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**HOMOTOPY COMMUTATIVITY AND THE MOORE SPECTRAL  
SEQUENCE**

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**This paper initiates the study of strong homotopy commutativity in both geometric and algebraic contexts in order to correct an error in a paper of J. C. Moore.**

The difficulty [8, § 7, Theorem II] lies in the tacit assumption here (and in the remark following Proposition 7.1) that the multiplication  $m: X \times X \rightarrow X$  on an  $H$ -space  $X$  induces a morphism of  $H$ -spaces  $\Omega X \times \Omega X \rightarrow \Omega X$  where  $\Omega X$  denotes the associative loop space of  $X$  defined in [6, Chapter 2]. Unfortunately the situation is more complex than this. A morphism of  $H$ -spaces  $\Omega m: \Omega(X \times X) \rightarrow \Omega X$  is induced by the product  $m$  on  $X$ . However for associative loop spaces,  $\Omega X \times \Omega X$  is not the same as  $\Omega(X \times X)$ , although it has the same homotopy type. Moreover there is no obvious morphism of  $H$ -spaces from  $\Omega X \times \Omega X$  to  $\Omega(X \times X)$  with which the induced morphism  $\Omega m$  could be composed.

There are three ways to resolve this problem so that the proofs involved can be carried through. One is to assume that  $X$  is associative, use the product induced from  $X$  as the product in the ordinary loop space, and to take the usual loop product as the morphism of  $H$ -spaces. Another way would be to use the product induced on the ordinary loop space as a morphism of Stasheff's  $A_\infty$  structures [10]. The third way, the one used in this paper, is to show that there is a strongly homotopy multiplicative map of  $H$ -spaces  $\Omega X \times \Omega X \rightarrow \Omega X$ , and that this is sufficient for the proofs desired. The second and third alternatives are homotopy equivalent, and the third is preferred in order to use the bar construction rather than the less familiar tilde construction of Stasheff [10].

The exposition is organized as follows: § 1 sets up the geometry and discusses strong homotopy commutativity; § 2 recalls the bar construction and the definition of the Moore spectral sequence; § 3 defines the algebraic analogue of strong homotopy commutativity for a differential graded algebra  $A$  and uses it to construct a product in the bar construction  $\bar{B}(A)$ , and consequently to introduce a Hopf algebra structure into the Moore spectral sequence; § 4 proves a homology suspension theorem for a contractible fibre space over an  $H$ -space, a slight improvement on a theorem of Browder [1, Theorem 5.13] which contains the original result of Moore in which the trouble began.

The comments of the referee resulted in substantial improvement

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in the author's exposition, and the author is indebted to W. Browder for helpful conversations.

1. **The geometric situation.**  $X$  will denote a pathwise connected and simply connected  $H$ -space with multiplication  $m: X \times X \rightarrow X$  which has a two-sided unit element  $e$ . In addition we assume that  $X$  has *finite homological type*, that is, that the singular homology groups of  $X$  are finitely generated in every degree.

$\Omega X$  will denote the associative loop space of  $X$  as defined in [6, Chapter 2], and  $\phi: \Omega X \times \Omega X \rightarrow \Omega X$  will denote the loop multiplication.  $\phi$  gives  $\Omega X$  the structure of an associative  $H$ -space with unit element  $\Omega e$ , the unique loop of length zero. Clearly  $\Omega X \times \Omega X$  is also an associative  $H$ -space with multiplication  $(\phi \times \phi)(1 \times T \times 1)$  where  $T$  denotes the standard twisting map. Note that (unlike the ordinary loop space)  $\Omega X \times \Omega X$  is not the same as  $\Omega(X \times X)$ .

**DEFINITION 1.1.** A map  $f: Y \rightarrow Z$  where  $Y$  and  $Z$  are associative  $H$ -spaces is *strongly homotopy multiplicative* if there exist maps

$$M_n: Y \times (I \times Y)^n \rightarrow Z$$

for every nonnegative integer  $n$ , such that  $M_0 = f$ , and such that

$$\begin{aligned} &M_n(y_0, t_1, y_1, \dots, t_n, y_n) \\ &= \begin{cases} M_{n-1}(y_0, t_1, \dots, t_{i-1}, y_{i-1}y_i, t_{i+1}, \dots, t_n, y_n) & \text{for } t_i = 0 \\ M_{i-1}(y_0, t_1, \dots, t_{i-1}, y_{i-1})M_{n-i}(y_i, t_{i+1}, \dots, t_n, y_n) & \text{for } t_i = 1. \end{cases} \end{aligned}$$

The definition is due to Sugawara [11].

