

Pacific Journal of Mathematics

RINGS OF ARITHMETIC FUNCTIONS

L. CARLITZ

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1. **Introduction.** Let F denote a fixed but arbitrary field and let Z denote the set of positive integers. By an *arithmetic function* f is meant a function from Z to F , that is to say $f(n) \in F$ for all $n \in Z$. If f, g are two arithmetic functions, the sum $h = f + g$ is defined by means of

$$(1) \quad h(n) = f(n) + g(n) \quad (n \in Z).$$

There are two products that are of interest, the *ordinary* product defined by

$$(2) \quad h(n) = f(n)g(n) \quad (n \in Z),$$

and the Dirichlet product defined by

$$(3) \quad h(n) = \sum_{rs=n} f(r)g(s) \quad (n \in Z),$$

where the summation on the right is extended over all factorizations $rs = n$. We shall denote the ordinary product by $f \circ g$ and the Dirichlet product by $f * g$.

Let S denote the set of arithmetic functions as defined above. It is well known and easy to prove that the system

$$(4) \quad \Omega = (S, f, \circ)$$

is a commutative ring. The multiplicative identity of Ω is defined by

$$(5) \quad v(n) = 1 \quad (n \in Z).$$

Clearly Ω is not a domain of integrity; note however that there are no nilpotent elements in Ω . On the other hand the system

$$(6) \quad \mathcal{A} = (S, f, *)$$

is a domain of integrity. The multiplicative identity of \mathcal{A} is given by

$$(7) \quad u(n) = \begin{cases} 1 & (n = 1) \\ 0 & (n > 1). \end{cases}$$

Moreover the function f has an inverse (relative to $*$) if and only if

$$(8) \quad f(1) \neq 0;$$

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the set of functions that satisfy (8) evidently constitute an abelian group with respect to $*$.

If $\lambda \in F$ we define the function λf by means of

$$(9) \quad (\lambda f)(n) = \lambda \cdot f(n) \quad (n \in Z).$$

It follows at once that S is a vector space over F of infinite dimension. Also we have

$$\lambda(f \circ g) = (\lambda f) \circ g = f \circ (\lambda g), \quad \lambda(f * g) = (\lambda f) * g = f * (\lambda g).$$

If in place of Z we employ a semigroup J that has no units except the identity, a countable infinity of primes, and which has the unique factorization property, the resulting systems Ω and \mathcal{A} are not essentially different. Indeed if $\bar{p}_1, \bar{p}_2, \bar{p}_3, \dots$ denote the primes of J we may set up the correspondence $f \rightleftharpoons \bar{f}$ by means of $f(n) = \bar{f}(\bar{n})$, where

$$(10) \quad n = \prod p_j^{e_j}, \quad \bar{n} = \prod \bar{p}_j^{e_j},$$

where the first half of (10) is the usual factorization of n into primes. There is therefore little loss in generality in restricting the discussion to Z .

In view of the above it is of interest to consider the system

$$(11) \quad \mathcal{O} = (S, +, \circ, *)$$

with three binary operations and in particular to attempt to give an abstract formulation of such systems. Since \circ and $*$ do not combine in any very obvious way, it is perhaps not clear how this can be done. We shall obtain such a characterization by making use of *minimal* functions. A function f is minimal provided there exists an integer k (depending on f) such that

$$(12) \quad f(n) = 0 \quad (n \neq k); \quad f(k) \neq 0.$$

We remark that Cashwell and Everett [1] have proved that \mathcal{A} is a unique factorization domain. However this result will not be required in what follows.

2. As above let F denote a fixed but arbitrary field. Let \bar{S} denote a vector space over F . The elements of \bar{S} will be denoted by small italic letters, the elements of F by small Greek letters; addition in \bar{S} will be denoted by $+$. Moreover we have two "multiplications" denoted by \circ and $*$. The following assumptions will be made.

S1. The system

$$(13) \quad \Omega = (\bar{S}, +, \circ)$$

is a commutative ring with multiplicative identity \bar{v} . Moreover

$$\alpha(\bar{f} \circ \bar{g}) = (\alpha\bar{f}) \circ \bar{g} = \bar{f} \circ (\alpha\bar{g}) \quad (\bar{f}, \bar{g} \in \bar{S}, \alpha \in F).$$

S2. The system

$$(14) \quad \bar{A} = (\bar{S}, +, *)$$

is a domain of integrity with multiplicative identity \bar{u} . Moreover

$$\alpha(\bar{f} * \bar{g}) = (\alpha\bar{f}) * \bar{g} = \bar{f} * (\alpha\bar{g}) \quad (\bar{f}, \bar{g} \in \bar{S}, \alpha \in F).$$

DEFINITION. Two elements $\bar{f}, \bar{g} \in \bar{S}$ are *associates* provided $\bar{f} = \lambda\bar{g}$, where $\lambda \in F$, $\lambda \neq 0$.

