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1. Introduction. We shall operate in Euclidean k-space, E_k , $k \ge 2$, and use the following notation:

$$egin{aligned} &x=(x_1,\,\cdots,\,x_k)\;; \quad y=(y_1,\,\cdots,\,y_k)\;;\ &lpha x+eta y=(lpha x_1+eta y_1,\,\cdots,\,lpha x_k+eta y_k)\;;\ &(x,\,y)=x_1y_1+\cdots+x_ky_k\;; \quad |x|=(x,\,x)^{1/2}\;. \end{aligned}$$

 T_k will designate the k-dimensional torus $\{x; -\pi < x_j \le \pi, j = 1, \dots, k\}$, v will always designate a point a distance one from the origin, i.e., |v| = 1, and m will always designate an integral lattice point. If f is in L^1 on T_k , then $\hat{f}(m)$ will designate the mth Fourier coefficient of f, i.e., $(2\pi)^{-k} \int_{\pi_k} f(x) e^{-i(m,x)} dx$.

We shall say that $f(x) \stackrel{i^*}{\text{in}} L^1$ on T_k is a generalized analytic function on T_k if there exists v such that f is in A_v , where $A_v = A_v^+ \cup A_{-v}^+$, and A_v^+ is defined as follows:

f is in A_v^+ if there exists an m_0 such that if $m \neq m_0$ and $(m - m_0, v) \leq 0$, then $\widehat{f}(m) = 0$.

We shall say that f(x) in L^1 on T_k is a strictly generalized anaic function on T_k if there exists a v such that f is in B_v , where $B_v = B_v^+ \cup B_{-v}^+$, and B_v^+ is defined as follows:

 $f ext{ is in } B_v^+ ext{ if there exists an } m_0 ext{ and } a ext{ } \gamma ext{ with } 0 < \gamma < 1 ext{ such that if } (m - m_0, v) < \gamma | m - m_0 |, ext{ then } \widehat{f}(m) = 0.$

It is quite clear that $B_v \subset A_v$. In this paper, we shall obtain a result which is false for bounded functions in A_v but which is true for bounded functions in B_v . It is primarily with the class B_v and its extension to finite complex measures that the classical paper of Bochner [2, p. 718] is concerned. On T_k , it is essentially with the class A_v that the papers of Helson and Lowdenslager [5], [6], and de Leeuw and Glicksberg [4] are concerned.

We shall be concerned in this paper with a class of functions C_v which for bounded functions is intermediate between the two classes B_v and A_v .

We first note that if f is in B_v^+ , then $\sum_m |\hat{f}(m)| e^{(m,v)\sigma} < \infty$ for every $\sigma < 0$. For with $||f||_p$, $1 \leq p \leq \infty$, designating the L^p -norm of f on T_k , we see that there exists a γ with $0 < \gamma < 1$ and an m_0 such that

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$$\sum_{m} |\widehat{f}(m)| e^{(m,v)\sigma} \leq ||f||_{1} \sum_{\gamma \mid m-m_0 \mid \leq (m-m_0,v)} e^{(m,v)\sigma},$$

and

$$\sum_{\gamma\mid m-m_0\mid\leq (m-m_0,v)} e^{(m,v)\sigma} \leq e^{(m_0,v)\sigma} \sum_m e^{\gamma\mid m-m_0\mid\sigma} < \infty \ .$$

Next, we note that if $\sum_{m} |\hat{f}(m)| e^{(m,v)\sigma_0} < \infty$, then

(1) there exists a function g(x) in L^1 on T_k which is continuous in an open subset of T_k and which furthermore has $\sum_m \hat{f}(m)e^{(m,v)\sigma_0}e^{i(m,x)}$ as its Fourier series.

We use (1) to define the class $C_v = C_v^+ \cup C_{-v}^+$. In particular we say that f is in C_v^+ if the following three conditions are met:

- (i) f is in L^{∞} on T_k ,
- (ii) f is in A_v^+ ,
- (iii) there exists a $\sigma_0 < 0$ such that (1) holds.

We note once again that if (ii) is replaced by

(ii') f is in B_v^+ ,

then (iii) follows automatically.