If  $X$  is an associative  $H$ -space, then  $B_X$  will denote the classifying space of  $X$  as constructed in Dold and Lashof [3].

**THEOREM 1.2.** (Sugawara [11, Lemma 2.2]). *If  $f: Y \rightarrow Z$  is a strongly homotopy multiplicative map, then  $f$  induces a map  $B_f: B_Y \rightarrow B_Z$ .*

**DEFINITION 1.3.** An associative  $H$ -space  $X$  is *strongly homotopy commutative* if there exists a strongly homotopy multiplicative map  $f: X \times X \rightarrow X$ , such that  $f(e, x) = x = f(x, e)$ .

**THEOREM 1.4.** (Sugawara [11, Theorem 4.3]). *If  $X$  is an  $H$ -space with associative multiplication, then  $B_X$  is an  $H$ -space if and only if the multiplication on  $X$  is strongly homotopy commutative.*

**COROLLARY 1.5.**  *$X$  is an  $H$ -space if and only if  $\Omega X$  is strongly homotopy commutative.*

**PROPOSITION 1.6.** For any pathwise connected spaces  $X_1, \dots, X_m$

there exists a strongly homotopy multiplicative map

$$\psi: \Omega X_1 \times \cdots \times \Omega X_m \rightarrow \Omega(X_1 \times \cdots \times X_m).$$

*Proof.*  $\Omega(X_1 \times \cdots \times X_m)$  may be considered to be the subspace of  $\Omega X_1 \times \cdots \times \Omega X_m$  consisting of  $m$ -tuples of loops of equal length. If  $(\omega_1, \dots, \omega_m) \in \Omega X_1 \times \cdots \times \Omega X_m$  and  $\omega_i: [0, s_i] \rightarrow X_i$ , then we define  $\psi(\omega_1, \dots, \omega_m) = (\bar{\omega}_1, \dots, \bar{\omega}_m)$  where  $\bar{\omega}_i = \omega_i \cdot \{s - s_i\}$  where  $s = \max \{s_i\}$  and  $\{s - s_i\}$  denotes the constant loop of length  $s - s_i$ .

The homotopies  $M_n$  are complicated to define. For  $i = 0, \dots, n$  suppose that  $\omega^i = (\omega_1^i, \dots, \omega_m^i) \in \Omega X_1 \times \cdots \times \Omega X_m$  and  $\omega_j^i: [0, s_j^i] \rightarrow X_j$ . Then we want to define

$$M_n(\omega^0, t_1, \omega^1, \dots, t_n, \omega^n) = \psi(\omega_1(t), \dots, \omega_m(t))$$

for  $t = (t_1, \dots, t_n) \in I^n$ , the unit  $n$ -cube, and with

$$\omega_j(t) = \omega_j^0 \cdot \{t_1 \delta_j^1(t)\} \cdot \omega_j^1 \cdot \cdots \cdot \{t_n \delta_j^n(t)\} \cdot \omega_j^n.$$

(As above  $\{r\}$  denotes the constant loop of length  $r$  and  $\cdot$  denotes the loop product.) Then for  $t_k = 1$  we must have that the loops  $\omega_j^0 \cdot \{t_1 \delta_j^1(t)\} \cdot \omega_j^1 \cdot \cdots \cdot \omega_j^{k-1} \cdot \{\delta_j^k(t)\}$  have the same lengths for different  $j$ 's. Setting  $\lambda_j^k(t) = s_j^0 + t_1 \delta_j^1(t) + \cdots + t_{k-1} \delta_j^{k-1}(t) + s_j^{k-1}$  and  $\lambda^k(t) = \max(\lambda_j^k(t))$ , we must have  $\delta_j^k(t) = \lambda^k(t) - \lambda_j^k(t)$ . This gives an inductive definition for  $\delta_j^k(t)$  and we note that  $\delta_j^k(t)$  actually depends only on  $t_1, \dots, t_{k-1}$ . This completes the definition of  $M_n$ .

