THE ASYMPTOTIC NATURE OF THE SOLUTIONS OF CERTAIN LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Suppose $y'(t) = [A + V(t) + R(t)]y(t)$ is a system of differential
equations defined on $[0, \infty)$, where A is a constant matrix,
$V(t) \to 0$ as $t \to \infty$ and the norms of the matrices $V'(t)$ and $R(t)$
are summable. If the roots of the characteristic polynomial
of A are simple, then under suitable conditions on the real
parts of the roots of the characteristic polynomials of $A + V(t)$
a theorem of N. Levinson gives an asymptotic estimate of the
behavior of the solutions of the differential system as $t \to \infty$.
In this paper Levinson's theorem is improved by removing the
condition that the characteristic roots of A are simple. Under
suitable conditions on $V(t)$ and $R(t)$ and the characteristic roots
of $A + V(t)$, which reduce to Levinson's conditions when the
characteristic roots of A are simple, asymptotic estimates are
obtained for the solutions of the given system.

The proof given here, with essential modifications, will follow the
proof given by Levinson [3] [2, p. 92]. One interest in the improved
theorem is in its application to the problem of finding the deficiency
index of an ordinary self-adjoint differential operator, which will appear
in a subsequent paper. We shall establish the following.

THEOREM.[1] *Let A be a constant $n \times n$ matrix whose minimal
polynomial is of degree n and is of the form*

$$\chi(\lambda) = \prod_{k=1}^{m} (\lambda - \lambda_k)^{n_k}, \ \lambda_j \neq \lambda_k \ for \ j \neq k, \ \sum_{k=1}^{m} n_k = n \ .$$

*Let $q + 1 = \max n_k$, $V(t)$ an $n \times n$ matrix with $(q + 1)$-times continu-
ously differentiable elements satisfying $t^{2q} | v_{ij}^{(r)}(t) |^{1/r} \in L^1$ for $1 \leq r \leq
q + 1$ and $V(t) \to 0$ as $t \to \infty$. Let the roots of $\det (A + V(t) - \lambda I) = 0$
be $\{\lambda_k(t)\}_1^m$ and for $t \geq \tau_0$ we suppose the minimal polynomial of
$A + V(t)$ is*

$$\chi(\lambda, t) = \prod_{k=1}^{m} (\lambda - \lambda_k(t))^{n_k} \ ,$$

where $\lambda_k(t) \to \lambda_k$ as $t \to \infty$. For a given k, let

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    [1] If A is an $n \times n$ matrix with entries a_{ij} we shall write $| A | = \sum_{ij} | a_{ij} |$. If x
is a vector with entries x_i we shall write $| x | = \sum_i | x_i |$.

$$d_{kj}(t) = Re(\lambda_k(t) - \lambda_j(t)) \,,$$

and suppose that all $j, 1 \leq j \leq n$, *fall into one of two classes* I_1 *and* I_2, *where* $j \in I_1$, *if and only if* $t^{-q} \exp \int_0^t d_{kj} \to \infty$ *as* $t \to \infty$ *and*

$$(|t - \tau|^q + 1) \exp - \int_\tau^t d_{kj} < M < \infty \quad for \quad t \geq \tau \geq 0 \,,$$

$j \in I_2$ *if and only if* $\int_\tau^t d_{kj} < \log M$ *for* $t \geq \tau \geq 0$. *Let* $R(t)$ *be a matrix valued function with measurable elements such that* $t^{2q} |R(t)| \in L^1$. *Let* $\{q_{kj}; 1 \leq j \leq n_k\}$ *be a set of "principal vectors" for* λ_k; *i.e.,* $q_{kj} = (A - \lambda_k I)^{n_k - j} g_{kn_k}$, $(A - \lambda_k I)^{n_k - 1} g_{kn_k} \neq 0$ *and* $(A - \lambda_k I)^{n_k} g_{kn_k} = 0$. *Then, given the differential equation*

$$(1.1) \qquad\qquad y'(t) = [A + V(t) + R(t)]y(t)$$

there exists a t_0 *and a fundamental system of solutions* $\{y_{kj}(t); 1 \leq j \leq n_k, 1 \leq k \leq m\}$ *such that*

$$\left[\frac{t^{j-1}}{(j-1)!} \exp \int_{t_0}^t \lambda_k(\tau)d\tau \right]^{-1} y_{kj}(t) - q_{kj} \to 0, \, t \to \infty \,.$$

