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DECOMPOSITION THEOREMS FOR FREDHOLM OPERATORS

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This paper is devoted to proving and discussing several consequences of the following decomposition theorem:

Let A and B be closed densely-defined linear operators from the Banach space X to the Banach space Y such that $D(B) \supseteq D(A)$, $D(B^*) \supseteq D(A^*)$, the range $R(A)$ of A is closed, and the dimension of the null-space $N(A)$ of A is finite. Then X and Y can be decomposed into direct sums $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$, where X_1 and Y_1 are finite dimensional, $X_1 \subseteq D(A)$, $X_0 \cap D(A)$ is dense in X , and (X_0, Y_0) and (X_1, Y_1) are invariant pairs of subspaces for both A and B . Let A_i and B_i be the restrictions of A and B respectively to X_i . For all integers k , $(B_0 A_0^{-1})^k(0) \subseteq R(A_0)$, and

$$\dim (B_0 A_0^{-1})^k(0) = k \dim (B_0 A_0^{-1})(0) = k \dim N(A_0).$$

Also, the action of A_1 and B_1 from X_1 to Y_1 can be given a certain canonical description.

The object of this paper is to study the operator equation $Ax - \lambda Bx = y$, where A and B are (unbounded) linear operators from a Banach space X to a Banach space Y . In §1, an integer $\mu(A:B)$ is defined, which expresses a certain interrelationship between the null space of A and the null space of B . In §1 and 2, decomposition theorems are proved which refine theorem 4 of [2]. The theorems allow us to split off certain finite dimensional invariant pairs of subspaces of X and Y so that A and B are well-behaved with respect to $\mu(A:B)$ on the remainder.

In §4, the stability of these decompositions under perturbation of A by λB is investigated. In §5, relations between the dimensions of certain subspaces of X and Y are given, and a formula for the Fredholm index of $A - \lambda B$ is obtained. These extend results of Kaniel and Schechter [1], who consider the case $X = Y$ and B the identity operator.

It should be noted that the results of Kaniel and Schechter referred to here follow from theorems 3 and 4 of [2]. The results of this paper properly refine Kato's results only when the null space of B is not $\{0\}$.

1. We will be considering linear operators T defined on a dense linear subset $D(A)$ of a Banach space X , and with values in a Banach space Y . $N(T)$ and $R(T)$ will denote the null space and range of T respectively, while $\alpha(T)$ is the dimension of $N(T)$, and $\beta(T)$ is the

codimension of $\overline{R(T)}$ in Y . T is a Fredholm operator if T is closed, $R(T)$ is closed, and both $\alpha(T)$ and $\beta(T)$ are finite. The index of a Fredholm operator is the integer.

$$\kappa(T) = \alpha(T) - \beta(T).$$

Let P be a subspace of X , Q a subspace of Y . (P, Q) is an *invariant pair of subspaces* for T if $T(P \cap D(T)) \subseteq Q$.

Standing assumptions: In the remainder of the paper, A and B are closed linear operators from X to Y , $D(A)$ is dense in X , $D(B) \supseteq D(A)$, and $D(B^*) \supseteq D(A^*)$; A is semi-Fredholm, in the sense that $R(A)$ is closed and $\alpha(A) < \infty$.

The assumption $D(B^*) \supseteq D(A^*)$ seems necessary for the proof of the decomposition theorems. It is often met when A and B are differential operators on some domain in Euclidean space, and the order of B is less than the order of A . It is always met when B is bounded.

The linear manifolds $N_k = N_k(A:B)$ and $M_k = M_k(A:B)$ are defined by induction as follows:

$$\begin{aligned} N_1 &= N(A) \\ N_k &= A^{-1}(BN_{k-1}), \quad k > 1 \\ M_k &= BN_k. \end{aligned}$$

N_k and M_k are increasing sequences of linear manifolds in X and Y respectively.

The smallest integer n such that N_n is not a subset of $B^{-1}R(A)$ will be denoted by $\nu(A:B)$. If N_n is a subset of $B^{-1}R(A)$ for all n , then we define $\nu(A:B) = \infty$. (cf. [2])

The dimension of N_k will be denoted by $\pi_k = \pi_k(A:B)$, and the dimension of M_k by $\rho_k = \rho_k(A:B)$. Then $\pi_1 = \alpha(A)$, and, in general, $\pi_k \leq k\alpha(A)$. $\mu(A:B)$ will denote the first integer n such that $\pi_n < n\alpha(A)$. If $\pi_n = n\alpha(A)$ for all integers n , then we define $\mu(A:B) = \infty$.