REMARK. If  $\omega_j^i = \Omega e$  for all  $j \neq k$ , then, setting  $s = s_1^k + \cdots + s_n^k$ ,

$$\begin{aligned} M_n(\omega^0, t_1, \dots, t_n, \omega^n) &= \psi(\Omega e, \dots, \Omega e, \omega_1^k \cdots \omega_n^k, \Omega e, \dots, \Omega e) \\ &= (\{s\}, \dots, \{s\}, \omega_1^k \cdots \omega_n^k, \{s\}, \dots, \{s\}). \end{aligned}$$

COROLLARY 1.7. *If  $X$  is an  $H$ -space, there are strongly homotopy multiplicative maps*

$$\psi^{(k)}: (\Omega X)^k \rightarrow \Omega X.$$

*Proof.* Let  $m^k: X^k \rightarrow X$  be given by some fixed way of associating the product of  $k$  elements of  $X$ . (Unless otherwise specified we shall assume that  $m^k(x_1, \dots, x_k) = x_1(x_2(\cdots(x_{k-1}(x_k))\cdots))$ ). Let  $M_n^k$  denote the homotopies defined above for the map  $\psi: (\Omega X)^k \rightarrow \Omega(X^k)$ . Then  $\psi^{(k)} = \Omega(m^k) \circ \psi$  and the homotopies for  $\psi^{(k)}$  are given by  $\tilde{M}_n^k = \Omega(m^k) \circ M_n^k$ . When  $k = 2$ , this shows that  $\Omega X$  is strongly homotopy commutative.

2. **The bar construction.** For the convenience of the reader and

to fix notation we recall the principal parts of the bar construction (Eilenberg and MacLane [4]).

$K$  will denote a commutative ring with unit and  $A$  will denote an associative DGA algebra over  $K$  with augmentation  $\varepsilon: A \rightarrow K$ . Then  $\bar{A} = \text{Ker } \varepsilon$  and  $s\bar{A}$  denotes the *suspension* of  $\bar{A}$ , the graded module formed from  $\bar{A}$  by raising degrees by one.  $\bar{B}_n(A) = (s\bar{A})^n$ , the  $n$ -fold tensor power of  $s\bar{A}$ , with the convention  $(s\bar{A})^0 = K$ . The (*normalized*) *bar construction*  $\bar{B}(A)$  is the graded  $K$ -module with component  $\bar{B}_n(A)$  in degree  $n$ , and with the obvious augmentation. Elements of  $\bar{B}_n(A)$  are written as linear combinations of elements  $[a_1 | \cdots | a_n] = [a_1] \otimes \cdots \otimes [a_n]$  where  $[a_i]$  denotes the suspension of  $a_i \in \bar{A}$ . Then  $\bar{B}(A)$  is graded by assigning the element  $[a_1 | \cdots | a_n]$  *external degree*  $n$  and *internal degree*  $m = \sum_{i=1}^n \text{deg}(a_i)$ , and bidegree  $(n, m)$ . It will be convenient to abbreviate by  $\bar{B}^n(A)$  the graded module with component  $\bar{B}_k(A)$  in (external) degree  $k$  for  $0 \leq k \leq n$ , and the 0 module in all other degrees. As a differential  $K$ -module  $\bar{B}(A)$  has a total differential  $d_T = d_E + d_I$  where the *external and internal differentials*,  $d_E$  and  $d_I$  are given by the formulas

$$d_E([a_1 | \cdots | a_n]) = \sum_{i=1}^{n-1} (-1)^{\sigma(i)} [a_1 | \cdots | a_i a_{i+1} | \cdots | a_n]$$

$$d_I([a_1 | \cdots | a_n]) = \sum_{i=1}^n (-1)^{\sigma(i-1)} [a_1 | \cdots | da_i | \cdots | a_n]$$

in which  $\sigma(i) = \text{deg}([a_1 | \cdots | a_i])$  and  $a_i a_{i+1}$  and  $da_i$  indicate the product and differential taken in  $A$ .  $\bar{B}(A)$  is an associative co-algebra in a natural way with coproduct

$$\Delta([a_1 | \cdots | a_n]) = \sum_{i=0}^n [a_1 | \cdots | a_i] \otimes [a_{i+1} | \cdots | a_n]$$

where in the extreme terms  $i = 0$  and  $i = 1$ ,  $[ ] = 1 \in \bar{B}_0(A) = K$ .

When  $K$  is a principal ideal domain, the homology of the bar construction  $\bar{B}(A)$  will be denoted  $\text{Tor}^4(K, K)$ , and extension of the usual use of this notation. (See Moore [8].)

