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## **MEASURABLE SETS OF MEASURES**

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1. **Introduction.** Let  $M$  be the set of all countably additive, finite, signed measures on a  $\sigma$ -field  $\Sigma$  of subsets of a set  $X$ . There is a natural definition of measurability in  $M$ , namely, a subset of  $M$  is *measurable* if it is an element of  $\Sigma^*$ , the smallest  $\sigma$ -field of subsets of  $M$  such that: for each  $A \in \Sigma$  the function  $\mu \rightarrow \mu(A)$  is measurable from  $M$  to the Borel line. The purpose of this note, motivated by questions arising from (Dubins and Freedman, 1963) is to investigate the measurability and category of interesting subsets of  $M$ , under the assumption that  $\Sigma$  is countably generated.

Here are some results. If  $X$  is compact metric, and  $\Sigma$  is the  $\sigma$ -field of Borel subsets of  $X$ , then any subset of  $X$  with the Baire property is measurable for a residual set of probability measures (3.17). If also  $X$  is uncountable, there are weakly open, but not  $\Sigma^*$ -measurable, subsets of  $M$ ; see (3.2). There is a  $G_\delta$  in the three-dimensional unit cube whose convex hull is not Borel (3.22). If  $F$  is a continuous, strictly monotone, purely singular distribution function on the unit interval, then  $F$  is differentiable only on a set of the first category (4.8).

2. **The abstract case.** Let  $X$  be a nonempty set,  $\mathcal{F}$  a countable field of subsets of  $X$ , and  $\Sigma$  the smallest  $\sigma$ -field including  $\mathcal{F}$ .

2.1. *Let  $\mathcal{A}$  be a  $\sigma$ -field of subsets of a set  $\Omega$ , and let  $\varphi$  map  $\Omega$  into  $M$ . Then  $\varphi$  is measurable from  $(\Omega, \mathcal{A})$  to  $(M, \Sigma^*)$  if and only if the function  $\omega \rightarrow \varphi(\omega)(A)$  is measurable from  $(\Omega, \mathcal{A})$  to the Borel line for each  $A \in \mathcal{F}$ .*

*Proof.* Routine.

2.2. *If  $\varphi$  is a measurable map from  $(\Omega, \mathcal{A})$  to  $(M, \Sigma^*)$ , and  $f$  is a bounded, measurable function from  $(\Omega \times X, \mathcal{A} \times \Sigma)$  to the Borel line, then  $\omega \rightarrow \int_x f(\omega, x)\varphi(\omega)(dx)$  is a measurable function from  $(\Omega, \mathcal{A})$  to the Borel line.*

*Proof.* Extend from indicators of measurable rectangles.

2.3. *The  $\sigma$ -field  $\Sigma^*$  is countably generated.*

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*Proof.* Use (2.1).

2.4. For each  $\mu \in M$ , the set  $\{\mu\}$  is measurable.

*Proof.*  $\{\mu\} = \bigcap_{A \in \mathcal{F}} \{\nu \mid \nu \in M, \nu(A) = \mu(A)\}$ .

2.5. If  $\varphi_1$  and  $\varphi_2$  are measurable maps from  $(\Omega, \mathcal{A})$  to  $(M, \Sigma^*)$ , then so are  $\varphi_1 + \varphi_2$  and  $c\varphi_1$  for any real number  $c$ , and  $\{\omega \mid \omega \in \Omega, \varphi_1(\omega) = \varphi_2(\omega) \in \mathcal{A}\}$ .

*Proof.* Use (2.1) for the first two assertions, and (2.4) for the third.

2.6. If  $\mu \in M$ , and  $A \in \Sigma$ , for any  $\delta > 0$  there is a set  $A(\mu, \delta) \in \mathcal{F}$  whose symmetric difference with  $A$  has  $\mu$ -measure less than  $\delta$ .

*Proof.* (Halmos 1958, Theorem D, page 56.)

Let  $M^+$  be the set of nonnegative measures on  $(X, \Sigma)$ .

2.7. The set of nonnegative measures is measurable; so is the set of probability measures.

*Proof.*  $M^+ = \bigcap_{A \in \mathcal{F}} \{\mu \mid \mu \in M, \mu(A) \geq 0\}$ .

