

# Pacific Journal of Mathematics

**ON ABSTRACT AFFINE NEAR-RINGS**

HARRY GONSHOR

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1. **Introduction.** We shall limit ourselves to near-rings for which addition is commutative. They will be known as abelian near-rings. We assume that the distributive law  $(b + c)a = ba + ca$  holds, but the law  $a(b + c) = ab + ac$  does not necessarily hold. (This is consistent with the usual convention that the product  $AB$  of two operators  $A$  and  $B$  stands for  $B$  followed by  $A$ , e.g., consider the near-ring of all mappings of a group into itself.) Our aim is to generalize the results of [1] and [2] to a class of near-rings which we call abstract affine near-rings.

2. **Abelian near-rings.** We first define two subsets  $L(R)$  and  $C(R)$  of a near-ring  $R$ . (When convenient, we call these sets  $L$  and  $C$ .  $L(R)$  is the set of all elements  $a \in R$  which satisfy  $a(b + c) = ab + ac$  for all  $b$  and  $c$  in  $R$ .  $C(R)$  is the set of all elements  $a \in R$  which satisfy  $ab = a$  for all  $b$  in  $R$ . Note that, in general,  $0 \cdot a = 0$  and  $(-a)b = -(ab)$ .

PROPOSITION 1.  $L$  is a subring of  $B$ .

*Proof.* If  $a, b \in L$ , then

$$\begin{aligned}(a + b)(x + y) &= a(x + y) + b(x + y) = ax + ay + bx + by \\ &= (ax + bx) + (ay + by) = (a + b)x + (a + b)y,\end{aligned}$$

hence  $a + b \in L$ . Since  $0 \cdot a = 0$  for all  $a$ ,  $0 \in L$ . Also if  $a \in L$ , then

$$\begin{aligned}(-a)(x + y) &= -[a(x + y)] = -[ax + ay] = (-ax) + (-ay) \\ &= (-a)x + (-a)y,\end{aligned}$$

hence  $-a \in L$ . Furthermore if  $a, b \in L$ , then  $ab(x + y) = a(bx + by) = abx + aby$ , hence  $ab \in L$ . This completes the proof. Note that if  $R$  contains an identity  $e$ , then  $e \in L$ .

DEFINITION. An  $r$ -ideal is a subgroup closed under multiplication on the left and right by arbitrary elements of  $R$ . An ideal  $I$  is a subgroup closed under right multiplication by elements of  $R$  and which furthermore satisfies  $y(x + a) - yx \in I$  for all  $a \in I$ ,  $x \in R$ ,  $y \in R$ .

PROPOSITION 2.  $C$  is an  $r$ -ideal of  $R$ .

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*Proof.* If  $a, b \in C$ , then  $(a + b)x = ax + bx = a + b$ , hence  $a + b \in C$ .  $0 \cdot x = 0$ , hence  $0 \in C$ . If  $a \in C$ , then  $(-a)x = -(ax) = -a$ , hence  $-a \in C$ . If  $a \in C$ , then  $(ax)y = a(xy) = a = ax$  and  $(xa)y = x(ay) = xa$ . This proves the result.

**PROPOSITION 3.**  $L \cap C = 0$ .

*Proof.* Let  $a \in L \cap C$ . Let  $x$  be arbitrary in  $R$ . Then  $a = a(x + x) = ax + ax = a + a$ . Thus  $a = 0$ .

### 3. Abstract affine near-rings.

**DEFINITION.** An abstract affine near-ring  $R$  is an abelian near-ring  $R$  which satisfies  $R = C + L$ .  $C$  can be regarded as a module over  $L$ . If  $r \in L$  and  $a \in C$  define  $r \circ a = ra$ . The axioms for a module are clearly satisfied. Also if  $l_1, l_2 \in L$ , and  $c_1, c_2 \in C$ , then

$$\begin{aligned}(l_1 + c_1)(l_2 + c_2) &= l_1(l_2 + c_2) + c_1(l_2 + c_2) \\ &= l_1l_2 + l_1c_2 + c_1 = l_1l_2 + l_1 \circ c_2 + c_1\end{aligned}$$

Thus multiplication can be expressed in terms of the ring and module operations. Conversely, let  $M$  be any left  $R$  module. We make the group direct sum  $R \oplus M$  into a near-ring as follows. Let  $r_1, r_2 \in R$  and  $m_1, m_2 \in M$ . Define  $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1)$ .

**PROPOSITION 4.** With this definition for multiplication  $R \oplus M$  is an abstract affine near-ring with  $L(R \oplus M) = R, 0$  and  $C(R \oplus M) = (0, M)$ .

*Proof.*

$$\begin{aligned}[(r_1, m_1)(r_2, m_2)](r_3, m_3) &= (r_1r_2, r_1m_2 + m_1)(r_3, m_3) \\ &= (r_1r_2r_3, r_1r_2m_3 + r_1m_2 + m_1) \\ (r_1, m_1)[(r_2, m_2)(r_3, m_3)] &= (r_1, m_1)(r_2r_3, r_2m_3 + m_2) \\ &= (r_1r_2r_3, r_1r_2m_3 + r_1m_2 + m_1).\end{aligned}$$

This verifies the associative law.

$$\begin{aligned}[(r_1, m_1) + (r_2, m_2)](r_3, m_3) &= (r_1 + r_2, m_1 + m_2)(r_3, m_3) \\ &= [(r_1 + r_2)r_3, (r_1 + r_2)m_3 + m_1 + m_2] \\ (r_1, m_1)(r_3, m_3) + (r_2, m_2)(r_3, m_3) &= (r_1r_3, r_1m_3 + m_1) + (r_2r_3, r_2m_3 + m_2) \\ &= (r_1r_3 + r_2r_3, r_1m_3 + r_2m_3 + m_1 + m_2).\end{aligned}$$