If  $\bar{B}(A)$  is filtered by external degree, we obtain the *Moore spectral sequence*,  $\{E^r(A), d^r\}$ , in which  $E^1 \approx \bar{B}(H(A))$ , and in which  $E^2(A) \approx \text{Tor}^{H(A)}(K, K)$  provided that  $H_n(A)$  is  $K$ -projective for all  $n$ . If  $H(A)$  is of finite type, then  $E^r \Rightarrow E^\infty \approx E^0(\text{Tor}^4(K, K))$ , the graded module associated with the filtration induced on  $\text{Tor}^4(K, K)$ .

**3. Homotopy commutativity and the bar construction.** In this section are given algebraic analogues for some parts of the geometric situation discussed in §1 and a product is introduced into the Moore spectral sequence turning it into a spectral sequence of Hopf algebras.

**DEFINITION 3.1.** Let  $A$  and  $A'$  be associative DGA algebras over  $K$ .

A homomorphism of *DGA* modules over  $K$ ,  $h: A \rightarrow A'$ , is *strongly homotopy multiplicative* if there exists for each nonnegative integer  $n$ , a homomorphism of  $K$ -modules of degree  $n$ ,

$$h_n: A \otimes \cdots (n + 1) \cdots \otimes A \rightarrow A'$$

such that  $h_0 = h$  and

$$\begin{aligned} dh_n(a_1 \otimes \cdots \otimes a_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{\sigma(i-1)} h_n(a_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^{\sigma(i)} [h_{n-1}(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &- h_i(a_1 \otimes \cdots \otimes a_{i+1}) h_{n-i+1}(a_{i+2} \otimes \cdots \otimes a_{n+1})] \end{aligned}$$

where as before  $\sigma(i) = \text{deg}([a_1 | \cdots | a_i])$ .

Clearly this condition implies that  $h$  is (chain) homotopy multiplicative. A homomorphism of algebras,  $h: A \rightarrow A'$ , is automatically strongly homotopy multiplicative taking  $h_n$  to be the zero homomorphism for  $n > 0$ .

**PROPOSITION 3.2.** If  $h: A \rightarrow A'$  is strongly homotopy multiplicative, then  $h$  induces a homomorphism of *DGA* coalgebras over  $K$

$$\bar{B}(h): \bar{B}(A) \rightarrow \bar{B}(A') .$$

Furthermore  $\bar{B}(h)$  induces a homomorphism of spectral sequences

$$E^r(h): E^r(A) \rightarrow E^r(A')$$

such that  $E^1(h) = \bar{B}(h_*)$  where  $h_*: H(A) \rightarrow H(A')$ . In other words  $E^1(h)$  is given by

$$E^1(h)([x_1 | \cdots | x_n]) = [h_*(x_1) | \cdots | h_*(x_n)] .$$

*Proof.* Let  $S(n, k)$  denote the set of  $k$ -tuples of nonnegative integers whose sum is  $n$ . Then

$$\begin{aligned} \bar{B}(h)([a_1 | \cdots | a_n]) &= \sum_{k=1}^n \sum_{S(n,k)} [h_{i_1-1}(a_1 \otimes \cdots \otimes a_{i_1}) | \cdots | h_{i_k-1}(a_{n-i_k+1} \otimes \cdots \otimes a_n)] . \end{aligned}$$

All the properties required of  $\bar{B}(h)$  are easily checked by direct computation.

**DEFINITION 3.3.** An associative *DGA* algebra  $A$  is said to be *strongly homotopy commutative* if there is a strongly homotopy multiplicative homomorphism  $h: A \otimes A \rightarrow A$  such that  $h(a \otimes 1) = a = h(1 \otimes a)$ .

**PROPOSITION 3.4.** If  $A$  is a strongly homotopy commutative, asso-

ciative *DGA* algebra, then  $\bar{B}(A)$  is a *DGA* Hopf algebra, and the terms of the Moore spectral sequence  $E^r(A)$  are Hopf algebras provided the ground ring  $K$  is a field. Furthermore as  $1 \in A_0$  is a unit for the map  $h: A \otimes A \rightarrow A$ , the Hopf algebras  $E^r(A)$  are commutative, associative, and have a unit.