With every unit point $v = (v_1, \dots, v_k)$ there is also associated a one-parameter subgroup of T_k which we shall call G_v where

$$G_{m{v}} = \{x; \, -\pi < x_j \leq \pi, \, x_j \equiv t v_j \, ext{mod} \, 2\pi, \, -\infty < t < \infty\}$$
 .

If v is linearly independent with respect to rational coefficients, then G_v is dense on T_k . If v is linearly dependent with respect to rational coefficients, G_v is not dense on T_k . (We say $v = (v_1, \dots, v_k)$ is linearly dependent with respect to rational coefficients if there exist rational numbers r_1, \dots, r_k with $r_1^2 + \dots + r_k^2 \neq 0$ such that $\sum_{j=1}^k r_j v_j = 0$.) In either case, however, the statement that a set $E \subset G_v$ is of positive linear measure is well-defined. In particular, we set $E^* = \{t; \text{ there exists an } x \text{ in } E \text{ such that } x_j \equiv tv_j \mod 2\pi \text{ for } j = 1, \dots, k\}$. Then E^* is a set on the real line $-\infty < t < \infty$. We say that E is of positive linear measure if E^* is a set with positive 1-dimensional Lebesgue measure.

In the sequel, we shall work primarily with functions f in L^{∞} on T_k . Also, all functions initially defined in T_k will be understood to be extended to all of E_k by periodicity of period 2π in each variable.

Given a function f in L^{∞} on T_k , we shall set

(2)
$$f(x, h) = \sum_{m} \hat{f}(m) e^{i(m,x)} e^{-|m|h}$$
 for $h > 0$.

We shall say that f vanishes at x_0 if

(3)
$$\lim_{h\to 0^+} f(x_0, h) = 0$$
.

We note that the changing of f on a set of k-dimensional measure zero does not affect its vanishing at the point x_0 . (In classical terminology, (3) says that the Fourier series of f is Abel summable to zero at x_{0} .)

We shall say that f vanishes on a set E if f vanishes at all points of E.

With B(x, h) representing the open k-ball with center x and radius h and |B(x, h)| representing the k-dimensional volume of B(x, h), we set

$$(\ 4\) \qquad \qquad f_h(x) = |\ B(x,\ h)\ |^{-1} {\int_{B(x,\ h)}} f(y) dy$$

and note that if $\lim_{h\to 0} f_h(x_0) = 0$, then f vanishes at x_0 , i.e., $\lim_{h\to 0^+} f(x_0, h) = 0$ (See [10, p. 55]).

The theorem that we shall prove is the following:

THEOREM. A necessary and sufficient condition that every f in C_v which vanishes on a subset of G_v of positive linear measure be zero almost everywhere on T_k is that v be linearly independent with respect to rational coefficients.

We first note that the sufficiency of the above theorem is false for bounded functions in A_v . This fact will be established in §4.

We next note that if f(x) is in C_v , so is $f(x + x_0)$. Consequently, the above theorem implies that if f is in C_v , v linearly independent with respect to rational coefficients, and f vanishes on a subset of $x_0 + G_v$ of positive linear measure, then f is zero almost everywhere on T_k .

We finally note that for k = 1 the above theorem reduces to the well-known theorem of F. and M. Riesz for holomorphic functions on the unit disc in the form that they first proved it, i.e., for bounded functions, [9]. There have been other extensions of the F. and M. Riesz Theorem to higher dimensions (see [5, p. 176] and [4, p. 188]), but these always involve the vanishing of f on sets of positive k-dimensional measure. Here, we only ask that f vanish on particular sets of positive 1-dimensional measure, but on the other hand, we deal with a more restricted class of functions.

2. Proof of sufficiency. Since $C_v = C_{-v}$ and $G_v = G_{-v}$ with no loss in generality, we can assume from the start that f is in C_v^+ .

Since f is in C_v^+ , it is in A_v^+ . Consequently there exists an m_0 such that $\hat{f}(m) = 0$ if $m \neq m_0$ and $(m - m_0, v) \leq 0$. If we set $a(x) = e^{-i(m_0,x)}f(x)$, then a(x) is in A_v^+ with $m_0 = 0$. Furthermore, it is clear that since f(x) satisfies (1), a(x) does also. If we can show that

(5) if
$$\lim_{h\to 0+} f(x_0, h) = 0$$
, then $\lim_{h\to 0+} a(x_0, h) = 0$,

it will be sufficient to prove the theorem for a(x).

