

Pacific Journal of Mathematics

ON ABSTRACT AFFINE NEAR-RINGS

HARRY GONSHOR

ON ABSTRACT AFFINE NEAR-RINGS

HARRY GONSHOR

1. **Introduction.** We shall limit ourselves to near-rings for which addition is commutative. They will be known as abelian near-rings. We assume that the distributive law $(b + c)a = ba + ca$ holds, but the law $a(b + c) = ab + ac$ does not necessarily hold. (This is consistent with the usual convention that the product AB of two operators A and B stands for B followed by A , e.g., consider the near-ring of all mappings of a group into itself.) Our aim is to generalize the results of [1] and [2] to a class of near-rings which we call abstract affine near-rings.

2. **Abelian near-rings.** We first define two subsets $L(R)$ and $C(R)$ of a near-ring R . (When convenient, we call these sets L and C . $L(R)$ is the set of all elements $a \in R$ which satisfy $a(b + c) = ab + ac$ for all b and c in R . $C(R)$ is the set of all elements $a \in R$ which satisfy $ab = a$ for all b in R . Note that, in general, $0 \cdot a = 0$ and $(-a)b = -(ab)$).

PROPOSITION 1. L is a subring of B .

Proof. If $a, b \in L$, then

$$\begin{aligned}(a + b)(x + y) &= a(x + y) + b(x + y) = ax + ay + bx + by \\ &= (ax + bx) + (ay + by) = (a + b)x + (a + b)y,\end{aligned}$$

hence $a + b \in L$. Since $0 \cdot a = 0$ for all $a, 0 \in L$. Also if $a \in L$, then

$$\begin{aligned}(-a)(x + y) &= -[a(x + y)] = -[ax + ay] = (-ax) + (-ay) \\ &= (-a)x + (-a)y,\end{aligned}$$

hence $-a \in L$. Furthermore if $a, b \in L$, then $ab(x + y) = a(bx + by) = abx + aby$, hence $ab \in L$. This completes the proof. Note that if R contains an identity e , then $e \in L$.

DEFINITION. An r -ideal is a subgroup closed under multiplication on the left and right by arbitrary elements of R . An ideal I is a subgroup closed under right multiplication by elements of R and which furthermore satisfies $y(x + a) - yx \in I$ for all $a \in I, x \in R, y \in R$.

PROPOSITION 2. C is an r -ideal of R .

Received October 7, 1963.

Proof. If $a, b \in C$, then $(a + b)x = ax + bx = a + b$, hence $a + b \in C$. $0 \cdot x = 0$, hence $0 \in C$. If $a \in C$, then $(-a)x = -(ax) = -a$, hence $-a \in C$. If $a \in C$, then $(ax)y = a(xy) = a = ax$ and $(xa)y = x(ay) = xa$. This proves the result.

PROPOSITION 3. $L \cap C = 0$.

Proof. Let $a \in L \cap C$. Let x be arbitrary in R . Then $a = a(x + x) = ax + ax = a + a$. Thus $a = 0$.

3. Abstract affine near-rings.

DEFINITION. An abstract affine near-ring R is an abelian near-ring R which satisfies $R = C + L$. C can be regarded as a module over L . If $r \in L$ and $a \in C$ define $r \circ a = ra$. The axioms for a module are clearly satisfied. Also if $l_1, l_2 \in L$, and $c_1, c_2 \in C$, then

$$\begin{aligned} (l_1 + c_1)(l_2 + c_2) &= l_1(l_2 + c_2) + c_1(l_2 + c_2) \\ &= l_1l_2 + l_1c_2 + c_1 = l_1l_2 + l_1 \circ c_2 + c_1 \end{aligned}$$

Thus multiplication can be expressed in terms of the ring and module operations. Conversely, let M be any left R module. We make the group direct sum $R \oplus M$ into a near-ring as follows. Let $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Define $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1)$.

PROPOSITION 4. With this definition for multiplication $R \oplus M$ is an abstract affine near-ring with $L(R \oplus M) = R, 0$ and $C(R \oplus M) = (0, M)$.

