

Pacific Journal of Mathematics

ON THE DEFINING RELATIONS OF A FREE PRODUCT

ELVIRA RAPAPORT STRASSER

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If $F_n = F(x_1, \dots, x_n)$ is the free group generated by the symbols x_1, \dots, x_n , and $R_i = R_i(x_1, \dots, x_n)$, $i: 1, \dots, k$ are element in it, let

$$R = \{R_1, \dots, R_k\}$$

be their normal closure in F_n , and let

$$F/R = (x_1, \dots, x_n; R_1, \dots, R_k)$$

be the factor group of F_n by R , with n and k assumed finite.

My object is to prove the following theorem and corollaries.

THEOREM. *If*

$$F/R = (x_1, \dots, x_n; R_1, \dots, R_k)$$

and

$$F^*/R^* = (x_1^*, \dots, x_{n^*}^*; R_1^*, \dots, R_{k^*}^*)$$

are isomorphic groups and $n^ + k < n + k^*$, then in the free group F_{n+n^*} generated by $x_1^*, \dots, x_{n^*}^*, z_1, \dots, z_n$ the normal subgroup $\{R_1^*(x_1^*, \dots, x_{n^*}^*), \dots, R_{k^*}^*(x_1^*, \dots, x_{n^*}^*), z_1, \dots, z_n\}$ is normal closure of $n^* + k < n + k^*$ elements.*

The two corollaries concern the cases $k \leq 1$ and $k^* \leq 1$ — that is, free groups and groups possessing presentations on a single defining relation. (Deficiency is defined in Remark 1. below.)

COROLLARY 1. *If a group possesses the two presentations of the theorem, then the one with the lesser deficiency has at least two defining relations ($k^* > 1$).*

COROLLARY 2. *If $G = F/R$ and $k \leq 1$, then $n - k = d$ is maximal for all presentations of G .*

The theorem is trivial if for example certain of the k^* defining relations are redundant. It becomes interesting when the number k^* is least possible for the subgroup R^* . For suppose that k^* is minimal for R^* :

$$R^* = \{R_1^*, \dots, R_{k^*}^*\} \neq \{S_1, \dots, S_{k^*-1}\}$$

for any elements S_1, \dots, S_{k^*-1} of F_{n^*} . Then if z_1, \dots, z_n are new symbols and F_{n+n^*} the free group on the symbols $x_1^*, \dots, x_{n^*}^*, z_1, \dots, z_n$, the normal closure of $R_1^*, \dots, R_{k^*}^*$ in F_{n+n^*} still requires k^* elements to generate it. Consider however the normal subgroup

$$T = \{R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n\}$$

of F_{n+n^*} . The R_j^* do not involve the symbols z_i ; k^* is least possible for R^* , and, of course, n is least possible for the group

$$(z_1, \dots, z_n; z_1, \dots, z_n) = 1.$$

Now the group

$$(x_1^*, \dots, x_{n^*}^*, z_1, \dots, z_n; R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n)$$

is the free product $G_1 * G_2$ of $G_1 = F^*/R^*$ and G_2 the trivial group $(z_1, \dots, z_n; z_1, \dots, z_n)$. The theorem claims that the sum of the (minimal) numbers of defining relations for the G_i is not always minimal for $G_1 * G_2$.

Compare this with Grushko's theorem, which implies that the number of generators of a free product is the sum of those for the factors.

REMARK 1. If one takes the number k in the presentation F/R (of any group) to be least possible for R in F , then this presentation is said to have the deficiency

$$d = n - k.$$

Thus, setting $d^* = n^* - k^*$, the inequality $n^* + k < n + k^*$ is the same as

$$d^* < d,$$

provided that k is minimal for R and k^* is minimal for R^* in their respective free groups. The deficiency of a presentation is not a group invariant [1].

REMARK 2. The group with noninvariant deficiency given in [1] happens to be non-Hopfian (it is isomorphic to a proper factorgroup of itself: $G \simeq G/N$, $N \neq 1$). It is not known whether such groups must be non-Hopfian; however, two presentations, G and G/N of a non-Hopfian group may have identical deficiencies even if N is not trivial. That is, if a group is given by the presentation

$$G_1 = F/R = (x_1, \dots, x_n; R_1, \dots, R_k)$$

and is isomorphic to its own proper factorgroup given by

$$G_2 = F/R^* = (G_1; R_{k+1}) ,$$

it need not follow that $k + 1$ is minimal for R^* whenever k is minimal for R . For example, the first known pair of presentations of a non-Hopfian group, due to Higman and quoted in [2], is

$$\begin{aligned} G_1 &= (a, b, c; a^{-1}bab^{-2}, bcb^{-1}c^{-1}) \\ G_2 &= (G_1; ab^{-1}a^{-1}c^{-1}aba^{-1}c) , \end{aligned}$$

but $R^* = \{R_1, R_2, R_3\} = \{R_1, R_3\}$ (the R_i being the three defining words above in the order given).