To establish (5), set $b(x) = a(x) - e^{-i(m_0, x_0)}f(x)$. Then $a(x, h) = b(x, h) + e^{i(m_0, x_0)}f(x, h)$, and by the remark after (4), (5) will follow once it is shown that $b_k(x_0) \to 0$ as $h \to 0$. But

$$\begin{aligned} | b_{h}(x_{0}) | &\leq 0(h^{-k}) || f ||_{\infty} \int_{B(x_{0},h)} | e^{-i(m_{0},x)} - e^{-i(m_{0},x_{0})} | dx \\ &\leq 0(h^{-k}) || f ||_{\infty} | m_{0} | \int_{B(x_{0},h)} | x - x_{0} | dx \\ &\leq o(1) \quad \text{as } h \to 0 , \end{aligned}$$

and (5) is established.

We now replace a(x) by f(x) and proceed, i.e., we set

(6)
$$M = \{m; (m, v) \ge 0\}$$

and assume

(7) if m is not in M, then
$$\hat{f}(m) = 0$$
.

Setting $P(x,h) = \sum_{m} e^{i(m,x)-|m|h}$ for h > 0 and noticing that P(x,h) > 0for x on T_k and h > 0, [3, p. 32], and that $(2\pi)^{-k} \int_{T_k} P(x,h) dx = 1$ we see that f(x,h) defined in (2) is given by

$$f(x, h) = (2\pi)^{-k} \int_{T_k} f(x - y) P(y, h) dy$$
.

Consequently,

(8)
$$|f(x,h)| \leq ||f||_{\infty}$$
 for $h > 0$ and x on T_k .

Next, with $z = \sigma + it$ and $\sigma \leq 0$, we set

(9)
$$F(z, h) = \sum_{m} \hat{f}(m) e^{i(tv,m)} e^{\sigma(v,m)} e^{-|m|h}$$
$$= \sum_{m \ln M} \hat{f}(m) e^{\lambda_{m} z} e^{-|m|h}$$

where

(10)
$$\lambda_m = (m, v)$$
 for m in M .

By (6), (7), (9), and (10), F(z, h) is, for fixed h > 0, analytic in the left half-plane $\sigma < 0$ and continuous in the closed half-plane $\sigma \leq 0$. Furthermore, since F(it, h) = f(tv, h), we have by (8) that

(11)
$$\sup_{-\infty < t < \infty} |F(it, h)| \leq ||f||_{\infty} \quad \text{for } h > 0.$$

Also, it is clear that for $\sigma \leq 0$, $|F(\sigma+it,h)| \leq \sum_{m \text{ in } M} |\hat{f}(m)| e^{-|m|h} < \infty$ and therefore that

 $\lim_{\sigma \to -\infty} \sup_{-\infty < t < \infty} |F(\sigma + it, h)| \leq |\widehat{f}(0)| \leq ||f||_{\infty} .$

Consequently, it follows from the Phragmen-Lindelof theorem, [1, p. 137], that

(12)
$$||F(z, h)|| \leq ||f||_{\infty}$$
 for $\sigma \leq 0$ and $h > 0$.

But then by Montel's theorem]1, p. 132],

(13) there exists a function F(z), analytic for $\sigma < 0$, and a sequence $h_1 > h_2 > \cdots > h_j > \cdots \to 0$ such that $\lim_{j\to\infty} F(z, h_j) = F(z)$ uniformly on any compact subset of the open left half-plane $\sigma < 0$.