In general, $\mu(A:B) \geq \nu(A:B) + 1$. This inequality is trivial if $\nu = \infty$. If $\nu < \infty$, then $M_{\nu-1} \subseteq R(A)$, while $M_\nu \not\subseteq R(A)$. Consequently, $\pi_{\nu+1} < \pi_\nu + \alpha(A) \leq (\nu + 1)\alpha(A)$, and so $\mu(A:B) \leq \nu + 1$.

We define $\sigma_k(A:B) = \pi_k - \pi_{k-1}$. Then σ_k is the dimension of the quotient space N_k/N_{k-1} . $\{\sigma_k\}$ is a decreasing sequence of nonnegative integers, and so the limit

$$\sigma(A:B) = \lim_{k \rightarrow \infty} \sigma_k(A:B) \quad \text{exists.}$$

If $\mu(A:B) = \infty$, then $\sigma(A:B) = \alpha(A)$.

2. THEOREM 1. *Assume, in addition to the standing assumptions on A and B , that $\nu(A:B) = \infty$. Then X and Y can be decomposed*

into direct sums

$$\begin{aligned} X &= X_0 \oplus X_1 \\ Y &= Y_0 \oplus Y_1, \end{aligned}$$

where X_1 and Y_1 are finite dimensional, $X_1 \subseteq D(A)$, $X_0 \cap D(A)$ is dense in X_0 , and (X_0, Y_0) and (X_1, Y_1) are invariant pairs for both A and B . If A_i and B_i are the restrictions of A and B respectively to X_i , then $\mu(A_0, B_0) = \infty$, while A_1 and B_1 map X_1 onto Y_1 .

Furthermore, X_1 and Y_1 can be decomposed as direct sums

$$\begin{aligned} X_1 &= P_1 \oplus \cdots \oplus P_p \\ Y_1 &= Q_1 \oplus \cdots \oplus Q_p, \end{aligned}$$

where A_1 and B_1 map P_j onto Q_j . Bases $\{x_j^i: 1 < i \leq \eta(j)\}$ and $\{y_j^i: 1 \leq i \leq \eta(j) - 1\}$ can be chosen for P_j and Q_j respectively so that

$$\begin{aligned} Ax_j^{i+1} &= Bx_j^i = y_j^i, & 1 \leq i \leq \eta(j) - 1 \\ Ax_j^1 &= 0 = Bx_j^{\eta(j)}. \end{aligned}$$

Although the decomposition is not, in general, unique, the integers p and $\eta(j)$, $1 \leq j \leq m$, are uniquely determined by A and B . In fact,

$$p = \alpha(A) - \sigma(A : B).$$

Proof. Let $n = \alpha(A)$, and suppose that $\{z_1^1, \dots, z_n^1\}$ is a basis for $N(A)$. Since $\nu(A : B) = \infty$, z_j^i can be chosen by induction so that $Az_j^i = Bz_j^{i-1}$. $\{z_j^i: 1 \leq j \leq n, 1 \leq i \leq m\}$ is a spanning set for N_m , while $\{Bz_j^i: 1 \leq j \leq n, 1 \leq i \leq m\}$ is a spanning set for M_m . Also, $\{z_i^m: 1 \leq i \leq n\}$ span N_m modulo N_{m-1} .

Recall that $\sigma_m = \sigma(m) = \dim(N_m/N_{m-1})$. By induction, the order of the z_j^i can be chosen so that $\{z_{n-\sigma(m)+1}^m, \dots, z_n^m\}$ span N_m modulo N_{m-1} . Then

$$G_m = \{z_j^i: n - \sigma(i) + 1 \leq j \leq n, 1 \leq i \leq m\}$$

is a basis for N_m .