If  $\mathcal{A}$  is a  $\sigma$ -field of subsets of the set  $\Omega$  and  $W \subset \Omega$ , then  $W\mathcal{A}$  is the  $\sigma$ -field of subsets of  $W$  having the form  $W \cap A$ ,  $A \in \mathcal{A}$ . Abbreviate  $M^+\Sigma^*$  to  $\Sigma^+$ . Recall that  $\mu \in M$  is the difference of two unique, nonnegative, mutually singular measures  $\mu^+$  and  $\mu^-$ . Let  $|\mu| = \mu^+ + \mu^-$  and  $\|\mu\| = |\mu|(X)$ .

2.8. THEOREM. The maps  $\mu \rightarrow \mu^+$  and  $\mu \rightarrow \mu^-$  are measurable from  $(M, \Sigma^*)$  to  $(M^+, \Sigma^+)$ .

*Proof.* By (2.5), it is enough to check the first assertion. By (2.6), if  $A \in \Sigma$ , then  $\mu^+(A) = \sup \{\mu(A \cap B) : B \in \mathcal{F}\}$ . Hence  $\mu \rightarrow \mu^+(A)$  is measurable and (2.1) applies.

2.9. The map  $\mu \rightarrow |\mu|$  is measurable from  $(M, \Sigma^*)$  to  $(M^+, \Sigma^+)$ . The function  $\mu \rightarrow \|\mu\|$  is measurable from  $(M, \Sigma^*)$  to the Borel line.

*Proof.* (2.5) and (2.8) imply the first assertion, and it implies the second.

Recall that for  $\mu$  and  $\nu$  in  $M$  there are two unique elements  $S(\mu, \nu)$  and  $A(\mu, \nu)$  of  $M$  with:

$$(i) \quad \mu = S(\mu, \nu) + A(\mu, \nu);$$

- (ii)  $S(\mu, \nu)$  and  $\nu$  are mutually singular;
- (iii)  $A(\mu, \nu)$  is absolutely continuous with respect to  $\nu$ .

2.10. THEOREM. *The maps  $S: (\mu, \nu) \rightarrow S(\mu, \nu)$  and  $A: (\mu, \nu) \rightarrow A(\mu, \nu)$  are measurable from  $(M \times M, \Sigma^* \times \Sigma^*)$  to  $(M, \Sigma^*)$ .*

*Proof.* If  $\mu, \nu \in M^+$  and  $A \in \Sigma$ , then  $S(\mu, \nu)(A) = \lim_{n \rightarrow \infty} \sup\{\mu(A \cap B) : B \in \mathcal{F}, \nu(B) < n^{-1}\}$ , by (2.6) and (Halmos, 1958, Theorem B, page 125). By (2.1),  $S$  restricted to  $M^+ \times M^+$  is  $\Sigma^+ \times \Sigma^+$ -measurable; apply (2.8) and (2.5).

2.11. *The set of  $(\mu, \nu)$  in  $M \times M$  with  $\mu$  absolutely continuous with respect to  $\nu$  is in  $\Sigma^* \times \Sigma^*$ , as are the set of pairs  $(\mu, \nu)$  with  $\mu$  equivalent to  $\nu$ , and the set with  $\mu$  and  $\nu$  mutually singular.*

*Proof.* Use (2.10) and (2.5).

Recall that an *atom* of  $\Sigma$  is a nonempty  $\Sigma$ -measurable set with no proper nonempty  $\Sigma$ -measurable subset. Write  $\alpha(\Sigma)$  for the collection of atoms of  $\Sigma$ . If  $\mu \in M$ , then  $\{A: A \in \alpha(\Sigma) \text{ and } \mu(A) > 0\}$  is countable. If this set of atoms is empty,  $\mu$  is *continuous*; if  $\|\mu\| = \Sigma\{\mu(A) : A \in \alpha(\Sigma)\}$ , then  $\mu$  is *atomic*. Any  $\mu \in M$  is the sum of a unique atomic  $\mu_a \in M$  and a unique continuous  $\mu_c \in M$ .