We propose to show that F(z) is identically zero. To do this we look at $F(it, h_j)$. By (11), $\{F(it, h_j)\}_{j=1}^{\infty}$ is a bounded sequence of continuous functions on the interval $-\infty < t < \infty$. Consequently, it follows from the notion of weak* convergence that there exists a function q(t) in L^{∞} on $-\infty < t < \infty$, with $|q(t)| \leq ||f||_{\infty}$ for almost every t and a subsequence $\{h_{j_n}\}_{n=1}^{\infty}$ of $\{h_j\}_{j=1}^{\infty}$ with $\lim_{n\to\infty} h_{j_n} = 0$ such that for every $\xi(t)$ in $L^{\infty} \cap L^1$ on $-\infty < t < \infty$,

(14)
$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\xi(t)^{F}(it, h_{j_{n}})dt = \int_{-\infty}^{\infty}\xi(t)q(t)dt .$$

Choosing ξ in (14) to be the function

$$\xi(u)=-\sigma[\sigma^2+(u-t)^2]^{-1}\pi^{-1}$$
 where $\sigma<0$,

we see from (13) that

(15)
$$F(\sigma + it) = \lim_{n \to \infty} F(\sigma + it, h_{j_n})$$
$$= \lim_{n \to \infty} -\pi^{-1} \sigma \int_{-\infty}^{\infty} F(iu, h_{j_n}) [\sigma^2 + (u - t)^2]^{-1} du$$
$$= -\pi^{-1} \sigma \int_{-\infty}^{\infty} q(u) [\sigma^2 + (u - t)^2]^{-1} du.$$

Since $|F(\sigma + it, h)| \leq ||f||_{\infty}$ for h > 0 and $\sigma \leq 0$, it follows from (13) that $|F(\sigma + it)| \leq ||f||_{\infty}$ for $\sigma < 0$, and consequently from (15) and [7, p. 447] that

(16)
$$\lim_{\sigma \to 0^-} F(\sigma + it) = q(t) \text{ for almost every } t$$

If we can show that q(t) = 0 on a set of positive measure, then it will follow from (16) and the F. and M. Riesz Theorem for a halfplane, [7, p. 449], that $F(\sigma + it)$ is identically zero for $\sigma < 0$.

To show that q(t) = 0 on a set of positive measure we set

$$E^* = \left\{t, \lim_{h \to 0} f(tv, h) = 0\right\}$$
.

By hypothesis, E^* is a set of positive linear measure in the infinite interval $-\infty < t < \infty$. Let B^* be any measurable subset of E^* of finite measure and let $\xi_{B^*}(t)$ be the indicator function of B^* . Then by (14)

(17)
$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\xi_{B^*}(t)F(it,\,h_{j_n})dt=\int_{B^*}q(t)dt\,.$$

However, $F(it, h_{i_n}) = f(tv, h_{j_n})$, $f(tv, h_{j_n}) \to 0$ as $n \to \infty$ for t in B^* , and $|f(tv, h_{j_n})| \leq ||f||_{\infty}$. We conclude from the Lebesgue dominated convergence theorem that

(18)
$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\xi_{B^*}(t)F(it, h_{j_n})dt = 0$$

From (17) and (18), we obtain that $\int_{B^*} q(t)dt = 0$. B^* , however, is an arbitrary subset of E^* of finite measure. Therefore q(t) must equal zero almost everywhere in E^* . Consequently, q(t) = 0 on a set of positive measure, and we have that

(19)
$$F(\sigma + it) = 0$$
 for $\sigma < 0$.

By hypothesis, there exist a $\sigma_0 < 0$, an open set $U \subset T_k$ and a function g(x) in L^1 on T_k such that the following facts prevail:

(21)
$$\widehat{g}(m) = \widehat{f}(m)e^{(v,m)\sigma_0}$$
 for every m :

(22) g is continuous in U.

From (9), (13), and (19), it follows that

(23)
$$\lim_{j \to \infty} \sum_m \widehat{f}(m) e^{(v,m)\sigma_0} e^{i(tv,m)} e^{-|m|h_j} = 0 \quad ext{for} \ -\infty < t < \infty$$
.

On the other hand, as is well-known (see [10, p. 55]), (21) and (22) imply

(24)
$$\lim_{j\to\infty}\sum \widehat{f}(m)e^{(m,v)\sigma_0}e^{i(m,x)}e^{-|m|h_j} = g(x) \quad \text{for } x \text{ in } U.$$

We conclude from (23) and (24) that g(x) = 0 for x in $U \cap G_v$. However, since G_v is dense in T_k and U is open, $U \cap G_v$ is dense in U, and consequently, g(x) = 0 in all of U.