Let $\eta(j)$ be the greatest integer k such that $z_j^k \in G_k$. If $z_j^k \in G_k$ for all k , let $\eta(j) = \infty$. Then $1 \leq \eta(1) \leq \eta(2) \leq \cdots \leq \eta(n)$. Let p be the greatest integer k such that $\eta(k) < \infty$. By definition of σ , it is clear that

$$p = \alpha(A) - \sigma.$$

Suppose $1 \leq j \leq p$. $z_j^{\eta(j)+1}$ is linearly dependent on the set $G_{\eta(j)+1}$, and so we can write

$$z_j^{\eta(j)+1} = \sum \alpha_{ik} z_k^i,$$

where the sum is taken over all pairs of integers (i, k) , with the understanding that $z_k^i = 0$ if $i \leq 0$ and $\alpha_{ik} = 0$ if $z_k^i \notin G_{\eta(j)+1}$. For $-1 \leq q \leq \eta(j)$ define

$$x_j^{\eta(j)-q} = z_j^{\eta(j)-q} - \sum \alpha_{ik} z_k^{i-q-1}.$$

For $0 \leq q \leq \eta(j)$,

$$\begin{aligned} Bx_j^{\eta(j)-q} &= Bz_j^{\eta(j)-q} - \sum \alpha_{ik} Bz_k^{i-q-1} \\ &= Az_j^{\eta(j)-q+1} - \sum \alpha_{ik} Az_k^{i-q} \\ &= Ax_j^{\eta(j)-q+1} \end{aligned}$$

In particular, $Bx_j^{\eta(j)} = 0$.

Since the sum for $x_j^{\eta(j)-q}$ involves $z_j^{\eta(j)-q}$ only in the first term, the $z_j^{\eta(j)-q}$ may be replaced by the $x_j^{\eta(j)-q}$, $0 \leq q \leq \eta(j)$, to obtain another basis for $N_{\eta(j)+1}$. Repeating this process for $1 \leq j \leq p$, and making other appropriate replacements, we arrive at vectors x_j^i such that.

$$(1) \quad x_1^1, \dots, x_1^n \text{ are a basis for } N(A)$$

$$(2) \quad Bx_j^i = Ax_j^{i+1}, \quad 1 \leq i \leq \eta(j)$$

$$(3) \quad Bx_j^{\eta(j)} = 0, \quad 1 \leq j \leq p.$$

For convenience, it is assumed that

$$(4) \quad x_j^i = 0 \quad \text{if } i > \eta(j).$$

If $1 \leq j \leq p$, let P_j be the subspace of X with basis $\{x_j^1, \dots, x_j^{\eta(j)}\}$. Let Q_j be the subspace of Y with basis $\{y_j^1, \dots, y_j^{\eta(j)-1}\}$, where $y_j^i = Bx_j^i = Ax_j^{i+1}$. Let $X_1 = P_1 \oplus \dots \oplus P_p$ and $Y_1 = Q_1 \oplus \dots \oplus Q_p$. Then X_1 and Y_1 satisfy all the conclusions of the theorem. To conclude the proof, it suffices to produce complementary subspaces to X_1 and Y_1 which also form an invariant pair.

We will construct functionals

$$\begin{aligned} \{g_j^i: 1 \leq i \leq \eta(j), \quad 1 \leq j \leq p\} \text{ on } X \text{ and} \\ \{f_j^i: 1 \leq i \leq \eta(j) - 1, \quad 1 \leq j \leq p\} \text{ on } Y \text{ such that} \end{aligned}$$

the f_j^i are in the domain of A^* and

$$(5) \quad g_j^{i+1} = A^* f_j^i, \quad 1 \leq i \leq \eta(j) - 1$$

$$(6) \quad g_j^i = B^* f_j^i, \quad 1 \leq i \leq \eta(j) - 1$$

$$(7) \quad f_j^i(y_k^q) = \delta_{iq} \delta_{jk}, \quad 1 \leq j, k \leq n \\ 1 \leq q \leq i$$

$$(8) \quad \begin{aligned} g_j^i(x_k^q) &= \delta_{iq} \delta_{jk}, & 1 \leq j, k \leq n \\ & & 1 \leq q \leq i. \end{aligned}$$

Let $g_j^{\eta(j)}$ be any functional on X which satisfies (8). The other g_j^i will be chosen by induction.