2. We begin the proof by first considering a differential system of the form

$$(2.1) \qquad\qquad y'(t) = (A(t) + R(t))y(t) \,,$$

where $A(t)$ is a matrix with blocks $\{J_j(t)\}_1^m$ down the main diagonal and zeros elsewhere, $J_j(t)$ being an $n_j \times n_j$ matrix with the same number $\lambda_j(t)$ down the main diagonal, 1 down the superdiagonal and zeros elsewhere, and $R(t)$ has measurable entries with $t^{2q} |R(t)| \in L^1$, where $q + 1 = \max \{n_j, 1 \leq j \leq m\}$.

One fundamental matrix Ψ for the system

$$(2.2) \qquad\qquad y'(t) = A(t)y(t)$$

has blocks $\{P_j\}_1^m$ down the main diagonal and zeros elsewhere, where P_j is an $n_j \times n_j$ matrix of the form

$$(2.3) \qquad P_j(t) = \exp \int_{t_0}^t \lambda_j \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n_j-1}/(n_j-1)! \\ 0 & 1 & t & \cdots & t^{n_j-2}/(n_j-2)! \\ \vdots & \cdot & & \cdot & \\ 0 & \cdot & \cdot & \cdot & 0 \quad 1 \end{bmatrix}.$$

This may be checked by a direct computation. Again, it may be easily checked that

$$P_j^{-1}(t) = \exp - \int_{t_0}^{t} \lambda_j \begin{bmatrix} 1 & -t & t^2/2! & -t^3/3! & \cdots & (-1)^{n_j-1}t^{n_j-1}/(n_j-1)! \\ 0 & 1 & -t & t^2/2! & \cdots & (-1)^{n_j-2}t^{n_j-2}/(n_j-2)! \\ \vdots & \cdot & \cdot & \cdot & & \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

and

(2.4)

$$P_j(t)P_j^{-1}(\tau) = \exp \int_{\tau}^{t} \lambda_j \begin{bmatrix} 1 & (t-\tau) & (t-\tau)^2/2! & \cdots & (t-\tau)^{n_j-1}/(n_j-1)! \\ 0 & 1 & (t-\tau) & \cdots & (t-\tau)^{n_j-2}/(n_j-2)! \\ \vdots & \cdot & \cdot & & \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}.$$

Let us fix k and let Ψ_1 be that matrix with zeros everywhere except for diagonal blocks $\{P_j; j \in I_1\}$, where each such P_j has the same position as in the matrix Ψ. Let Ψ_2 be the corresponding type matrix with diagonal blocks $\{P_j; j \in I_2\}$. Clearly $\Psi = \Psi_1 + \Psi_2$.

Let e_i be the vector with jth component equal to δ_{ij}, δ_{ij} being the Kronecker symbol. Now set $i = l + \sum_{j=1}^{k-1} n_j$, where $1 \leqq l \leqq n_k$, and consider the equation

(2.5)
$$\phi(t) = \Psi(t)e_i + \int_{t_0}^{t} \Psi_1(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau$$
$$- \int_{t}^{\infty} \Psi_2(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau .$$

It may be checked by a straightforward computation that, at least formally, ϕ is a solution to (2.1). Hence, if it can be shown that a solution to (2.5) exists, where the integrands are in L^1, then this solution will also be a solution to (2.1).

We proceed by successive approximations. Choose $\phi^0 = 0$ and hence $\phi^1 = \Psi(t)e_i$. It follows that

(2.6)
$$|\phi^1 - \phi^0| \leqq \left[\exp \int_{t_0}^{t} Re\lambda_k \right] \sum_{j=0}^{l-1} t^j/j!$$

Now, the matrix $\Psi_1(t)\Psi^{-1}(\tau)$ has blocks along the main diagonal which are zero in those positions for which $j \in I_2$ and of the form (2.4) in those positions for which $j \in I_1$. Hence, using the hypothesis of the theorem of § 1, for $t_0 \leqq \tau \leqq t$ we have