Suppose that $B(x_0, h_0) \subset U$. Then for $0 < h < h_0$ and $g_h(x)$ defined by (4), we have that $g_h(x)$ is a continuous periodic function which for each fixed h is zero on an open set. In particular, $g_h(x + x_0)$ is zero on a subset of G_v of positive linear measure. Since

$${\widehat g}_{h}(m)={\widehat f}(m)e^{(m,v)\sigma_{0}}\,|\,B(0,\,h)\,|^{-1}\!\int_{B^{(0,\,h)}}\!\!\!\!e^{i(m,x)}dx$$
 ,

we conclude from the argument previously given that $g_h(tv + x_0) = 0$ for $-\infty < t < \infty$ and $0 < h < h_0$. But then the continuous function $g_h(x)$ is zero on a dense subset of T_k , and therefore for $0 < h < h_0$, $g_h(x) = 0$ for all x on T_k . Consequently, g(x) = 0 almost everywhere on T_k . We conclude from (21) that $\hat{f}(m) = 0$ for every m. Therefore f(x) = 0almost everywhere, and the proof of the sufficiency is complete.

3. Proof of necessity. Let $v = (v_1, \dots, v_k)$ be linearly dependent over the rationals with $v_1^2 + \dots + v_k^2 = 1$. We shall show that there exists a nonzero trigonometric polynomial f(x) in B_v^+ (and therefore in C_v^+) such that f(x) = 0 for x in G_v .

Two cases present themselves. Either there exists a coordinate v_{j_0} of v which is zero or all the coordinates of v are different from zero. We handle the former case first.

Since |v| = 1, there exists a coordinate v_{j_1} of v which is different from zero. Let m' be the integral lattice point with 1 in the j_0 coordinate, $\operatorname{sgn} v_{j_1}$ in the j_1 -coordinate, and zero at all other coordinates. Similarly define m'' to be the integral lattice point with 2 in the j_0 coordinate, $\operatorname{sgn} v_{j_1}$ in the j_1 -coordinate, and zero at all other coordinates. Then $(m', v) = (m'', v) = |v_{j_1}| > 0$, and the trigonometric polynomial $f(x) = e^{i(m',x)} - e^{i(m'',x)}$ is clearly in B_v^+ . Also, $f(tv) = e^{it(m',v)} - e^{it(m'',v)} = 0$ for $-\infty < t < \infty$; f(x) is zero on G_v , and the first case is settled.

Next, suppose that all the coordinates of v are different from zero. Since by assumption v is linearly dependent with respect to rational coefficients, there exists a nonzero integral lattice point m such that (m, v) = 0. Let m_{j_0} be the first coordinate of m which is different from zero. We can assume $\operatorname{sgn} m_{j_0} = \operatorname{sgn} v_{j_0}$ for otherwise we can replace m by -m. Let m' be the integral lattice point with $\operatorname{sgn} v_{j_0}$ in the j_0 -coordinate and zero elsewhere. Set m'' = m + m'. Then

$$(m'', v) = (m + m', v) = (m', v) = |v_{j_0}| > 0$$
,

and the trigonometric polynomial $f(x) = e^{i(m',x)} - e^{i(m'',x)}$ is in B_v^+ and is zero on G_v . The second case is settled, and the proof of the theorem is complete.

4. Counter-example for A_v . Given v linearly independent with respect to rational coefficients, we shall exhibit a function f(x) in L^{∞} on T_k and in A_v^+ such that

(25)
$$\lim_{h \to 0} f_h(x) = 0 \quad \text{for every } x \text{ in } G_v$$

and such that $f(x) \neq 0$ in a set of positive measure on T_k .

We note once again that (25) implies that f vanishes on all of G_{v} .

We start in the classical manner (see [11, p. 276 and p. 105]). Observing that G_v is of k-dimensional measure zero, we see that there exists a sequence of sets $\{G_n\}_{n=1}^{\infty}$ each open in the torus sense on T_k with the following properties:

$$(26) T_k \supset G_1 \supset G_2 \supset \cdots \supset G_n \cdots \supset G_v;$$

(27) the k-dimensional measure of G_n is $\leq n^{-4}$.