2.12. THEOREM. *The maps  $\mu \rightarrow \mu_a$  and  $\mu \rightarrow \mu_c$  are measurable from  $(M, \Sigma^*)$  to  $(M, \Sigma^*)$ .*

*Proof.* As usual, it suffices to verify that, for a fixed  $A \in \Sigma$ , the function  $\mu \rightarrow \mu_c(A)$  is  $\Sigma^+$ -measurable on  $M^+$ . For this purpose, let  $\{\Pi_n: n = 1, 2, \dots\}$  have the following properties:

(i) each  $\Pi_n$  is a partition of  $A$  into a finite number of elements of  $\Sigma$ ;

(ii)  $\Pi_{n+1}$  is a refinement of  $\Pi_n$ ;

(iii)  $A\Sigma$  is the smallest  $\sigma$ -field of subsets of  $A$  which includes  $\bigcup_n \Pi_n$ . For  $\delta > 0$ , let  $\varphi_{n,\delta}(\mu) = \Sigma\{\mu(B) : B \in \Pi_n, \mu(B) < \delta\}$ . Clearly,  $\varphi_{n,\delta}$  is  $\Sigma^+$ -measurable on  $M^+$ , and increases to a  $\Sigma^+$ -measurable function  $\varphi_\delta$  on  $M^+$  as  $n$  increases to  $\infty$ . As  $\delta$  decreases to 0 through a fixed sequence,  $\varphi_\delta$  decreases to a  $\Sigma^+$ -measurable function  $\varphi$  on  $M^+$ .

The argument will be completed by showing that  $\varphi(\mu) = \mu_c(A)$  for  $\mu \in M^+$ . If  $A_n \in \Pi_n$  and  $A_n \supset A_{n+1}$  for  $1 \leq n < \infty$ , then  $\bigcap_{n=1}^\infty A_n$  is empty or an atom of  $\Sigma$ , and in either case has  $\mu_c$ -measure 0. The famous lemma of König (1936, Theorem 6, page 81) then implies  $\lim_{n \rightarrow \infty} \max\{\mu_c(B) : B \in \Pi_n\} = 0$ ; so  $\varphi_\delta(\mu) \geq \mu_c(A)$ . For the reverse

inequality, if  $\varepsilon > 0$  there is a positive  $\delta$  so small that  $\Sigma\{\mu_a(B) : B \in \alpha(\Sigma), B \subset A, \mu_a(B) \leq \delta\} < \varepsilon$ , which implies  $\varphi_\delta(\mu) \leq \mu_c(A) + \varepsilon$ .

2.13 *Both the set of atomic measures and the set of continuous measures are measurable.*

*Proof.* (2.12) and (2.5).

2.14. *The set  $G$  of probability measures with  $\max\{\mu(A) : A \in \alpha(\Sigma)\} > 9/10$  is measurable. Let  $g$  be any function from  $G$  to  $X$  such that, for all  $\mu \in G$ , the  $\mu$ -measure of the  $\Sigma$ -atom containing  $g(\mu)$  is greater than  $9/10$ . Then  $g$  is  $G\Sigma^*$ -measurable.*

*Proof.* Adapt the argument for (2.12).

3. The compact metric case. If  $\Omega$  is a topological space,  $\sigma(\Omega)$  means the smallest  $\sigma$ -field of subsets of  $\Omega$  which includes the topology. In this section,  $X$  is a nonempty compact metric space, and  $\Sigma$  is  $\sigma(X)$ , the  $\sigma$ -field of Borel subsets of  $X$ . According to a famous theorem of Riesz,  $M$  can be identified with the dual of  $C(X)$ , the Banach space of all continuous real-valued functions on  $X$  with the sup norm:  $\|f\| = \max\{|f(x)| : x \in X\}$ . Unless otherwise noted,  $M$  has the weak \* topology; and subsets of  $M$  have the relative weak \* topology.

3.1. *The smallest  $\sigma$ -field including the weak \* topology of  $M$  is  $\Sigma^*$ ; that is,  $\Sigma^* = \sigma(M)$ .*

*Proof.* Easy.