(2.7)

$$|\Psi_1(t)\Psi^{-1}(\tau)R(\tau)| \leqq C[|t-\tau|^q + 1] \exp\left(-\int_{\tau}^{t} d_{kj}\right) \exp\left(\int_{\tau}^{t} Re\lambda_k\right) |R(\tau)|$$

$$\leqq CM |R(\tau)| \exp \int_{\tau}^{t} Re\lambda_k ,$$

where C is a suitable constant dependent only of q. In the same way, for $t \leqq \tau < \infty$,

$$(2.8) \qquad | \Psi_2(t)\Psi^{-1}(\tau)R(\tau) | \leqq CM\,[\,|\,t - \tau\,|^q + 1\,]\,|\,R(\tau)\,|\,\exp - \int_t^\tau Re\lambda_k\,.$$

Using the estimates (2.6), (2.7) and (2.8) we arrive at the estimate

$$(2.9) \qquad \begin{aligned} &|\,\phi^2 - \phi^1\,|\,\exp - \int_{t_0}^t Re\lambda_k \\ &\leqq CM\Big\{\int_{t_0}^t |\,R(\tau)\,|\sum_{j=0}^{l-1} \tau^j/j!\,d\tau + \int_t^\infty |\,R(\tau)\,|\,[\,|\,t - \tau\,|^q + 1\,]\sum_{j=0}^{l-1}\tau^j/j!\,d\tau\Big\}\,. \end{aligned}$$

Now using the fact that $\tau^{2q}\,|\,R(\tau)\,| \in L^1$ we can choose t_0 so large so that

$$(2.10) \qquad |\,\phi^2 - \phi^1\,|\,\exp - \int_{t_0}^t Re\lambda_k \leqq 1/2 \quad \text{for} \quad t \geqq t_0\,.$$

Using (2.7), (2.8) and (2.10) and proceeding by induction we find that for $j \geqq 1$,

$$(2.11) \qquad \begin{aligned} &|\,\phi^{j+1} - \phi^j\,|\,\exp - \int_{t_0}^t Re\lambda_k \\ &\leqq (1/2)^{j-1}CM\Big\{\int_{t_0}^t |\,R(\tau)\,|\,d\tau + \int_t^\infty [\,|\,t - \tau\,|^q + 1\,]\,|\,R(\tau)\,|\,d\tau\Big\} \\ &\leqq (1/2)^j\,. \end{aligned}$$

This means that there exists a function ϕ so that on every compact subinterval of $[t_0, \infty)$, ϕ^j goes uniformly to ϕ, and indeed, using (2.6),

$$(2.12) \qquad |\,\phi - \phi^j\,| \leqq (1/2)^{j-1}\exp\int_{t_0}^t Re\lambda_k,\ |\,\phi\,| \leqq C[t^q + 1]\exp\int_{t_0}^t Re\lambda_k\,.$$

The estimates (2.12) taken together with the estimate (2.8) shows that the integrands in (2.5) are in L^1 and that indeed ϕ is a solution of that equation.

We claim that

$$(2.13) \qquad [\phi(t) - \Psi(t)e_i]\exp - \int_{t_0}^t \lambda_k \to 0 \quad \text{as} \quad t \to \infty\,.$$

To show this, it is enough to show that

$$(2.14) \qquad \exp\Big(-\int_{t_0}^t Re\lambda_k\Big)\int_{t_0}^t \Psi_1(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau \to 0 \quad \text{as} \quad t \to \infty,\ \text{and}$$

$$(2.15) \qquad \exp\Big(-\int_{t_0}^t Re\lambda_k\Big)\int_t^\infty \Psi_2(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau \to 0 \quad \text{as} \quad t \to \infty\,.$$

Using (2.12) and (2.8) we see that the norm of (2.15) is less than or equal to

$$C^2 M \int_t^\infty [|\,t - \tau\,|^q + 1][\tau^q + 1] \,|\,R(\tau)\,|\,d\tau,$$

which goes to zero as $t \to \infty$. To prove (2.14) we use the fact that $t^{-q} \exp \int_{t_0}^t d_{kj} \to \infty$. Choose t_1 so that $CM \int_{t_1}^\infty |\,R(\tau)\,|\,|\,\phi(\tau)\,|\,d\tau < \varepsilon$. Then the norm of (2.14) is less than or equal to

$$\varepsilon + \exp\left(-\int_{t_0}^t Re\lambda_k\right) |\,\Psi_1(t)\,| \int_{t_0}^{t_1} |\,\Psi^{-1}(\tau)\,|\,|\,R(\tau)\,|\,|\,\phi(\tau)\,|\,d\tau\,.$$

Now,

$$\exp\left(-\int_{t_0}^t Re\lambda_k\right) |\,\Psi_1(t)\,| \leqq Ct^q \sum_{j \in I_1} \exp -\int_{t_0}^t d_{kj} \to 0 \quad \text{as } t \to \infty\,.$$

Hence we see that (2.14) is valid.