Proof.

$$\begin{aligned} [(r_1, m_1)(r_2, m_2)](r_3, m_3) &= (r_1r_2, r_1m_2 + m_1)(r_3, m_3) \\ &= (r_1r_2r_3, r_1r_2m_3 + r_1m_2 + m_1) \\ (r_1, m_1)[(r_2, m_2)(r_3, m_3)] &= (r_1, m_1)(r_2r_3, r_2m_3 + m_2) \\ &= r_1r_2r_3, r_1r_2m_3 + r_1m_2 + m_1. \end{aligned}$$

This verifies the associative law.

$$\begin{aligned} [(r_1, m_1) + (r_2, m_2)](r_3, m_3) &= (r_1 + r_2, m_1 + m_2)(r_3, m_3) \\ &= [(r_1 + r_2)r_3, (r_1 + r_2)m_3 + m_1 + m_2] \\ (r_1, m_1)(r_3, m_3) + (r_2, m_2)(r_3, m_3) &= (r_1r_3, r_1m_3 + m_1) + (r_2r_3, r_2m_3 + m_2) \\ &= (r_1r_3 + r_2r_3, r_1m_3 + r_2m_3 + m_1 + m_2). \end{aligned}$$

This verifies the distributive law. Hence $R \oplus M$ is an abelian near-ring. Furthermore,

$$\begin{aligned}
 (r_1, 0)[(r_2, m_2) + (r_3, m_3)] &= (r_1, 0)(r_2 + r_3, m_2 + m_3) \\
 &= [r_1(r_2 + r_3), r_1(m_2 + m_3)]. \\
 (r_1, 0)(r_2, m_2) + (r_1, 0)(r_3, m_3) &= (r_1r_2, r_1m_2) + (r_1r_3, r_1m_3) \\
 &= (r_1r_2 + r_1r_3, r_1m_2 + r_1m_3).
 \end{aligned}$$

Hence $(r_1, 0) \in L$, $(0, m_1)(r_2, m_2) = (0r_2, 0m_2 + m_1) = (0, m_1)$. Hence $(0, m_1) \in C$. Since $L \cap C = 0$. This completes the proof.

We are now ready to discuss the connection with [1] and [2]. Embed M in a module M_1 so that R is faithful, i.e., $rm = 0$ for all $m \in M_1$ implies $r = 0$. This can always be done. If the element $(r, m) \in R \oplus M$ is identified with the map of M_1 into itself defined by $x \rightarrow rx + m$ for all $x \in M_1$, we obtain an isomorphism of the abstract affine near-ring and a near-ring of maps of M_1 into M_1 . (It is easily verified that the operations are preserved.) Furthermore each map is the sum of an endomorphism and a constant map. Thus the near-ring considered in [2] corresponds to the special case where M is a vector space and R is the ring of all linear transformations.

4. The Ideals in $R \oplus M$. Henceforth we write $r + m$ for (r, m) . We now classify the ideals and r -ideals of $R \oplus M$. Let J be an ideal or an r -ideal and let $r + m \in J$. Then $(r + m)0 \in J$, i.e., $m \in J$. Thus $r \in J$. This shows that $J = R_1 \oplus M_1$ where R_1 and M_1 are subgroups of R and M respectively. If $r_1 \in R_1$ and $r \in R$, then r_1r and rr_1 are in J , hence in R_1 . Thus R_1 is an ideal in R . (Note that an ideal is closed under left multiplication by elements of $L(R \oplus M)$.) If $m_1 \in M_1$ and $r \in R$, then $rm_1 \in J$. Hence $rm_1 \in M_1$. Thus M_1 is a submodule of M .

At this point we consider the ideals and r -ideals separately. Let J be an r -ideal. Let $m \in M$. Since $0 \in J$, $m = m \cdot 0 \in J$. Hence $M_1 = M$. Thus all r -ideals have the form $R_1 \oplus M$ where R_1 is an ideal in R . Conversely, let J be any set of the form $R_1 \oplus M$ where R_1 is an ideal of R . Clearly, J is a subgroup. Let $r_1 \in R_1$, $r \in R$, $m_1 \in M_1$ and $m \in M$. Then $(r_1 + m_1)(r + m) = r_1r + r_1m + m_1 \in R_1 \oplus M$ and $(r + m)(r_1 + m_1) = rr_1 + rm_1 + m \in R_1 \oplus M$. Thus J is an r -ideal.

Now let J be an ideal. Let $r_1 \in R_1$ and $m \in M$. Then $r_1m \in J$. Hence $R_1M \subset M_1$. (Note that left multiplication by elements of M give no new information since $m(y + x) - mx = 0$ for all $m \in M$ and $x, y \in R$.) Conversely, let J be of the form $R_1 \oplus M_1$ where R_1 is an ideal of R and M_1 is a submodule of M containing R_1M . Again J is a subgroup. Let $r_1 \in R_1$, $m_1 \in M_1$, $r \in R$ and $m \in M$. Then

$$(r_1 + m_1)(r + m) = r_1r + r_1m + m_1 \in R_1 + R_1M + M_1 \subset R_1 \oplus M_1 = J.$$

On the left it suffices to check with r and m separately. For r we

may use left multiplication. Thus $r(r_1 + m_1) = rr_1 + rm_1 \in R_1 \oplus M_1$. For m the result is automatically 0 since $mx = my$ for all $x, y \in R$. Thus J is an ideal.