Let  $P$  be the set of probability measures on  $(X, \Sigma)$ . Then  $P$  is a compact metrizable subset of  $M^+$ ; and  $M^+$  is a closed subset of  $M$ . It is less widely known that  $M^+$  is metrizable; it is complete and separable in this metric:

$$\rho^*(\mu, \nu) = \sum_{j=1}^{\infty} 2^{-j} \|f_j\|^{-1} \left| \int f_j d\mu - \int f_j d\nu \right|,$$

where  $\{f_j : 1 \leq j < \infty\}$  is dense in  $C(X)$ . Thus  $\Sigma^+ = \sigma(M^+)$  is the Borel  $\sigma$ -field of  $M^+$ , and  $P\Sigma^* = \sigma(P)$  is the Borel  $\sigma$ -field of  $P$ .

3.2. **THEOREM.** *If  $X$  is uncountable, there is a weakly open subset of  $M$  which is not  $\Sigma^*$ -measurable.*

*Proof.* Let  $N$  be a nonanalytic subset of  $X$ , and  $E = \{\mu : \mu \in M, \mu\{x\} > 9/10 \text{ for some } x \in N\}$ . Then  $E$  is weakly open. If  $EP$  were

an analytic subset of  $P$ , then—using the notation and result of (2.14)  $EP \subset G$  and  $N = g(EP)$  would be analytic, a contradiction.

Recall that the *support*  $C(\mu)$  of  $\mu \in M$  is the smallest closed subset  $K$  of  $X$  with  $|\mu|(X - K) = 0$ . It is familiar that a closed subset  $E$  of  $X$  includes  $C(\mu)$  if and only if  $\int f d\mu = 0$  for each  $f \in C(X)$  vanishing on  $E$ .

3.3. *If  $\mu_n \in M$ ,  $1 \leq n < \infty$  and  $\mu_n \rightarrow \mu \in M$ , then  $C(\mu)$  is a subset of the closure of  $\bigcup_{n=1}^{\infty} C(\mu_n)$ .*

*Proof.* Easy.

Let  $2^X$  be the space of nonempty closed subsets of  $X$ , endowed with the usual compact metric topology (Hausdorff, 1927, Section 28).

3.4. *If  $M_0$  is a metrizable subset of  $M$  and does not contain the zero measure, the restriction of  $C$  to  $M_0$  is lower semi-continuous in the sense of (Kuratowski, 1932, page 148).*

*Proof.* Use (3.3).

Let  $M_0$  be the set of nonzero elements of  $M$ .

3.5. *The map  $C$  is measurable from  $(M_0, \sigma(M_0))$  to  $(2^X, \sigma(2^X))$ .*

*Proof.*  $M_0$  is the countable union of metrizable sets. Then use (3.4) and (Kuratowski, 1932, page 152).

3.6. *For each  $K \in 2^X$ , the set of probability measures whose support is  $K$  is a  $G_\delta$  in  $P$ .*

*Proof.* Use (3.4) and (Kuratowski, 1932, page 151).

3.7. *The set of nonnegative measures whose supports have no isolated points is an  $F_{\sigma\delta}$  in  $M^+$ , as is the set of nonnegative measures whose supports have no interior.*

*Proof.* Since the set of perfect, nonempty subsets of  $X$  is a  $G_\delta$  in  $2^X$ , as is the set of closed, nowhere dense, nonempty subsets, (3.4) and (Kuratowski, 1932, page 152) apply.

3.8 *The real-valued function  $(\mu, K) \rightarrow \mu(K)$  is upper semi-continuous on  $M^+ \times 2^X$  with the product topology.*

*Proof.* Endow  $C(X)$  with the norm topology. There is a natural

embedding of  $2^X$  into  $C(X)$ : assign to  $K \in 2^X$  the function  $\hat{K} \in C(X)$  whose value at  $x \in X$  is

$$1 - [(\text{distance from } x \text{ to } K)/(\text{diameter of } X)].$$

As is easily verified,  $K \rightarrow \hat{K}$  is continuous (and 1-1, although this will not be used); moreover,  $\hat{K}^n$  decreases pointwise to the indicator of  $K$  as  $n$  increases to  $\infty$ . Since the function  $(\mu, f) \rightarrow \int f d\mu$  is continuous on  $M^+ \times C(X)$ , the functions  $(\mu, K) \rightarrow \int \hat{K}^n d\mu$  are continuous on  $M^+ \times 2^X$ . This sequence decreases pointwise to the function  $(\mu, K) \rightarrow \mu(K)$  as  $n$  increases to  $\infty$ .