The vector $\left[\exp -\int_{t_0}^t \lambda_k\right] \Psi(t)e_i$ has the entry $t^{l-j-1}/(l - j - 1)!$ in the $i + j$ position, $0 \leqq j \leqq l - 1$, and zero elsewhere. Hence

(2.16) $$\left\{\frac{t^{l-1}}{(l - 1)!} \exp \int_{t_0}^t \lambda_k\right\}^{-1} \phi(t) - e_i \to 0 \quad \text{as } t \to \infty\,.$$

Let us designate the solution we have obtained in the previous considerations by ϕ_i. Then the set of solutions $\{\phi_i\}_1^n$ is a fundamental system for (2.1). Indeed, it is clear that the determinant of the matrix Φ with the vectors ϕ_i as columns is nonzero for t sufficiently large.

3. In order to use the results of § 2 to prove the theorem of § 1 it will be necessary to establish the following.

LEMMA. *Suppose the matrix $A + V(t)$ satisfies the conditions of the theorem of § 1. Then for all sufficiently large t there exists a differentiable and invertible matrix $P(t)$ such that $t^{2q}\,|\,P^{-1}(t)P'(t)\,| \in L^1$, $P(t)[A + V(t)]P^{-1}(t)$ is a Jordan canonical form, $P(t) \to P$ and $P^{-1}(t) \to P^{-1}$ as $t \to \infty$, where PAP^{-1} is a corresponding Jordan canonical form for A, and the columns of P^{-1} are a given set of principal vectors for A.*

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the distinct eigenvalues of A. Since the coefficients of the characteristic polynomial of $A + V(t)$ are continuous functions of t in a neighborhood of ∞, using the hypothesis of the theorem, there exists a neighborhood of ∞ so that $A + V(t)$ has eigenvalues $\lambda_1(t), \cdots, \lambda_m(t)$ which are continuous for all t in that neighborhood. In particular, this means that $\lambda_k(t) \to \lambda_k$ as $t \to \infty$.

In fact, for t sufficiently large, each $\lambda_k(t)$ is $(q + 1)$-times continu-

ously differentiable. To see this, we consider the characteristic polynomial

$$(3.1) \qquad F(\lambda, t) = \sum_{j=0}^{n} f_j(t)\lambda^{n-j} = (-1)^n \prod_{j=1}^{m} (\lambda - \lambda_j(t))^{n_j} ,$$

where $f_j(t)$ is $(q + 1)$-times continuously differentiable. If we set $G_k(\lambda, t) = \partial^{n_k - 1} F(\lambda, t)/\partial\lambda^{n_k - 1}$, then $G_k(\lambda_k(\tau), \tau) = 0$, but $\partial G_k(\lambda_k(\tau), \tau)/\partial\lambda \neq 0$. Hence, the implicit function theorem tells us that there exists a neighborhood about τ and a $(q + 1)$-times continuously differentiable function μ_k, defined in this neighborhood, so that $\mu_k(\tau) = \lambda_k(\tau)$ and $G_k(\mu_k(t), t) = 0$. Moreover, if any other continuous function satisfies the last two conditions, then this other function coincides with μ_k in some neighborhood of τ. Hence $\lambda_k(t) = \mu_k(t)$ in some neighborhood of τ, which proves our assertion.

Let $\{q_{kj}; 1 \leq j \leq n_k\}$ be a given set of principal vectors for λ_k and let Q be the matrix whose columns are $\{q_{11}, \cdots, q_{1n_1}, q_{21}, \cdots, q_{2n_2}, \cdots, q_{m1}, \cdots, q_{mn_m}\}$, in the given order. Then, since the minimal and characteristic polynomials of A are of the same degree, $Q^{-1}AQ$ is in the Jordan canonical form (see e.g. [1], Ch. XVII). If V_k is the subspace generated by $\{g_{kj}; 1 \leq j \leq n_k\}$, then A is reduced by V_k. Hence, if we set

$$\pi_k(A) = \prod_{j \neq k} (A - \lambda_j)^{n_j} ,$$

then this matrix is reduced by V_k and the restriction of $\pi_k(A)$ to V_k has an inverse. Let us set $h_k = \pi_k^{-1}(A)q_{kn_k}$, where by $\pi_k^{-1}(A)$ we mean the inverse of the restriction of $\pi_k(A)$ to V_k.