*Proof.* The shuffling map  $\Sigma: \bar{B}(A) \otimes \bar{B}(A) \rightarrow \bar{B}(A \otimes A)$  is a homomorphism of coalgebras and therefore  $\Phi = B(h)\Sigma$  provides a product for  $\bar{B}(A)$  which is a homomorphism of coalgebras, and hence  $\bar{B}(A)$  becomes a Hopf algebra. Let  $E^0(A)$  denote the associated graded *DGA* Hopf algebra for the filtration of  $\bar{B}(A)$  by external degree. Then  $E^0(A) \approx \bar{B}(A)$  as a coalgebra, and as  $h(a \otimes 1) = a = h(1 \otimes a)$ , it follows that  $E^0(\Phi)$  is just the well known shuffle product of Eilenberg and MacLane [4]. The remaining conclusions follow immediately since the shuffle product is commutative, associative, and has a unit.

**COROLLARY 3.5.** *If  $X$  is a pathwise connected and simply connected  $H$ -space of finite homological type, then there exists a spectral sequence of commutative and associative Hopf algebras with unit over  $Z_p$ ,  $\{E^r(\Omega X; Z_p), d^r\}$ , such that*

$$\begin{aligned} E^2(\Omega X; Z_p) &\approx \text{Tor}^{H_*(\Omega X; Z_p)}(Z_p, Z_p) && \text{(as Hopf algebras)} \\ E^\infty(\Omega X; Z_p) &\approx E^0(H_*(X; Z_p)) && \text{(as Hopf algebras)} \end{aligned}$$

where  $E^0(H_*(X; Z_p))$  denotes the associated graded Hopf algebra under a filtration of  $H_*(X; Z_p)$ .

*Proof.* Let  $A = C_N(\Omega X; Z_p)$ , the normalized singular chains of  $\Omega X$  mod  $p$ . Then by 1.7  $\Omega X$  is strongly homotopy commutative. It will follow that  $A$  is strongly homotopy commutative in the algebraic sense if we set (for  $a_i \in A \otimes A$ )

$$h_n(a_1 \otimes \cdots \otimes a_{n+1}) = \tilde{M}_{n\#}^2(a_1 \otimes e \otimes \cdots \otimes e \otimes a_{n+1})$$

where  $\#$  indicates the induced chain homomorphism and  $e$  denotes the singular 1-chain of  $I$  given by the identity map on  $I$ .<sup>1</sup> From Moore [8, Theorem 7.1] it follows that  $H_*(X; Z_p) \approx \text{Tor}^4(Z_p, Z_p)$  as  $Z_p$ -coalgebras, and the rest is immediate from 3.4 by setting  $E^r(\Omega X; Z_p) = E^r(A)$ .

**REMARK.** The results of this paper up to this point could be generalized as follows: an  $H$ -space  $Y$  is  $n$ -homotopy commutative if and only if there exists for  $k = 0, 1, \dots, n$  maps

$$M_k^2: (Y \times Y) \times (I \times Y \times Y)^k \rightarrow Y$$

<sup>1</sup> The only nontriviality involved is to know that  $h_n(a_1 \otimes \cdots \otimes a_{n+1}) = 0$  when each  $a_i = c_i \otimes 1$  (or  $1 \otimes c_i$ ) and this follows from the remark which precedes 1.7.

with the appropriate properties as given in § 1. Then strongly homotopy commutative would be the same as  $\infty$ -homotopy commutative. The definitions and proofs above could be modified to obtain a Hopf algebra structure in the terms  $E^1(Y; Z_p), \dots, E^{n+1}(Y; Z_p)$  of the Moore spectral sequence converging to  $E^0(H_*(B_Y; Z_p))$  using as hypothesis only that  $Y$  is  $n$ -homotopy commutative.

**4. The suspension theorem.** Using the results above we prove a suspension theorem which is a slight improvement on Browder [1, Theorem 5.13]. The notation and terminology is that of [5]; in particular  $Q$  is the functor which assigns to an algebra its module of decomposable elements, and  $P$  is the functor which assigns to a coalgebra its submodule of primitive elements.  $\Gamma(x)$  denotes the ring with divided powers of  $x$  as defined in [5, Chapter 5] and  $\gamma_k(x) = x^k/k!$ .