3.9. *The function  $(\mu, K) \rightarrow \mu(K)$  is measurable from  $(M \times 2^X, \sigma(M) \times \sigma(2^X))$  to the Borel line.*

*Proof.* Use (2.8) and (3.8).

3.10. *The function  $(\mu, \nu) \rightarrow \mu[C(\nu)]$  is measurable from  $[M \times M, \sigma(M) \times \sigma(M)]$  to the Borel line.*

*Proof.* Use (3.5) and (3.9).

3.11. *The set of  $(\mu, K)$  in  $M^+ \times 2^X$  with  $\mu(K) = 0$  is a  $G_\delta$ .*

*Proof.* Use (3.8).

3.12. *For each dense subset  $G$  of  $X$  the set of  $\mu$  in  $P$  with  $\mu(G) = 1$  is dense in  $P$ .*

*Proof.* Approximate  $\mu \in P$  by a finite linear combination of point masses.

3.13. *The set  $P_+$  of  $\mu$  in  $P$  assigning positive probability to all nonempty open subsets of  $X$  is a dense  $G_\delta$  in  $P$ .*

*Proof.* For each open subset  $V$  of  $X$ ,  $\{\mu: \mu \in P, \mu(V) = 0\}$  is closed. Let  $\{V_n: 1 \leq n < \infty\}$  be a basis for the topology of  $X$ . Then  $P - P_+$  is  $\bigcup_{n=1}^{\infty} \{\mu: \mu \in P, \mu(V_n) = 0\}$ , an  $F_\sigma$ . Plainly,  $P_+$  is dense.

3.14. *The set of continuous  $\mu$  in  $P$  is a  $G_\delta$ . It is dense in  $P$  if and only if  $X$  has no isolated points.*

*Proof.* For the first assertion, if  $\delta > 0$ , then  $\{\mu: \mu \in P, \text{ and } \mu\{x\} \geq \delta \text{ for some } x \in X\}$  is closed. For the second, if  $X$  has no

isolated points, then each open subset of  $X$  has cardinality  $c$  and supports a continuous  $\mu \in P$ . The converse is easy.

3.15. *If  $G$  is a dense  $G_\delta$  in  $X$ , then the set  $G_1$  of  $\mu$  in  $P$  with  $\mu(G) = 1$  is a dense  $G_\delta$  in  $P$ .*

*Proof.* Let  $\{U_n: 1 \leq n < \infty\}$  be open sets whose intersection is  $G$ . Then  $G_1 = \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{\infty} \{\mu: \mu \in P, \mu(U_n) > 1 - j^{-1}\}$ , and (3.12) applies.

Any superset of a dense  $G_\delta$  is *residual*. The complement of a residual set is of the *first category*. A set not of the first category is of the *second category*.

3.16. *If  $F$  is of the first category in  $X$ , then  $F$  has outer measure 0 for a residual set of  $\mu$  in  $P$ .*

*Proof.* (3.15).

Recall that  $B \subset X$  has the *property of Baire* if there is an open subset of  $X$  whose symmetric difference with  $B$  is of the first category. For a discussion, see (Kuratowski, 1958, Section 11). If  $X$  is uncountable and  $\mu \in P$ , there are  $\mu$ -measurable sets without the property of Baire; if  $\mu$  is continuous, there are sets with the property of Baire whose inner  $\mu$ -measure is 0, and whose outer  $\mu$ -measure is 1. According to (Kuratowski, 1958, pages 421–423), there is a subset of  $X$  which is  $\mu$ -measurable for no continuous  $\mu \in P$ . There is, however, a connection between measurability and the property of Baire:

3.17. **THEOREM.** *If  $B$  is of the second category in  $X$  and has the property of Baire, then  $B$  is  $\mu$ -measurable and of positive  $\mu$ -measure for a residual set of  $\mu$  in  $P$ .*

*Proof.*  $B$  differs from a nonempty open set by a set of the first category. Apply (3.16) and (3.13).

3.18. *If  $\mu \in P$  and either  $X$  has no isolated points or  $\mu$  is continuous, then there is a dense  $G_\delta$  in  $X$  of  $\mu$ -measure 0.*

*Proof.* As in (Halmos, 1958, (4) on page 66).