We have proved the following theorem.

THEOREM. *The r -ideals of $R \oplus M$ are exactly the sets of the form $R_1 \oplus M$ where R_1 is an ideal of R . The ideals of $R \oplus M$ are exactly the sets of the form $R_1 \oplus M_1$ where R_1 is an ideal of R and M_1 is a submodule of M containing R_1M . Thus every r -ideal is an ideal.*

In the special case considered in [2], M is a simple R module and $R_1M = M$ for all ideals $R \neq 0$. Thus the result there that classifies all ideals other than (0) as those sets which have the form $R_1 \oplus M$ where R_1 is an ideal of R follows from our theorem.

BIBLIOGRAPHY

1. D. W. Blackett, *The near-ring of affine transformations*, Proc. Amer. Math. Soc., **7** (1956), 517-519.
2. K. G. Wolfson, *Two-sided ideals of the affine near-ring*, Amer. Math. Monthly, **65** (1958), 29-30.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

ROBERT OSSERMAN

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

J. DUGUNDJI

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$3.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal but they are not owners or publishers and have no responsibility for its content or policies.

Homer Franklin Bechtell, Jr., <i>Pseudo-Frattini subgroups</i>	1129
Thomas Kelman Boehme and Andrew Michael Bruckner, <i>Functions with convex means</i>	1137
Lutz Bungart, <i>Boundary kernel functions for domains on complex manifolds</i>	1151
L. Carlitz, <i>Rings of arithmetic functions</i>	1165
D. S. Carter, <i>Uniqueness of a class of steady plane gravity flows</i>	1173
Richard Albert Dean and Robert Harvey Oehmke, <i>Idempotent semigroups with distributive right congruence lattices</i>	1187
Lester Eli Dubins and David Amiel Freedman, <i>Measurable sets of measures</i>	1211
Robert Pertsch Gilbert, <i>On class of elliptic partial differential equations in four variables</i>	1223
Harry Gonshor, <i>On abstract affine near-rings</i>	1237
Edward Everett Grace, <i>Cut points in totally non-semi-locally-connected continua</i>	1241
Edward Everett Grace, <i>On local properties and G_δ sets</i>	1245
Keith A. Hardie, <i>A proof of the Nakaoka-Toda formula</i>	1249
Lowell A. Hinrichs, <i>Open ideals in $C(X)$</i>	1255
John Rolfe Isbell, <i>Natural sums and abelianizing</i>	1265
G. W. Kimble, <i>A characterization of extremals for general multiple integral problems</i>	1283
Nand Kishore, <i>A representation of the Bernoulli number B_n</i>	1297
Melven Robert Krom, <i>A decision procedure for a class of formulas of first order predicate calculus</i>	1305
Peter A. Lappan, <i>Identity and uniqueness theorems for automorphic functions</i>	1321
Lorraine Doris Lavallee, <i>Mosaics of metric continua and of quasi-Peano spaces</i>	1327
Mark Mahowald, <i>On the normal bundle of a manifold</i>	1335
J. D. McKnight, <i>Kleene quotient theorems</i>	1343
Charles Kimbrough Megibben, III, <i>On high subgroups</i>	1353
Philip Miles, <i>Derivations on B^* algebras</i>	1359
J. Marshall Osborn, <i>A generalization of power-associativity</i>	1367
Theodore G. Ostrom, <i>Nets with critical deficiency</i>	1381
Elvira Rapaport Strasser, <i>On the defining relations of a free product</i>	1389
K. Rogers, <i>A note on orthogonal Latin squares</i>	1395
P. P. Saworotnow, <i>On continuity of multiplication in a complemented algebra</i>	1399
Johan Schoneheim, <i>On coverings</i>	1405
Victor Lenard Shapiro, <i>Bounded generalized analytic functions on the torus</i>	1413
James D. Stafney, <i>Arens multiplication and convolution</i>	1423
Daniel Sterling, <i>Coverings of algebraic groups and Lie algebras of classical type</i>	1449
Alfred B. Willcox, <i>Šilov type C algebras over a connected locally compact abelian group. II</i>	1463
Bertram Yood, <i>Faithful $*$-representations of normed algebras. II</i>	1475
Alexander Zabrodsky, <i>Covering spaces of paracompact spaces</i>	1489