Suppose that f_k^q and g_k^q are chosen, for $q > i \geq 1$, to satisfy (5) through (8). By (8), g_k^{i+1} is orthogonal to $N(A)$, and so g_k^{i+1} is in the closure of $R(A^*)$. Since $R(A)$ is closed, $R(A^*)$ is closed, and there is an $f_k^i \in D(A^*)$ for which $A^* f_k^i = g_k^{i+1}$. Let $g_k^i = B^* f_k^i$. Then (5) and (6) hold by definition.

To verify (7), we have for $q \leq i$,

$$\begin{aligned} f_j^i(y_k^q) &= f_j^i(Ax_k^{q+1}) \\ &= (A^* f_j^i)(x_k^{q+1}) \\ &= g_j^{i+1}(x_k^{q+1}) = \delta_{iq} \delta_{jk}. \end{aligned}$$

(8) is an immediate consequence of (7).

$$\begin{aligned} \text{Let } X_0 &= \cap \{N(g_j^i): 1 \leq i \leq \eta(j), \quad 1 \leq j \leq p\} \\ Y_0 &= \cap \{N(f_j^i): 1 \leq i \leq \eta(j) - 1, \quad 1 \leq j \leq p\}. \end{aligned}$$

From (7) and (8), it is clear that $X_0 \cap X_1 = \{0\}$ and $Y_0 \cap Y_1 = \{0\}$. Since the codimension of X_0 in X is no greater than the number of functionals g_j^i defining it, and since this number is the dimension of X_1 , we must have $X = X_0 \oplus X_1$. Similarly, $Y = Y_0 \oplus Y_1$.

Suppose $x \in D(A) \cap X_0$. Then $f_j^i(Ax) = (A^* f_j^i)(x) = g_j^{i+1}(x) = 0$, and so $Ax \in Y_0$. Similarly, $Bx \in Y_0$, and (X_0, Y_0) is an invariant pair for both A and B .

Since (X_0, Y_0) and (X_1, Y_1) are invariant pairs, $N_k(A : B) \cap X_0 = N_k(A_0 : B_0)$. For k sufficiently large, $X_1 \subseteq N_k(A : B)$, and so

$$\begin{aligned} \dim \{N_{k+1}(A_0 : B_0) / N_k(A_0 : B_0)\} &= \dim \{N_{k+1}(A : B) / N_k(A : B)\} \\ &= \sigma \\ &= \alpha(A) - p \\ &= \alpha(A_0). \end{aligned}$$

This can occur only if $\dim N_k(A_0 : B_0) = k\alpha(A_0)$ for all integers k . Hence $\mu(A_0 : B_0) = \infty$.

3. Let (P, Q) be an invariant pair of finite dimensional subspaces for A and B . (P, Q) is an *irreducible invariant pair of type ν* if there are bases $\{x_i\}_{i=1}^n$ for P and $\{y_i\}_{i=1}^n$ for Q such that $Bx_i = y_i$, $Ax_1 = 0$, and $Ax_i = y_{i-1}$, $2 \leq i \leq n$.

(P, Q) is an *irreducible invariant pair of type μ* if there are bases $\{x_i\}_{i=1}^n$ for P and $\{y_i\}_{i=1}^n$ for Q such that

$$\begin{aligned} Ax_1 &= 0 = Bx_n \\ Ax_{i+1} &= y_i = Bx_i, \quad 1 \leq i \leq n-1. \end{aligned}$$

(P, Q) is an *irreducible invariant pair of type μ^** if there are bases $\{x_i\}_{i=1}^{n-1}$ for P and $\{y_i\}_{i=1}^n$ for Q such that

$$\begin{aligned} Bx_i &= y_i, \quad 1 \leq i \leq n-1 \\ Ax_i &= y_{i+1}, \quad 1 \leq i \leq n-1. \end{aligned}$$

(P, Q) is an *invariant pair of type ν* if $P = P_1 \oplus \cdots \oplus P_k$ and $Q = Q_1 \oplus \cdots \oplus Q_k$, where (P_j, Q_j) is an irreducible invariant pair of type ν , $1 \leq j \leq k$. *Invariant pairs of type μ or type μ^** are defined similarly.