3.19. *The set  $P_\perp$  of pairs  $(\mu, \nu)$  with  $\mu$  and  $\nu$  mutually singular is a  $G_\delta$  in  $P \times P$ . It is dense if and only if  $X$  has no isolated points.*

*Proof.* Let  $\{f_n: 1 \leq n < \infty\}$  be dense in the unit ball of  $C(X)$ ,



and let  $F_n(\mu, \nu) = \max_{1 \leq j \leq n} \left| \int f_j d\mu - \int f_j d\nu \right|$ . Then  $F_n$  is continuous on  $P \times P$  for each  $n$ , and the sequence  $\{F_n\}$  increases pointwise to  $F: (\mu, \nu) \rightarrow \|\mu - \nu\|$ . So  $F$  is lower semi-continuous, and  $P_1 = F^{-1}\{2\}$  is a  $G_\delta$ . For the second assertion, use (3.12) in one direction, and (3.13) in the other.

**3.20. THEOREM.** *For each  $\mu$  in  $P$ , the set  $\mu_\perp$  of  $\nu$  in  $P$  singular with respect to  $\mu$  is a  $G_\delta$ . If  $X$  has no isolated points or  $\mu$  is continuous, then  $\mu_\perp$  is dense in  $P$ .*

*Proof.*  $\mu_\perp$  is a  $G_\delta$  by (3.19), and dense by (3.12), (3.18).

There are reasonable sets of probability measures which are not Borel. A first example.

**3.21.** *If  $X$  is uncountable, the set of probability measures with uncountable support is analytic but not Borel; the set of probability measures with countably infinite support is analytic but not Borel.*

*Proof.* As reported in (Kuratowski and Szpilrajn, 1932, pages 166-169), the set of uncountable closed subsets of  $X$  is analytic but not Borel in  $2^X$ . To obtain the first assertion in (3.21), apply (Kuratowski and Szpilrajn, 1932, Proposition IV, page 163). The second follows from the first, because the set of probability measures whose support has  $k$  points or fewer is closed, for every natural number  $k$ .

*A second example:* it is natural to guess that the convex hull of a Borel set is Borel, especially since this happens to be true in two-dimensional Euclidean space. However,

**3.22. THEOREM.** *There is a  $G_\delta$  of the unit cube in three-dimensional Euclidean space whose convex hull is not Borel.*

*Proof.* Let  $A$  be a  $G_\delta$  of the unit square whose projection  $A^*$  on the  $x$ -axis is not Borel. Let  $A_n = \{(x, y) : 0 \leq x \leq 1, -\infty < y < \infty, (x, y - n) \in A\}$ , and  $A_\infty = \bigcup_{n=-\infty}^{\infty} A_{2n}$ . Let  $f$  be a homeomorphism of  $(0, 1)$  onto  $(-\infty, \infty)$ , and let  $B = \{(x, y) : 0 \leq x \leq 1, 0 < y < 1, (x, f(y)) \in A_\infty\}$ . For any  $\varepsilon$  in  $(0, 1)$ , the projection of  $B \cap \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \varepsilon\}$  or of  $B \cap \{(x, y) : 0 \leq x \leq 1, 1 - \varepsilon \leq y \leq 1\}$  onto the  $x$ -axis is  $A^*$ .

Let  $\varphi$  map the unit square into the unit cube by  $\varphi(x, y) = \{x, y, 1/2 - [1/4 - (x - 1/2)]^{1/2}\}$ . Thus  $\varphi$  maps the unit square homeo-

morphically onto a half-cylinder  $C$ , and  $\varphi(B)$  is a  $G_\delta$ . If its convex hull  $H$  were Borel, then  $\varphi^{-1}(C \cap H)$  would be Borel. Also the section of  $\varphi^{-1}(C \cap H)$  by the line  $y = 1/2, 0 \leq x \leq 1$ , namely  $A^*$  translated upward by  $1/2$ , would be Borel, a contradiction.

4. The unit interval. In this section,  $X$  is the closed unit interval.

4.1. *The set of  $\mu$  in  $P$  with well-ordered support is complementary analytic but not Borel in  $P$ .*

*Proof.* The set of closed, nonempty, well-ordered subsets of  $X$  is complementary analytic but not Borel in  $2^X$  (Kuratowski and Szpilrajn, 1932, page 166).