DEFINITION. An element $\bar{f} \in \bar{S}$, $\bar{f} \neq 0$, is *minimal* provided

$$(15) \quad \bar{f} \circ \bar{g} = \lambda(\bar{f}, \bar{g})\bar{f} \quad (\bar{g} \in \bar{S})$$

where \bar{g} is any element of \bar{S} and $\lambda(\bar{f}, \bar{g})$ is a number of F . It is evident that $\lambda(\bar{f}, \bar{g})$ is unique.

Clearly the associate of a minimal element is also minimal. Also it is evident that if \bar{f}, \bar{g} are two minimal elements that are not associates then

$$(16) \quad \bar{f} \circ \bar{g} = 0.$$

S3. For each minimal element \bar{f} there exists a nonzero number $\lambda(\bar{f})$ of F such that

$$(17) \quad \bar{f} \circ \bar{f} = \lambda(\bar{f})\bar{f}.$$

DEFINITION. A minimal element $\bar{f} \in \bar{S}$ is *normalized* provided

$$(18) \quad \bar{f} \circ \bar{f} = \bar{f}.$$

S4. If \bar{g} is an arbitrary nonzero element of \bar{S} there exists at least one minimal element \bar{f} such that $\lambda(\bar{f}, \bar{g}) \neq 0$, where $\lambda(\bar{f}, \bar{g})$ is defined by (15).

Let M denote the set of normalized minimal elements.

S5. M is a semigroup with respect to $*$; the identity element of M coincides with \bar{u} , the multiplicative identity of \bar{A} . Moreover M contains no units except the identity.

DEFINITION. An element \bar{f} of M , $\bar{f} \neq \bar{u}$, is *prime* provided $\bar{f} = \bar{g} * \bar{h}$ implies $\bar{g} = \bar{u}$ or $\bar{h} = \bar{u}$.

S6. M contains a countable number of primes. Any element of M , different from \bar{u} , can be expressed as a product of primes in essentially only one way.

DEFINITION. Let $\bar{f}_1, \bar{f}_2, \bar{f}_3, \dots$ denote the elements of M . If \bar{g} is an arbitrary element of \bar{S} the numbers

$$\lambda_j(\bar{g}) = \lambda(\bar{f}_j, \bar{g})$$

defined by

$$(19) \quad \bar{f}_j \circ \bar{g} = \lambda(\bar{f}_j, \bar{g})\bar{f}_j$$

may be called the (Dirichlet) coefficients of \bar{g} .

S7. If $\bar{g} \neq \bar{h}$ then for at least one value of j we have $\lambda_j(\bar{g}) \neq \lambda_j(\bar{h})$.

It evidently follows that two elements of \bar{S} are equal if and only if the respective sets of coefficients are equal.

S8. If \bar{g} and \bar{h} are arbitrary elements of \bar{S} while \bar{f} is an element of M , then

$$\bar{f} \circ (\bar{g} * \bar{h}) = \Sigma(\bar{f}_r \circ \bar{g}) * (\bar{f}_s \circ \bar{h})$$

where the summation is over all $\bar{f}_r, \bar{f}_s \in M$ such that $\bar{f}_r * \bar{f}_s = \bar{f}$.

Finally we have

S9. For every sequence $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_j \in F$, there exists a $\bar{g} \in \bar{S}$ such that

$$\bar{f}_j \circ \bar{g} = \lambda_j \bar{f}_j \quad (j = 1, 2, 3, \dots).$$

3. LEMMA 1. If \bar{f}_i, \bar{f}_j are distinct elements of M then

$$(20) \quad \bar{f}_i \circ \bar{f}_j = 0 \quad (i \neq j).$$

This is immediate from (16).