If  $\sigma_*: H_*(\Omega X; Z_p) \rightarrow H_*(X; Z_p)$  denotes the suspension in homology mod  $p$  for the contractible fibre space of paths over  $X$  as defined in [7], then  $\text{Ker } \sigma_*$  contains the decomposable elements of  $H_*(\Omega X; Z_p)$  and  $\text{Im } \sigma_*$  is contained in the submodule of primitive elements of  $H_*(X; Z_p)$  and  $\sigma_*$  has degree 1. (See [7] for proofs.) Therefore  $\sigma_*$  induces in each degree  $i \geq 1$  a homomorphism

$$s_i: Q(H_i(\Omega X; Z_p)) \rightarrow P(H_{i+1}(X; Z_p))$$

from the indecomposable elements of degree  $i$  of  $H_*(\Omega X; Z_p)$  to the primitive elements of degree  $i + 1$  of  $H_*(X; Z_p)$ . Our suspension theorem will be a statement about the  $s_i$ . The philosophy of the proof will be a bit more clear if the reader bears in mind that taking  $A = C_N(\Omega X; Z_p)$ ,  $H_*(X; Z_p) \approx \text{Tor}^A(Z_p, Z_p) \equiv H(\bar{B}(A))$  and under this isomorphism the suspension is given by  $x \rightarrow [x]$  on the chain level. More precisely if  $\sigma: A \rightarrow \bar{B}(A)$  is given by  $\sigma(x) = [x]$ , then we have a commutative diagram

$$\begin{array}{ccc} H_*(\Omega X; Z_p) & \xrightarrow{\sigma_*} & H_*(X; Z_p) \\ \approx \downarrow & & \downarrow \approx \\ H(A) & \xrightarrow{\sigma_*} & H(\bar{B}(A)) . \end{array}$$

In fact this is just [8, Proposition 7.2] applied to the case at hand.

**THEOREM 4.1.** *Let  $X$  be a pathwise connected and simply connected  $H$ -space of finite homological type and let  $s_i$  denote the homology mod  $p$  suspension in degree  $i$  as defined above. Then*

- (a) *if  $p \neq 2$ ,  $s_i$  is a monomorphism unless  $i = (2mp^k + 2)p^q - 2$  for  $k > 0, q > 0$ , and  $Q(H_{2m}(\Omega X; Z_p)) \neq 0$ , or unless  $i = (2m + 2)p^q - 2$  for  $q > 0$  and  $Q(H_{2m+1}(\Omega X; Z_p)) \neq 0$ ;*
- (b) *if  $p \neq 2$ ,  $s_i$  is an epimorphism unless  $i = 2mp^k + 1$  for  $k > 0$*



and  $Q(H_{2m}(\Omega X; Z_p)) \neq 0$ ;

(c) if  $p = 2$ ,  $s_i$  is a monomorphism unless  $i = 2^q(2^k m + 2) - 2$ , for  $q > 0, k > 0$ , and  $Q(H_m(\Omega X; Z_2)) \neq 0$ ;

(d) if  $p = 2$ ,  $s_i$  is an epimorphism unless  $i = 2^k m + 1$  for  $k > 0$  and  $Q(H_m(\Omega X; Z_2)) \neq 0$ .

*Proof.* Since  $X$  has finite homological type,  $\Omega X$  does also, and  $\Omega X$  is pathwise connected.  $H_*(\Omega X; Z_p)$  is a commutative and associative Hopf algebra over  $Z_p$  and is connected. From the Borel decomposition theorem for Hopf algebras it follows that as an algebra,  $H_*(\Omega X; Z_p)$  is a tensor product of exterior, polynomial and truncated polynomial algebras, each with a single generator. Since  $\text{Tor}^4(K, K)$  commutes with tensor products (as a functor of  $A$ ) [2, Chapter XI, Theorem 3.1, p. 209], to compute  $E^2 \approx \text{Tor}^{H_*(\Omega X; Z_p)}(Z_p, Z_p)$  in the spectral sequence of 3.5, we need only compute on sample factors. The results are listed in the table below. The first entry is given by [8, Proposition 4.1], and the others admit similar and very simple proofs.