We conjecture that the set of probabilities whose support has a given order-type is Borel, but have verified this only for well-ordered order-types. More generally, for any compact metric space  $X$ , the collection of elements of  $2^X$  homeomorphic to a fixed  $K \in 2^X$  may be Borel. These conjectures have been confirmed in: Dana Scott, Invariant Borel sets, *Fund. Math.* 41 (1964) C. Ryll-Nardjewski, On Borel measurability of orbits, to appear \_\_\_\_\_, On Freedman's problem, to appear \_\_\_\_\_.

Other questions arise from differentiation. For each  $x$  in  $[0, 1)$  and real-valued function  $f$  on  $[0, 1)$ , the upper and lower right derivatives of  $f$  at  $x$  are

$$f^*(x) = \limsup_{y \rightarrow 0^+} y^{-1}[f(x + y) - f(x)]$$

and

$$f_*(x) = \liminf_{y \rightarrow 0^+} y^{-1}[f(x + y) - f(x)].$$

The next main result is (4.5). For the preliminaries (4.2)–(4.4), let  $A \in \Sigma^*$  and for  $\mu \in A$  suppose the real-valued function  $f_\mu$  on  $[0, 1)$  is continuous, and for each  $x \in [0, 1)$  the function  $\mu \rightarrow f_\mu(x)$  is measurable from  $(A, \sigma(A))$  to the Borel line.

4.2. *The function  $(x, \mu) \rightarrow f_\mu^*(x)$  is measurable from  $\{[0, 1) \times A, \sigma[0, 1) \times \sigma(A)\}$  to the extended Borel line.*

*Proof.* By a familiar argument, the function  $(x, \mu) \rightarrow f_\mu(x)$  is measurable; and

$$f_\mu^*(x) = \limsup_{n \rightarrow \infty} \left\{ \frac{f_\mu(x + r) - f_\mu(x)}{r}; 0 < r < n^{-1}, r \text{ rational} \right\}.$$

4.3. *If  $0 \leq a < b < 1$ , then the functions  $S_{[a,b)} : \mu \rightarrow \sup \{f_\mu^*(x) : a \leq x < b\}$  and  $I_{[a,b)} : \mu \rightarrow \inf \{f_\mu^*(x) : a \leq x < b\}$  are measurable from*

$(A, \sigma(A))$  to the extended Borel line.

*Proof.* By a theorem of Dini (Saks, 1937, page 204),

$$S_{[a,b]}(\mu) = \sup \left\{ \frac{f_\mu(y) - f_\mu(x)}{y - x} : a \leq x < y < b \right\},$$

where  $x$  and  $y$  can be restricted to rational values. An identical argument shows that  $I_{[a,b]}$  is measurable.

4.4. The set  $A_1$  of  $\mu \in A$  with  $f_\mu$  continuously differentiable on  $[0, 1)$  is measurable.

*Proof.* By the same result of Dini,  $f_\mu$  is continuously differentiable on  $[0, 1)$  if and only if  $f_\mu^*$  is continuous there. Hence  $A_1 = \bigcap_{j=1}^{\infty} B_j$ , where  $B_j$  is the set of  $\mu \in A$  with  $-\infty < I_{[0,1-2^{-j}]}(\mu) \leq S_{[0,1-2^{-j}]}(\mu) < \infty$  and

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq 2^n - 2^{n-j}} \{S_{[(k-1)/2^n, k/2^n]}(\mu) - I_{[(k-1)/2^n, k/2^n]}(\mu)\} = 0.$$

If  $\mu \in M$ , its distribution function  $F_\mu$  is defined as  $F_\mu(x) = \mu[0, x]$  for  $x \in X$ .

4.5. THEOREM. The set  $C_k$  of  $\mu \in M$  whose distribution function  $F_\mu$  has a  $k$ th continuous derivative  $F_\mu^{(k)}$  on  $[0, 1)$  is measurable; and the function  $(x, F_\mu^{(k)}) \rightarrow F_\mu^{(k)}(x)$  is measurable from  $\{[0, 1) \times C_k, \sigma[0, 1) \times \sigma(C_k)\}$  to the Borel line.

*Proof.* For  $k = 0$ , use (2.12). Then apply (4.4) and (4.2) inductively.