It is straightforward to verify that if (P, Q) is an (irreducible) invariant pair of type $\mu(A : B)$ (resp. $\mu^*(A : B)$), then (P, Q) is an (irreducible) invariant pair of type $\mu(A - \lambda B : B)$ (resp. $\mu^*(A - \lambda B : B)$), for all complex numbers λ . If (P, Q) is an invariant pair of type μ , then $\nu(A|P, B|P) = \infty$ and $\mu((A|P)^*, (B|P)^*) = \infty$. If (P, Q) is of type μ^* , then $\nu(A|P, B|P) = \infty$ and $\mu(A|P, B|P) = \infty$.

THEOREM 2. *Suppose A and B satisfy the standing hypothesis. Then there exist decompositions*

$$\begin{aligned} X &= X_0 \oplus X_1 \oplus X_2 \\ Y &= Y_0 \oplus Y_1 \oplus Y_2 \end{aligned}$$

Where (X_0, Y_0) is an invariant pair, (X_1, Y_1) is an invariant pair of type μ , and (X_2, Y_2) is an invariant pair of type ν . If A_0 and B_0 are the restrictions of A and B respectively to X_0 , then $\nu(A_0, B_0) = \infty$ and $\mu(A_0, B_0) = \infty$.

Proof. Theorem 2 follows from Theorem 1 and Kato's Theorem 4 [1], after it is noted that the latter theorem, although stated only for bounded operators B , is valid under the less restrictive assumption that $D(B^*) \supseteq D(A^*)$.

THEOREM 3. *In addition to the standing hypotheses, suppose that A is a Fredholm operator. Then there exist decompositions*

$$\begin{aligned} X &= X_0 \oplus X_1 \oplus X_2 \oplus X_3 \\ Y &= Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3, \end{aligned}$$

where each (X_i, Y_i) is an invariant pair, (X_1, Y_1) is of type μ , (X_2, Y_2) is of type ν , and (X_3, Y_3) is of type μ^* . If A_0 and B_0 are the restrictions of A and B to X_0 , then $\nu(A_0 : B_0) = \infty$, $\mu(A_0 : B_0) = \infty$, $\mu(A_0^* : B_0^*) = \infty$, and $\nu(A_0^* : B_0^*) = \infty$.

If $X^* = X_0^* \oplus X_1^* \oplus X_2^* \oplus X_3^*$ and $Y^* = Y_0^* \oplus Y_1^* \oplus Y_2^* \oplus Y_3^*$ are the corresponding decompositions of the adjoint spaces, then (Y_1^*, X_1^*) is an invariant pair of type $\mu_*(A^* : B^*)$, (Y_2^*, X_2^*) is an invariant pair of type $\nu(A^* : B^*)$, and (Y_3^*, X_3^*) is an invariant pair of type $\mu(A^* : B^*)$.

Proof. In view of Theorem 2, we may assume that $\mu(A : B) = \infty$ and $\nu(A : B) = \infty$. Then $\nu(A^* : B^*) = \infty$, and we can proceed to decompose X^* and Y^* , as in the proof of Theorem 1. The only difficulty encountered is to produce vectors x_j^i to span X_3 which actually lie in $D(A)$. An induction argument similar to that used in Theorem 1 to produce the f_j^i and g_j^i can also be employed in this case.

4. Let $\Phi^+(A : B)$ be the set of complex numbers λ such that $A - \lambda B$ is a closed operator from $D(A)$ to Y , and such that $R(A - \lambda B)$ is closed and $\alpha(A - \lambda B) < \infty$. $\Phi^+(A : B)$ is an open subset of the complex plane which, by assumption, contains the point $\lambda = 0$.

For all $\lambda \in \Phi^+(A : B)$, Theorems 1 and 2 are applicable to the operators $A - \lambda B$ and B . Also, for $\lambda \in \Phi^+(A : B)$ we define

$$\begin{aligned} \sigma_k(\lambda) &= \sigma_k(A - \lambda B : B) \\ \pi_k(\lambda) &= \pi_k(A - \lambda B : B) \\ \rho_k(\lambda) &= \rho_k(A - \lambda B : B) \\ \sigma(\lambda) &= \sigma(A - \lambda B : B) . \end{aligned}$$

THEOREM 4. *Let A and B satisfy the standing hypotheses. There exists a decomposition*

$$\begin{aligned} X &= X_0 \oplus X_1 \\ Y &= Y_0 \oplus Y_1 \end{aligned}$$

such that (X_0, Y_0) is an invariant pair, and (X_1, Y_1) is an invariant pair of type $\mu(A - \lambda B : B)$ for all complex numbers λ . If A_0 and B_0 are the restrictions of A and B to X_0 , then $\mu(A_0 - \lambda B_0 : B_0) = \infty$ for all $\lambda \in \Phi^+(A : B)$ satisfying $\nu(A - \lambda B : B) = \infty$.