LEMMA 2. Let \bar{g}, \bar{h} be two arbitrary elements of \bar{S} and let $\lambda_j(\bar{g}), \lambda_j(\bar{h})$ denote the respective sets of coefficients of \bar{g} and \bar{h} . Then

$$(21) \quad \lambda_j(\bar{g} \circ \bar{h}) = \lambda_j(\bar{g})\lambda_j(\bar{h}) \quad (j = 1, 2, 3, \dots).$$

Indeed we have by (18) and (19)

$$\begin{aligned} \lambda_j(\bar{g} \circ \bar{h})\bar{f}_j &= \bar{f}_j \circ (\bar{g} \circ \bar{h}) = (\bar{f}_j \circ \bar{g}) \circ (\bar{f}_j \circ \bar{h}) = (\lambda_j(\bar{g})\bar{f}_j) \circ (\lambda_j(\bar{h})\bar{f}_j) \\ &= \lambda_j(\bar{g})\lambda_j(\bar{h})(\bar{f}_j \circ \bar{f}_j) = \lambda_j(\bar{g})\lambda_j(\bar{h})\bar{f}_j \end{aligned}$$

and (21) follows at once.

LEMMA 3. Let \bar{g}, \bar{h} be two arbitrary elements of \bar{S} and let $\lambda_j(\bar{g}), \lambda_j(\bar{h})$ denote the respective sets of coefficients of \bar{g} and \bar{h} . Then

$$(22) \quad \lambda_j(\bar{g} * \bar{h}) = \sum \lambda_r(\bar{g}) \lambda_s(\bar{h}) \quad (j = 1, 2, 3, \dots),$$

where the summation is over all pairs r, s such that

$$(23) \quad \bar{f}_r * \bar{f}_s = \bar{f}_j.$$

Proof. We have by S8

$$\begin{aligned} \lambda_j(\bar{g} * \bar{h}) \bar{f}_j &= \bar{f}_j \circ (\bar{g} * \bar{h}) = \sum_{\bar{f}_r * \bar{f}_s = \bar{f}_j} (\bar{f}_r \circ \bar{g}) * (\bar{f}_s \circ \bar{h}) \\ &= \sum_{\bar{f}_r * \bar{f}_s = \bar{f}_j} (\lambda_r(\bar{g}) \bar{f}_r) * (\lambda_s(\bar{h}) \bar{f}_s) \\ &= \left\{ \sum_{\bar{f}_r * \bar{f}_s = \bar{f}_j} \lambda_r(\bar{g}) \lambda_s(\bar{h}) \right\} \bar{f}_j. \end{aligned}$$

This evidently implies (22).

Let $\bar{p}_1, \bar{p}_2, \bar{p}_3, \dots$ denote the primes of M and let p_1, p_2, p_3, \dots denote the ordinary primes. We assume to begin with that the number of primes in M is infinite and set up the correspondence

$$(24) \quad p_j \leftrightarrow \bar{p}_j \quad (j = 1, 2, 3, \dots).$$

If

$$n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$$

is an arbitrary positive integer, we put

$$(25) \quad \bar{f}_n = \bar{p}_1^{e_1} * \bar{p}_2^{e_2} * \dots * \bar{p}_r^{e_r},$$

where

$$\bar{p}_e = \bar{p} * \dots * \bar{p},$$

with e factors on the right. By means of (25) we have the one-to-one correspondence between Z and M

$$(26) \quad n \leftrightarrow \bar{f}_n \quad (n = 1, 2, 3, \dots).$$

Let \bar{g} be an arbitrary element of \bar{S} and let $\lambda_j(\bar{g})$ denote the set of coefficients of \bar{g} . Corresponding to \bar{g} we have the function g in S defined by

$$(27) \quad g(n) = \lambda_n(\bar{g}).$$

Conversely if g is any function in S then by S9 and S7 the element \bar{g} of \bar{S} is uniquely determined by means of (27), so that we have obtained a one-to-one correspondence between S and \bar{S} .

Now if $\alpha \in F$ it follows at once from (27) that

$$(28) \quad \alpha g(n) = \lambda_n(\alpha \bar{g}) ,$$

so that scalar multiplication is consistent with the correspondence defined by (27). Again if $h \in S$ and $\bar{h} \in \bar{S}$ satisfy

$$(29) \quad h(n) = \lambda_n(\bar{h})$$

it is clear that

$$(30) \quad g(n) + h(n) = \lambda_n(\bar{g} + \bar{h}) .$$

In the next place, if (27) and (29) hold, it follows from Lemma 2 that

$$(31) \quad g(n)h(n) = \lambda_n(\bar{g})\lambda_n(\bar{h}) = \lambda_n(\bar{g} \circ \bar{h}) .$$

Thus if \bar{g} corresponds to g and \bar{h} corresponds to h then $\bar{g} \circ \bar{h}$ corresponds to the "ordinary" product of g and h .

