Robert Alan Melter, <i>Contributions to Boolean geometry of p-rings</i>	995
James Ronald Retherford, <i>Basic sequences and the Paley-Wiener criterion</i>	1019
Dallas W. Sasser, <i>Quasi-positive operators</i>	1029
Oved Shisha, <i>On the structure of infrapolynomials with prescribed coefficients</i>	1039
Oved Shisha and Gerald Thomas Cargo, <i>On comparable means</i>	1053
Maurice Sion, <i>A characterization of weak* convergence</i>	1059
Morton Lincoln Slater and Robert James Thompson, <i>A permanent inequality for positive functions on the unit square</i>	1069
David A. Smith, <i>On fixed points of automorphisms of classical Lie algebras</i>	1079
Sherman K. Stein, <i>Homogeneous quasigroups</i>	1091
J. L. Walsh and Oved Shisha, <i>On the location of the zeros of some infrapolynomials with prescribed coefficients</i>	1103
Ronson Joseph Warne, <i>Homomorphisms of d-simple inverse semigroups with identity</i>	1111
Roy Westwick, <i>Linear transformations on Grassman spaces</i>	1123