Let us write the minimal polynomial, $\chi(\lambda, t)$, of $A + V(t)$ as

$$\chi(\lambda, t) = (\lambda - \lambda_k(t))^{n_k}\pi_k(\lambda, t) ,$$

where

$$\pi_k(\lambda, t) = \prod_{j \neq k} (\lambda - \lambda_j(t))^{n_j} .$$

Set $q_{kn_k}(t) = \pi_k(A + V(t), t)h_k$; then since $\pi_k(A + V(t), t) \to \pi_k(A)$ as $t \to \infty$, it follows that if we set

$$q_{kj}(t) = (A + V(t) - \lambda_k(t))^{n_k - j}q_{kn_k}(t)$$

the set $\{q_{kj}(t)\}_1^{n_k}$ forms a set of principal vectors for the eigenvalue $\lambda_k(t)$, provided t is sufficiently large. Indeed for t sufficiently large,

$$(A + V(t) - \lambda_k(t))^{n_k - 1}q_{kn_k}(t) \neq 0 ,$$

but

$$(A + V(t) - \lambda_k(t))^{n_k}q_{kn_k}(t) = \chi(A + V(t), t)h_k = 0 .$$

If $Q(t)$ is the matrix whose columns are the vectors

$$\{q_{11}(t), \cdots, q_{1n_1}(t), q_{21}(t), \cdots, q_{2n_2}(t), \cdots, q_{m1}(t), \cdots, q_{mn_m}(t)\} ,$$

in the order given, then $Q^{-1}(t)[A + V(t)]Q(t)$ is in the Jordan canonical form ([1]).

Notice that the elements of $Q(t)$ are polynomial functions in $\{\lambda_k(t)\}_1^m$ and the elements of $A + V(t)$, and hence the elements of $Q^{-1}(t)$ are rational functions in these variables, where the denominator of each rational function is $\det Q(t)$. Hence, if we set $P(t) = [\det Q(t)]Q^{-1}(t)$, then the elements of $P(t)$ are polynomials in the previously mentioned variables and $P(t)[A + V(t)]P^{-1}(t)$ is in the Jordan canonical form. Further, from the assumptions of the lemma, and the manner of construction of $Q(t)$, it is clear that $Q(t) \to Q$, where $Q^{-1}AQ$ is in the Jordan canonical form. Hence $P(t) \to P$, where PAP^{-1} is in the Jordan canonical form.

Since $P^{-1}(t) \to P^{-1}$, it is clear that $P^{-1}(t)$ is bounded in a neighborhood of infinity. Hence, if we can show that $t^{2q} | P'(t) | \in L^1$ we will have proved the lemma. The elements of $P'(t)$ are linear functions of $\{\lambda_k'(t)\}_1^m$ and $\{v_{ij}'(t)\}$ (the entries of $V'(t)$) with coefficients which are bounded in a neighborhood of infinity. Since, by hypothesis $t^{2q} | v_{ij}'(t) | \in L^1$, if we can show that $t^{2q} | \lambda_k'(t) | \in L^1$ we will be done.

Use (3.1) to obtain

$$\frac{\partial^{n_k} F(\lambda_k(t), t)}{\partial t^{n_k}} = \sum_{j=1}^{n} f_j^{(n_k)}(t)\lambda_k^{n-j}(t)$$

$$= (-1)^{n+n_k}\left[n_k! \prod_{j \neq k} (\lambda_k(t) - \lambda_j(t))^{n_j}\right][\lambda_k'(t)]^{n_k} .$$

Since $\prod_{j \neq k} (\lambda_k(t) - \lambda_j(t))^{n_j}$ is uniformly bounded away from zero and $\lambda_k(t)$ is bounded, in a neighborhood of ∞, it follows that there exists a constant N such that

$$(3.2) \qquad | \lambda_k'(t) | \leqq N\left[\sum_{1}^{n} | f_j^{(n_k)}(t) |\right]^{1/n_k} \leqq N \sum_{1}^{n} | f_j^{(n_k)}(t) |^{1/n_k} .$$