Proof. The points $\lambda \in \Phi^+(A : B)$ for which $\nu(A - \lambda B : B) < \infty$ form a discrete subset of $\Phi^+(A : B)$, and so there is a $\lambda' \in \Phi^+$ such that $\nu(A - \lambda' B : B) = \infty$. Let $X = X_0 \oplus X_1$ be the decomposition of Theorem 1 with respect to $A - \lambda' B$ and B . Then (X_1, Y_1) is an invariant pair of type $\mu(A - \lambda B : B)$ for all complex numbers λ , as remarked earlier.

If $\lambda \in \Phi^+(A : B)$ and $\nu(A - \lambda B : B) = \infty$, then X_0 and Y_0 cannot be decomposed further as in Theorem 1, for such a decomposition would violate the fact that $\mu(A_0 - \lambda' B_0 : B) = \infty$. Hence $\nu(A - \lambda B : B) =$

∞ implies $\mu(A_0 - \lambda B_0 : B_0) = \infty$.

Let D be the subset of $\Phi^+(A : B)$ of complex numbers λ for which $\nu(A - \lambda B : B) < \infty$. D is a discrete subset of $\Phi^+(A : B)$ with no limit points in $\Phi^+(A : B)$ (cf [1]).

THEOREM 5. $\mu(A - \lambda B : B)$ is a constant, either finite or infinite, for $\lambda \in \Phi^+(A : B) - D$.

Proof. In view of Theorem 4, it suffices to prove the theorem when A and B are operators in an invariant pair of type μ . For this, it suffices to look at an irreducible invariant pair of type μ . This case is easy to verify.

THEOREM 6. $\sigma(\lambda)$ is constant on each component of $\Phi^+(A : B)$.

Proof. It suffices to show that $\sigma(\lambda)$ is constant in a neighborhood of an arbitrary point $\lambda' \in \Phi^+(A : B)$. Let $X = X_0 \oplus X_1 \oplus X_2$ and $Y = Y_0 \oplus Y_1 \oplus Y_2$ be the decomposition of Theorem 2 with respect to $A - \lambda' B$ and B . Then $\nu(A_0 - \lambda B_0 : B_0) = \infty$ for λ near λ' , and so $\sigma(\lambda) = \alpha(A_0 - \lambda B_0)$ for λ near λ' . By Theorem 3, [2], $\alpha(A_0 - \lambda B_0) = \alpha(A_0 - \lambda' B_0)$ for λ near λ' .

5. Let $X = X_0 \oplus X_1 \oplus X_2$ and $Y = Y_0 \oplus Y_1 \oplus Y_2$ be the decompositions of Theorem 2 with respect to A and B . Let $\pi_k = \pi_k^0 + \pi_k^1 + \pi_k^2$ and $\rho_k = \rho_k^0 + \rho_k^1 + \rho_k^2$ be the corresponding decompositions of π_k and ρ_k . Assume that r is chosen small that $0 < |\lambda| < r$ implies $\lambda \in \Phi^+(A : B)$ and $\nu(A - \lambda B : B) = \infty$. Then $\pi_k^2(\lambda) = k\sigma(\lambda)$ for $|\lambda| < r$. If k is sufficiently large,

$$\begin{aligned} \pi_k^1(\lambda) &= \dim X_1, & |\lambda| < r \\ \pi_k^2(\lambda) &= \begin{cases} \dim X_2, & \lambda = 0 \\ 0, & 0 < |\lambda| < r. \end{cases} \end{aligned}$$