	A	$\text{Tor}^4(Z_p, Z_p)$
	$E(x, 2m + 1)$	$\Gamma(sx, 1, 2m + 1)$
$p \neq 2$	$L(x, 2m)$	$E(sx, 1, 2m)$
	$L(x, 2m)/(x^{2^k})$	$E(sx, 1, 2m) \otimes \Gamma(tx, 2, 2mp^k)$
-----		
	$E(x, m) = L(x, m)/(x^2)$	$\Gamma(sx, 1, m)$
$p = 2$	$L(x, m)$	$E(sx, 1, m)$
	$L(x, m)/(x^{2^k}) \quad (k > 1)$	$E(sx, 1, m) \otimes \Gamma(tx, 2, 2^k m)$

where  $E(x, n)$ ,  $L(x, n)$ , and  $\Gamma(x, n)$  indicate exterior, polynomial, and divided polynomial algebras on a single generator of degree  $n$ . In the right hand column bidegrees are specified, and all entries are Hopf algebras with primitive generators.  $sx$  and  $tx$  indicate the suspension and transpotence of  $x$ . (For the definition of transpotence see [9].)

By induction on  $r$  we shall prove the following statements for  $E^r$ :

(1) The generators of odd degree have external degree 1 and are primitive.

(2) Generators of even degree are primitive if the external degree is 1 and nonprimitive for external degree greater than 2, and the nonprimitive generators have the form  $\gamma_{p^q}$  where  $x$  is a primitive generator.

(3) If  $\gamma_k(x) \neq 0$ , but  $\gamma_{k+1}(x) = 0$ , then  $k < r$ .

(4) As a differential Hopf algebra  $E^r$  is the tensor product of differential Hopf algebras with differential identically zero, and differential Hopf algebras of the form  $E(x, 1, m) \otimes \Gamma(y, u, v)$  where  $d^r(\gamma_{p^q}(y)) =$

$x$  and where (consequently)  $r = up^{q-1}$  for  $u = 1$  or  $u = 2$ .

Clearly (4) implies (1), (2), and (3) for  $E^{r+1}$ , and we need only show that (1), (2), and (3) imply (4). Suppose that  $d^r$  does not vanish completely. Let  $z$  be a generator of minimal degree such that  $d^rz \neq 0$ . Since  $r \geq 2$  and  $d^r$  has bidegree  $(-r, r - 1)$  it must be that  $z = \gamma_{p^q}(y)$  where  $y$  has external degree  $u = 1$  or  $u = 2$ . Furthermore  $z$  has even total degree and  $d^rz$  is therefore a primitive element of odd degree and therefore has external degree 1. We may therefore assume without loss of generality that  $x = d^rz$  is a generator. It follows that  $r = up^q - 1$ . Then setting  $A = E(x, 1, m) \otimes \Gamma(y, u, v)$  we have that  $E^r \approx A \otimes B$  where  $B$  is a differential Hopf algebra satisfying (1), (2), and (3) and the same argument may be repeated for  $B$ , and so on until  $E^r$  is exhausted. A small modification is necessary in the case that  $p = 2$  and  $x = d^rz$  generates a divided polynomial algebra  $\Gamma(x, 1, m)$ , but we leave this to the reader. It should also be remarked that in the notation of [5], the algebra  $B$  is usually denoted by  $E^r//A$ .

Since the ground ring is the field  $Z_p$ , we have  $E^\infty \approx H_*(X; Z_p)$  as a  $Z_p$ -module. Then the following diagram is commutative

$$\begin{array}{ccc}
 H_*(\Omega X; Z_p) & \xrightarrow{\sigma_*} & H_*(X; Z_p) \\
 \searrow \sigma_*^\infty & & \downarrow \approx \\
 & & E^\infty(\Omega X; Z_p) = E^0(H_*(X; Z_p))
 \end{array}$$

where  $\sigma_*^\infty$  is induced from the algebraic suspension  $\sigma$  defined above. Since the primitive elements of  $H_*(X; Z_p)$  are mapped into (but not necessarily onto) primitive elements of  $E^\infty(\Omega X; Z_p)$  by the vertical isomorphism, information about  $\sigma_*$  may be obtained from information about  $\sigma_*^\infty$  which can be calculated routinely using (4) above.

Evidently by purely algebraic considerations of the filtration on the bar construction it could be shown that the vertical isomorphism is a homomorphism of coalgebras. Since the primitive submodule of  $E^\infty$  gives us an upper bound on the primitive elements of  $H_*(X; Z_p)$ , we have not tried to carry this out.

*Added in proof.* Judging by P. J. Hilton's review [Math. Reviews, 29, 2809] a similar proof of a theorem similar to 4.1 appears in Samuel Gitler's paper, *Spaces fibered by H-spaces* (Spanish), Bol. Soc. Mat. Mexicana (2) 7 (1962), 71-84. On the evidence of the review, it appears that Gitler's proof contains the algebraic version of Moore's error.

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