Each function f_j is the sum of suitably signed products of elements of $A + V(t)$. A typical term in the sum representing f_j is say $a_1(t) \cdots a_j(t)$, where $a_i(t)$ is an entry of $A + V(t)$. The n_k derivative of this product is given by

$$\sum C_{i_1, \cdots, i_j} a_1^{(i_1)}(t) \cdots a_j^{(i_j)}(t) ,$$

where $C_{i_1, \cdots, i_j}$ are the constants which appear in the multinomial expansion of $(x_1 + \cdots + x_j)^{n_k}$ and the sum is taken over all j-tuples of nonnegative integers, $(i_1, \cdots, i_j)$, whose sum is n_k. Hence if

(3.3) $$t^{2q} \, | \, a_1^{(i_1)} \cdots a_j^{(i_j)} \, |^{1/n_k} \in L^1$$

it will follow that $t^{2q} \, | \, \lambda_k' \, | \in L^1$ and hence $t^{2q} \, | \, P'(t) \, | \in L$.

If $\sum_{r=1}^{j} i_r = n_k$, we may apply Holder's inequality to get,

(3.4) $$\int_{t_0}^{\infty} t^{2q} \left| \prod_{r=1}^{j} a_r^{(i_r)} \right|^{1/n_k} \leq \prod_{r=1}^{j} \left[\int_{t_0}^{\infty} t^{2q} \, | \, a_r^{(i_r)} \, |^{1/i_r} \right]^{i_r/n_k} ,$$

where we make the convention that if $i_r = 0$, then

$$\| \, a_r \, \|_{\infty} = \sup_{t \geq t_0} | \, a_r(t) \, | = \left[\int_{t_0}^{\infty} t^{2q} \, | \, a_r^{(i_r)} \, |^{1/i_r} \right]^{i_r/n_k} .$$

From the hypothesis of the lemma it follows from (3.4) that (3.3) is satisfied and hence lemma is proved.

4. Using the results of § 2 and § 3 it is now an easy matter to finish the proof of the theorem stated in § 1. Make the transformation $x(t) = P(t)y(t)$ in (1.1) and we get the equation

(4.1) $$x' = [P(A + V)P^{-1} - P^{-1}P' + PRP^{-1}]x .$$

The matrix $P(A + V)P^{-1}$ is in the Jordan form of the matrix $A(t)$ of (2.1) and $t^{2q} \, | \, PRP^{-1} - P^{-1}P' \, | \in L^1$. Hence, we may apply the results of § 2 and for $i = l + \sum_{j=1}^{k-1} n_j, 1 \leq l \leq n_k$, we find a solution x_i such that

$$\left[\frac{t^{l-1}}{(l-1)!} \exp \int_{t_0}^{t} \lambda_k \right]^{-1} x_i(t) - e_i \to 0 \quad \text{as } t \to \infty .$$

Hence, if $y_i(t) = P^{-1}(t)x_i$, we get

$$\left[\frac{t^{l-1}}{(l-1)!} \exp \int_{t_0}^{t} \lambda_k \right]^{-1} y_i(t) - P^{-1}e_i \to 0 \quad \text{as } t \to \infty ,$$

where $P^{-1} = \lim_{t \to \infty} P^{-1}(t)$.

The vector $P^{-1}e_i$ is the ith column of P^{-1} which by Lemma 3 can be taken to be the given principal vector q_{kl}. Since the vectors $\{q_{kl}; 1 \leq l \leq n_k, 1 \leq k \leq m\}$ are linearly independent, the vectors $\{y_i(t)\}_1^m$ form a fundamental set of solutions of (1.1). This completes the proof of the theorem.

Note added in proof. The theorem of this paper can be generalized in the following way. Using the same notation as in the theorem let p be a real number satisfying the inequality $0 \leq p \leq q$. Suppose further that for each given k all integers j, $1 \leq j \leq n$, fall into two classes I_1 and I_2 where I_1 is the same as in the hypothesis of the theorem but now I_2 is the collection of j so that

$$(|\,t - \tau\,|^p + 1) \exp \int_\tau^t d_{kj} < M < \infty \quad \text{for} \quad t \geqq \tau \geqq 0 \,.$$

Then under the hypothesis that $t^{2q-p}\,|\,v_{ij}^{(r)}(t)\,|^{1/r}$, $1 \leq r \leq q + 1$, and $t^{2q-p}\,|\,R(t)\,|$ are summable, the conclusion of the theorem holds. The proof of the generalized theorem follows the proof given in the text *mutatis mutandis.*

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