We set

(28)
$$g_n(x) = n^2 \quad \text{for } x \text{ in } G_n ,$$
$$= 0 \quad \text{for } x \text{ in } T_k - G_n ,$$

and

(29)
$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

Now $\int_{x_k} g(x) dx \leq \sum_{n=1}^{\infty} n^{-2}$. Consequently, g(x) is a nonnegative function on T_k , and the set $\{x; g(x) = +\infty\}$ is of k-dimensional measure zero.

Next, we set $a(x) = e^{-g(x)}$ and observe that a(x) is a Borel measurable function on T_k with the following properties:

$$(30) 0 \leq a(x) \leq 1 ext{ for } x ext{ in } T_k,$$

(31) $\{x; a(x) = 0\}$ is of k-dimensional measure zero.

Observing that $G_v \subset G_n$ for every *n* by (27) and that by (29), $a(x) \leq e^{-g_n(x)}$, we see from (28) that for fixed *n* and a fixed x_0 in G_v , $a_h(x_0) \leq e^{-n^2}$ for *h* sufficiently small. We conclude that

$$\lim_{h\to 0} a_h(x) = 0 \quad \text{for } x \text{ in } G_v \text{ .}$$

From (31) and (32), we see that there is no constant such that a(x) is equal to it almost everywhere on T_k . Consequently there exists an $m_0 \neq 0$ such that $\hat{a}(m_0) \neq 0$. Since a(-x) satisfies (30), (31), and (32), with no loss in generality, we can also assume that $(m_0, v) > 0$. Thus we have

(33)
$$\hat{a}(m_0) \neq 0 \text{ and } (m_0, v) > 0$$
.

Next, as in [8, p. 60], we introduce the complex Borel measure μ on T_k defined by

(34)
$$\int_{T_k} b(x) d\mu(x) = \int_{-\infty}^{\infty} b(tv) (1 - it)^{-2} dt$$

for every bounded Borel measurable function on T_k .

From the fact that

$$\int_{-\infty}^{\infty} e^{i\lambda t} (1-it)^{-2} dt = 0 \qquad \qquad ext{for } \lambda \geqq 0 \ = -(2\pi)\lambda e^{\lambda} \qquad \qquad ext{for } \lambda < 0$$
 ,

we see that $\hat{\mu}(m) = (2\pi)^{-k} \int_{T_k} e^{-i(m,x)} d\mu(x)$ is such that

(35)
$$\hat{\mu}(m) \neq 0 \text{ for } (m, v) > 0$$

= 0 for $(m, v) \leq 0$.

We set

(36)
$$f(x) = (2\pi)^{-k} \int_{T_k} a(x-y) d\mu(y)$$

and shall show that f has the requisite properties set forth at the beginning of this section.

In the first place, we see from (30), (34), and (36)

$$|f(x)| \leq (2\pi)^{-k} \int_{-\infty}^{\infty} (1+t^2)^{-1} dt$$
 for x in T_k ,

and consequently f(x) in L^{∞} on T_k .

In the second place, we observe from (36) that $\hat{f}(m) = \hat{a}(m)\hat{\mu}(m)$ and consequently by (35) that f(x) is in A_v^+ . Furthermore, by (33) and (35), $\hat{f}(m_0) \neq 0$. Consequently, $f(x) \neq 0$ on a set of positive measure on T_k .

All that remains to establish is (25). Let x_0 be a fixed point in G_v . Then by (36) and Fubini's theorem,

(37)
$$(2\pi)^{k} f_{h}(x_{0}) = \int_{x_{k}} a_{h}(x_{0} - y) d\mu(y) \\ = \int_{-\infty}^{\infty} a_{h}(x_{0} - tv)(1 - it)^{-2} dt$$

By (30), $|a_{k}(x)| \leq 1$ for all x on T_{k} . Furthermore, since x_{0} is in G_{v} , so is $x_{0} - tv$ for $-\infty < t < \infty$. Therefore, by (32), $\lim_{h\to 0} a_{k}(x_{0} - tv) = 0$ for $-\infty < t < \infty$. We consequently conclude from the Lebesgue dominated convergence theorem and (37) that $\lim_{h\to 0} f_{k}(x_{0}) = 0$, and (25) is established.

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