Also, $\rho_k^0(\lambda) = k\sigma(\lambda)$ for $|\lambda| < r$. For k sufficiently large,

$$\begin{aligned} \rho_k^1(\lambda) &= \dim Y_1 \\ \rho_k^2(\lambda) &= \begin{cases} \dim Y_2, & \lambda = 0 \\ 0, & 0 < |\lambda| < r. \end{cases} \end{aligned}$$

We define, for any $\lambda \in \Phi^+(A : B)$,

$$(1) \quad \pi(\lambda) = \lim_{k \rightarrow \infty} [\pi_k(\lambda) - k\sigma(\lambda)]$$

$$(2) \quad \rho(\lambda) = \lim_{k \rightarrow \infty} [\rho_k(\lambda) - k\sigma(\lambda)]$$

$\pi(\lambda)$ and $\rho(\lambda)$ correspond to $\tau(\lambda)$ defined in [1]. From the preced-

ing, we deduce that

$$(3) \quad \pi(\lambda) = \begin{cases} \dim X_1, & 0 < |\lambda| < r \\ \dim (X_1 \oplus X_2), & \lambda = 0 \end{cases}$$

$$(4) \quad \rho(\lambda) = \begin{cases} \dim Y_1, & 0 < |\lambda| < r \\ \dim (Y_1 \oplus Y_2), & \lambda = 0. \end{cases}$$

From these formulae, it follows that

$$(5) \quad \alpha(A - \lambda B) = \sigma(\lambda) + \pi(\lambda) - \rho(\lambda), \quad 0 < |\lambda| < r,$$

for both sides of this expression are equal to

$$\alpha(A_0 - \lambda B_0) + \dim X_1 - \dim Y_1.$$

We will assume in the remainder of the discussion that A is a Fredholm operator. The set of complex numbers λ such that $A - \lambda B$ is a Fredholm operator will be denoted by $\Phi(A : B)$. $\Phi(A : B)$ is an open subset of the complex plane, and consists of the union of those components of $\Phi^+(A : B)$ for which $R(A - \lambda B)$ is of finite codimension in Y , i.e., for which $\alpha(A^* - \lambda B^*) < \infty$.

The quantities $\pi_k^*(\lambda) = \pi_k(A^* - \lambda B^* : B^*)$, $\rho_k^*(\lambda)$, $\sigma^*(\lambda)$, $\pi^*(\lambda)$ and $\rho^*(\lambda)$ are then well-defined for $\lambda \in \Phi(A : B)$. The formula for the adjoint operators corresponding to (5) is

$$(6) \quad \alpha(A^* - \lambda B^*) = \sigma^*(\lambda) + \pi^*(\lambda) - \rho^*(\lambda), \quad 0 < |\lambda| < r.$$

Since $\alpha(A^* - \lambda B^*) = \beta(A - \lambda B)$, we have

$$(7) \quad \begin{aligned} \kappa(A - \lambda B) &= (\sigma(\lambda) - \sigma^*(\lambda)) \\ &+ (\pi(\lambda) - \pi^*(\lambda)) - (\rho(\lambda) - \rho^*(\lambda)) \quad 0 < |\lambda| < r. \end{aligned}$$

In view of the decomposition of Theorem 3, the jump discontinuity of π^* at $\lambda = 0$ is equal to that of π at $\lambda = 0$, i.e., they are both equal to $\dim X_2 = \dim Y_2$. Hence (7) holds also for $\lambda = 0$, and we arrive at the following theorem.

THEOREM 7. *For all $\lambda \in \Phi(A : B)$,*

$$\kappa(A - \lambda B) = (\sigma(\lambda) - \sigma^*(\lambda)) + (\pi(\lambda) - \pi^*(\lambda)) - (\rho(\lambda) - \rho^*(\lambda)).$$

Analogous formulae can be written down if it is assumed, further, that B is a Fredholm operator. If $M(B) = \{0\}$ and $R(B)$ is dense in Y_1 then $\rho(\lambda) = \rho^*(\lambda) = \pi(\lambda) = \pi^*(\lambda) = 0$, and Theorem 7 reduces to

$$(8) \quad \kappa(A - \lambda B) = \sigma(\lambda) - \sigma^*(\lambda), \quad \lambda \in \Phi(A : B).$$

This latter formula is due to Kaniel and Schechter [1], when $X = Y$ and B is the identity operator.

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