If a real-valued function  $f$  on  $[0, 1)$  is infinitely differentiable there, let  $S_n(f, x_0, x) = \sum_{j=0}^n f^{(j)}(x_0)/j! (x - x_0)^j$ . Then  $f$  is analytic on  $[0, 1)$  if each  $x_0 \in [0, 1)$  has a neighborhood  $N(x_0)$  in  $[0, 1)$  on which  $S_n(f, x_0, \cdot)$  converges uniformly to  $f$ .

4.6. THEOREM. The set of  $\mu$  in  $M$  with  $F_\mu$  analytic on  $[0, 1)$  is measurable.

*Proof.* The set  $C_\infty = \bigcap_{k=1}^{\infty} C_k$  is measurable, and for each  $x_0 \in X$ , the function  $(\mu, x) \rightarrow S_n(F_\mu, x_0, x)$  is measurable from  $\{C_\infty \times [0, 1), \sigma(C_\infty) \times \sigma[0, 1)\}$  to the Borel line. If  $J$  is an interval, then  $R_{n,J,x_0} : \mu \rightarrow \sup_{x \in J, 0 \leq x < 1} |F_\mu(x) - S_n(F_\mu, x_0, x)|$  is measurable on  $(C_\infty, \sigma(C_\infty))$ , since  $x$  can be restricted to rational values. Therefore, the

set  $A(x_0, J)$  of  $\mu \in C_\infty$  with  $\lim_{n \rightarrow \infty} R_{n,J,x_0}(\mu) = 0$  is in  $\sigma(C_\infty)$ . The set of  $\mu$  with  $F_\mu$  analytic on  $[0, 1)$  is

$$\bigcap_{h=1}^\infty \bigcap_{k=h}^\infty \bigcap_{j=0}^{2^k-2^{k-h}-1} A(j/2^k, [(j-1)/2^k, (j+1)/2^k]) .$$

Let  $Pr$  be the set of  $\mu$  in  $P$  which are continuous, singular with respect to Lebesgue measure, and assign positive measure to all nonempty open sets.

4.7. *The set  $Pr$  is a dense  $G_\delta$  in  $P$ .*

*Proof.* (3.14), (3.20), (3.13).

According to (Saks, 1937, Chapter IV), if  $\mu \in Pr$ , then  $F_\mu$  is differentiable with derivative 0 on a set of Lebesgue measure 1, and differentiable with derivative  $\infty$  on a set of  $\mu$ -measure 1. Topologically speaking, however,  $F_\mu$  is differentiable essentially nowhere:

4.8 THEOREM. *The set of pairs  $(x, \mu)$  with  $F_\mu^*(x) = \infty$  and  $F_{\mu^*}(x) = 0$  is a  $G_\delta$  in  $[0, 1) \times Pr$ . Each of its sections is dense.*

*Proof.* Let  $W$  be the set of pairs  $(x, \mu)$  in  $[0, 1) \times Pr$  with  $F_{\mu^*}(x) = 0$ , and  $W^*$  the set with  $F_\mu^*(x) = \infty$ . It is enough to prove that  $W$  and  $W^*$  are  $G_\delta$ 's with dense sections.

The complement of  $W$  in  $[0, 1) \times Pr$  is  $\bigcup_{n=1}^\infty C_n$ , where  $C_n$  is the intersection over all rational  $s$  in  $[0, 1)$  of  $\{(x, \mu): 0 \leq x < 1, \mu \in Pr, \text{ and either } x \geq s \text{ or } F_\mu(s) - F_\mu(x) \geq n^{-1}(s - x)\}$ . Since each  $C_n$  is closed,  $W$  is a  $G_\delta$ . Being disjoint from the dense set on which  $F_\mu$  has zero derivative, the section of  $C_n$  by  $\mu$  in  $Pr$  has no interior. The section of  $C_n$  by  $x$  in  $[0, 1)$  has no interior because, for  $\mu$  in  $Pr$ , arbitrarily small translates modulo 1 of  $\mu$  have distribution functions with 0 derivative at  $x$ . Then the sections of  $W$  are dense according to Baire's category argument.

The similar proof for  $W^*$  is omitted.

There is, of course, an analogous theorem for derivatives from the left.

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