Next we observe that if

$$r \leftrightarrow \bar{f}_r , \quad s \leftrightarrow \bar{f}_s$$

under the correspondence (26), then

$$(32) \quad rs \leftrightarrow \bar{f}_r * \bar{f}_s .$$

Thus, assuming (27) and (29), we get

$$\sum_{rs=n} g(r)h(s) = \sum_{rs=n} \lambda_r(\bar{g})\lambda_s(\bar{h}) = \sum_{\bar{f}_r * \bar{f}_s = \bar{f}_n} \lambda_r(\bar{g})\lambda_s(\bar{h}) .$$

Therefore, by Lemma 3,

$$(33) \quad \sum_{rs=n} g(r)h(s) = \lambda_n(\bar{g} * \bar{h}) .$$

Thus if \bar{g} corresponds to g and \bar{h} corresponds to h then $\bar{g} * \bar{h}$ corresponds to the Dirichlet product of g and h .

Combining (27), (28), (29), (30), (31), (32) and (33) we have the following result.

THEOREM 1. *Let Φ denote the system of arithmetic functions from the integers to an arbitrary but fixed field F as defined in § 1. Let $\bar{\Phi}$ be a structure with the three binary operations $+$, \circ , $*$ that satisfies the assumptions S1-S9 of § 2. Also let the number of primes in M be infinite. Then $\bar{\Phi}$ is isomorphic to Φ , all operations being preserved under the isomorphism.*

4. We have assumed in the above result that the number of

prime elements in M is infinite. The conclusion of the theorem is no longer valid when the number of primes is finite. However it is easily verified that in this case $\bar{\Phi}$ is isomorphic to a subset of Φ . More precisely, we have the following result.

Let $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k$ denote the primes of M and let p_1, p_2, \dots, p_k be a set of k distinct primes, for example the first k primes. Then the correspondence (26) holds except that n is now restricted to the set of integers Z_k whose prime divisors are in the set p_1, p_2, \dots, p_k . Consider the set of functions g such that

$$(34) \quad g(n) = 0 \quad (n \in Z - Z_k),$$

while $g(n)$ is an arbitrary number of F when $n \in Z_k$. It is easily verified that the set of functions satisfying (34) is closed under scalar, ordinary and Dirichlet multiplication. We denote the system by Φ_k . Then we have

THEOREM 2. *Let Φ_k denote the system of arithmetic functions that satisfy (34). Let Φ be a structure with three binary operations $+$, \circ , $*$ that satisfies the assumptions S1-S9 of § 2 but let the number of primes in M equal k . Then $\bar{\Phi}$ is isomorphic to Φ_k .*

It is evident that Φ_k is isomorphic to $F\{x_1, x_2, \dots, x_k\}$, the ring of formal power series in k indeterminates with coefficients in F .

REMARK. The referee has pointed out that S4 and S7 are equivalent, in the presence of the other assumptions. First, S7 implies S4. For $\bar{g} \neq 0$, by S7 there exists a j such that $\lambda_j(\bar{g}) \neq \lambda_j(0) = 0$. Hence S4 holds with $\bar{f} = \bar{f}_j$.

Conversely, S4 implies S7. For if $\bar{g} \neq \bar{h}$, then $\bar{d} = \bar{g} - \bar{h} \neq 0$. By S4 there exists a minimal \bar{f} such that $\bar{f} \circ \bar{d} = \lambda(\bar{f}, \bar{d})\bar{f}$, where $\lambda(\bar{f}, \bar{d}) \neq 0$. Since \bar{f} is minimal, $\bar{f} \circ \bar{f} = \lambda(\bar{f})\bar{f}$, where $\lambda(\bar{f}) \neq 0$ by S3. Hence there exists a minimal

$$\bar{f} = (\lambda(\bar{f}))^{-1}\bar{f}$$

(an associate of the minimal element \bar{f}) which is also normalized. Thus

$$\begin{aligned} \bar{f}_j \circ \bar{d} &= \lambda(\bar{f}, \bar{d})\bar{f}_j = \bar{f}_j \circ (\bar{g} - \bar{h}) = \bar{f}_j \circ \bar{g} - \bar{f}_j \circ \bar{h} \\ &= \lambda_j(\bar{g})\bar{f}_j - \lambda_j(\bar{h})\bar{f}_j = [\lambda_j(\bar{g}) - \lambda_j(\bar{h})]\bar{f}_j. \end{aligned}$$

Hence

$$\lambda_j(\bar{g}) - \lambda_j(\bar{h}) = \lambda(\bar{f}, \bar{d}) \neq 0.$$

REFERENCE

1. E. D. Cashwell and C. J. Everett, *The ring of number-theoretic functions*, Pacific J. Math., **9** (1959), 975-985.

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