This verifies the distributive law. Hence  $R \oplus M$  is an abelian near-ring. Furthermore,

$$\begin{aligned} (r_1, 0)[(r_2, m_2) + (r_3, m_3)] &= (r_1, 0)(r_2 + r_3, m_2 + m_3) \\ &= [r_1(r_2 + r_3), r_1(m_2 + m_3)] . \\ (r_1, 0)(r_2, m_2) + (r_1, 0)(r_3, m_3) &= (r_1r_2, r_1m_2) + (r_1r_3, r_1m_3) \\ &= (r_1r_2 + r_1r_3, r_1m_2 + r_1m_3) . \end{aligned}$$

Hence  $(r_1, 0) \in L$ ,  $(0, m_1)(r_2, m_2) = (0r_2, 0m_2 + m_1) = (0, m_1)$ . Hence  $(0, m_1) \in C$ . Since  $L \cap C = 0$ . This completes the proof.

We are now ready to discuss the connection with [1] and [2]. Embed  $M$  in a module  $M_1$  so that  $R$  is faithful, i.e.,  $rm = 0$  for all  $m \in M_1$  implies  $r = 0$ . This can always be done. If the element  $(r, m) \in R \oplus M$  is identified with the map of  $M_1$  into itself defined by  $x \rightarrow rx + m$  for all  $x \in M_1$ , we obtain an isomorphism of the abstract affine near-ring and a near-ring of maps of  $M_1$  into  $M_1$ . (It is easily verified that the operations are preserved.) Furthermore each map is the sum of an endomorphism and a constant map. Thus the near-ring considered in [2] corresponds to the special case where  $M$  is a vector space and  $R$  is the ring of all linear transformations.

4. The Ideals in  $R \oplus M$ . Henceforth we write  $r + m$  for  $(r, m)$ . We now classify the ideals and  $r$ -ideals of  $R \oplus M$ . Let  $J$  be an ideal or an  $r$ -ideal and let  $r + m \in J$ . Then  $(r + m)0 \in J$ , i.e.,  $m \in J$ . Thus  $r \in J$ . This shows that  $J = R_1 \oplus M_1$  where  $R_1$  and  $M_1$  are subgroups of  $R$  and  $M$  respectively. If  $r_1 \in R_1$  and  $r \in R$ , then  $r_1r$  and  $rr_1$  are in  $J$ , hence in  $R_1$ . Thus  $R_1$  is an ideal in  $R$ . (Note that an ideal is closed under left multiplication by elements of  $L(R \oplus M)$ .) If  $m_1 \in M_1$  and  $r \in R$ , then  $rm_1 \in J$ . Hence  $rm_1 \in M_1$ . Thus  $M_1$  is a submodule of  $M$ .

At this point we consider the ideals and  $r$ -ideals separately. Let  $J$  be an  $r$ -ideal. Let  $m \in M$ . Since  $0 \in J, m = m \cdot 0 \in J$ . Hence  $M_1 = M$ . Thus all  $r$ -ideals have the form  $R_1 \oplus M$  where  $R_1$  is an ideal in  $R$ . Conversely, let  $J$  be any set of the form  $R_1 \oplus M$  where  $R_1$  is an ideal of  $R$ . Clearly,  $J$  is a subgroup. Let  $r_1 \in R_1, r \in R, m_1 \in M_1$  and  $m \in M$ . Then  $(r_1 + m_1)(r + m) = r_1r + r_1m + m_1 \in R_1 \oplus M$  and  $(r + m)(r_1 + m_1) = rr_1 + rm_1 + m \in R_1 \oplus M$ . Thus  $J$  is an  $r$ -ideal.

Now let  $J$  be an ideal. Let  $r_1 \in R_1$  and  $m \in M$ . Then  $r_1m \in J$ . Hence  $R_1M \subset M_1$ . (Note that left multiplication by elements of  $M$  give no new information since  $m(y + x) - mx = 0$  for all  $m \in M$  and  $x, y \in R$ .) Conversely, let  $J$  be of the form  $R_1 \oplus M_1$  where  $R_1$  is an ideal of  $R$  and  $M_1$  is a submodule of  $M$  containing  $R_1M$ . Again  $J$  is a subgroup. Let  $r_1 \in R_1, m_1 \in M_1, r \in R$  and  $m \in M$ . Then

$$(r_1 + m_1)(r + m) = r_1r + r_1m + m_1 \in R_1 + R_1M + M_1 \subset R_1 \oplus M_1 = J.$$

On the left it suffices to check with  $r$  and  $m$  separately. For  $r$  we

may use left multiplication. Thus  $r(r_1 + m_1) = rr_1 + rm_1 \in R_1 \oplus M_1$ . For  $m$  the result is automatically 0 since  $mx = my$  for all  $x, y \in R$ . Thus  $J$  is an ideal.

We have proved the following theorem.

**THEOREM.** *The  $r$ -ideals of  $R \oplus M$  are exactly the sets of the form  $R_1 \oplus M$  where  $R_1$  is an ideal of  $R$ . The ideals of  $R \oplus M$  are exactly the sets of the form  $R_1 \oplus M_1$  where  $R_1$  is an ideal of  $R$  and  $M_1$  is a submodule of  $M$  containing  $R_1M$ . Thus every  $r$ -ideal is an ideal.*

In the special case considered in [2],  $M$  is a simple  $R$  module and  $R_1M = M$  for all ideals  $R \neq 0$ . Thus the result there that classifies all ideals other than (0) as those sets which have the form  $R_1 \oplus M$  where  $R_1$  is an ideal of  $R$  follows from our theorem.

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