Proof of the theorem. Set

$$x = (x_1, \dots, x_n), x^* = (x_1^*, \dots, x_n^*), f(x^*) = (f_1(x^*), \dots, f_n(x^*)) ,$$

etc., and let

$$\begin{aligned} G_1 &= F/R , \\ G_2 &= IG_1 = F^*/R^* , \end{aligned}$$

with the isomorphism I given by

$$\begin{aligned} I(x_i) &= f_i(x^*), \quad i : 1, \dots, n , \\ I(g_j(x)) &= x_j^*, \quad j : 1, \dots, n^* . \end{aligned}$$

Form the groups

$$H_1 = \overset{\circ}{F}/\overset{\circ}{R} = (x_1, \dots, x_n, y_1, \dots, y_{n^*}; R_1, \dots, R_k, y_j g_j^{-1}(x), \quad j : 1, \dots, n^*)$$

and

$$\begin{aligned} H_2 = \overset{\circ}{F^*}/\overset{\circ}{R^*} &= (x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*; R_1^*, \dots, R_{k^*}^*, y_i^* f_i^{-1}(x^*), \\ &\quad i : 1, \dots, n) . \end{aligned}$$

These are new presentations of the same group and the following mapping, J_1 , defines, and isomorphism between H_1 and H_2 [cf. 4]:

$$\begin{aligned} J_1(x_i) &= y_i^*, \quad i : 1, \dots, n \\ J_1(y_j) &= x_j^*, \quad j : 1, \dots, n^* . \end{aligned}$$

Since

$$x_j^* = I(g_j(x)) = g_j(I(x)) = g_j(f(x^*)) \text{ modulo } R^*$$

and

$$R(f(x^*)) = 1 \text{ modulo } R^*$$

in G_2 , and hence in H_2 , one gets the following identities in H_2 :

$$J_1(R) = R(y^*) = R(f(x^*)) = 1 ,$$

$$J_1(y_j g_j^{-1}(x)) = x_j^* g_j^{-1}(y^*) = x_j^* g_j^{-1}(f(x^*)) = 1 .$$

Clearly, J_1 maps not only H_1 on H_2 , but also $\overset{\circ}{F}$ on $\overset{\circ}{F}^*$, and $\overset{\circ}{R}$ on $\overset{\circ}{R}^*$, isomorphically.

Finally let

$$H_3 = F'/R' = (x_1^*, \dots, x_{n^*}^*, z_1, \dots, z_n ; R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n) .$$

Under the transformation J_2 given by

$$J_2(y_i^*) = z_i f_i(x^*), \quad i : 1, \dots, n ,$$

$$J_2(x_j^*) = x_j^*, \quad j : 1, \dots, n^* ,$$

H_2 is mapped isomorphically on H_3 and J_2 is also an isomorphism between the free groups involved :

$$J_2(\overset{\circ}{F}^*) = F' \quad J_2(\overset{\circ}{R}^*) = R' .$$

Writing $J = J_2 J_1$ gives $JH_1 = H_3$, $J\overset{\circ}{F} = F'$, $J\overset{\circ}{R} = R'$, since J maps two free groups of equal rank (namely $n + n^*$) onto each other. Hence

$$\overset{\circ}{JR} = \{JR_1, \dots, JR_k, J(y_1 g_1^{-1}(x)), \dots, J(y_{n^*} g_{n^*}^{-1}(x))\}$$

$$= R' = \{R_1^*, \dots, R_{k^*}^*, z_1, \dots, z_n\} ,$$

showing that R' is the normal closure of $n^* + k$ elements in spite of the possibility that k^* is minimal for R^* and that the R_i^* contain no z -symbols.

This concludes the proof.

Proof of Corollary 1. If $k^* = 0$ then R' becomes $\{z_1, \dots, z_n\}$, the normal closure of a free factor of F' , and so requires $n = n + k^* > n^* + k$ defining elements. This contradicts the theorem, so $k^* \neq 0$.

If $k^* = 1$, then $R' = \{R_1^*, z_1, \dots, z_n\} = \{s_1, \dots, s_n\}$ obtains. Now R' contains the free factor $F(z_1, \dots, z_n)$ of F' and is the closure of n elements in F' . Therefore, by [3], $R' = \{z_1, \dots, z_n\}$; and since the R_j^* contain no z -symbols, R_1^* is empty, so $k^* = 0$. Hence $k^* > 1$.

Proof of Corollary 2. Suppose, on the contrary, that there is a presentation with deficiency $d^{**} > d$. Then k can play the role of k^* above and Corollary 1 is contradicted.

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