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# DECOMPOSITION THEOREMS FOR FREDHOLM OPERATORS

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**This paper is devoted to proving and discussing several consequences of the following decomposition theorem:**

**Let $A$ and $B$ be closed densely-defined linear operators from the Banach space $X$ to the Banach space $Y$ such that $D(B) \supseteqq D(A)$, $D(B^*) \supseteqq D(A^*)$, the range $R(A)$ of $A$ is closed, and the dimension of the null-space $N(A)$ of $A$ is finite. Then $X$ and $Y$ can be decomposed into direct sums $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$, where $X_1$ and $Y_1$ are finite dimensional, $X_1 \subseteqq D(A)$, $X_0 \cap D(A)$ is dense in $X$, and $(X_0, Y_0)$ and $(X_1, Y_1)$ are invariant pairs of subspaces for both $A$ and $B$. Let $A_i$ and $B_i$ be the restrictions of $A$ and $B$ respectively to $X_i$. For all integers $k$, $(B_0A_0^{-1})^k(0) \subseteqq R(A_0)$, and**

$$\dim (B_0A_0^{-1})^k(0) = k \dim (B_0A_0^{-1})(0) = k \dim N(A_0) \, .$$

**Also, the action of $A_1$ and $B_1$ from $X_1$ to $Y_1$ can be given a certain canonical description.**

The object of this paper is to study the operator equation $Ax - \lambda Bx = y$, where $A$ and $B$ are (unbounded) linear operators from a Banach space $X$ to a Banach space $Y$. In § 1, an integer $\mu(A{:}B)$ is defined, which expresses a certain interrelationship between the null space of $A$ and the null space of $B$. In § 1 and 2, decomposition theorems are proved which refine theorem 4 of [2]. The theorems allow us to split off certain finite dimensional invariant pairs of subspaces of $X$ and $Y$ so that $A$ and $B$ are well-behaved with respect to $\mu(A{:}B)$ on the remainder.

In § 4, the stability of these decompositions under perturbation of $A$ by $\lambda B$ is investigated. In § 5, relations between the dimensions of certain subspaces of $X$ and $Y$ are given, and a formula for the Fredholm index of $A - \lambda B$ is obtained. These extend results of Kaniel and Schechter [1], who consider the case $X = Y$ and $B$ the indentity operator.

It should be noted that the results of Kaniel and Schechter referred to here follow from theorems 3 and 4 of [2]. The results of this paper properly refine Kato's results only when the null space of $B$ is not $\{0\}$.

1. We will be considering linear operators $T$ defined on a dense linear subset $D(A)$ of a Banach space $X$, and with values in a Banach space $Y$. $N(T)$ and $R(T)$ will denote the null space and range of $T$ respectively, while $\alpha(T)$ is the dimension of $N(T)$, and $\beta(T)$ is the

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codimension of $\overline{R(T)}$ in $Y$. $T$ is a Fredholm operator if $T$ is closed, $R(T)$ is closed, and both $\alpha(T)$ and $\beta(T)$ are finite. The index of a Fredholm operator is the integer.

$$\kappa(T) = \alpha(T) - \beta(T) .$$

Let $P$ be a subspace of $X$, $Q$ a subspace of $Y$. $(P, Q)$ is an *invariant pair of subspaces* for $T$ if $T(P \cap D(T)) \subseteqq Q$.

*Standing assumptions*: In the remainder of the paper, $A$ and $B$ are closed linear operators from $X$ to $Y$, $D(A)$ is dense in $X$, $D(B) \supseteqq D(A)$, and $D(B^*) \supseteqq D(A^*)$; $A$ is semi-Fredholm, in the sense that $R(A)$ is closed and $\alpha(A) < \infty$.

The assumption $D(B^*) \supseteqq D(A^*)$ seems necessary for the proof of the decomposition theorems. It is often met when $A$ and $B$ are differential operators on some domain in Euclidean space, and the order of $B$ is less than the order of $A$. It is always met when $B$ is bounded.

The linear manifolds $N_k = N_k(A:B)$ and $M_k = M_k(A:B)$ are defined by induction as follows:

$$\begin{aligned} N_1 &= N(A) \\ N_k &= A^{-1}(BN_{k-1}) , \qquad k > 1 \\ M_k &= BN_k . \end{aligned}$$

$N_k$ and $M_k$ are increasing sequences of linear manifolds in $X$ and $Y$ respectively.

The smallest integer $n$ such that $N_n$ is not a subset of $B^{-1}R(A)$ will be denoted by $\nu(A:B)$. If $N_n$ is a subset of $B^{-1}R(A)$ for all $n$, then we define $\nu(A:B) = \infty$. (cf. [2])

The dimension of $N_k$ will be denoted by $\pi_k = \pi_k(A:B)$, and the dimension of $M_k$ by $\rho_k = \rho_k(A:B)$. Then $\pi_1 = \alpha(A)$, and, in general, $\pi_k \leqq k\alpha(A)$. $\mu(A:B)$ will denote the first integer $n$ such that $\pi_n < n\alpha(A)$. It $\pi_n = n\alpha(A)$ for all intergers $n$, then we define $\mu(A:B) = \infty$.

In general, $\mu(A:B) \geqq \nu(A:B) + 1$. This inequality is trivial if $\nu = \infty$. If $\nu < \infty$, then $M_{\nu-1} \subseteqq R(A)$, while $M_\nu \nsubseteqq R(A)$. Consequently, $\pi_{\nu+1} < \pi_\nu + \alpha(A) \leqq (\nu + 1)\alpha(A)$, and so $\mu(A:B) \leqq \nu + 1$.

We define $\sigma_k(A:B) = \pi_k - \pi_{k-1}$. Then $\sigma_k$ is the dimension of the quotient space $N_k/N_{k-1}$. $\{\sigma_k\}$ is a decreasing sequence of nonnegative integers, and so the limit

$$\sigma(A:B) = \lim_{k\to\infty} \sigma_k(A:B) \qquad \text{exists.}$$

If $\mu(A:B) = \infty$, then $\sigma(A:B) = \alpha(A)$.

2. THEOREM 1. *Assume, in addition to the standing assumptions on $A$ and $B$, that $\nu(A:B) = \infty$. Then $X$ and $Y$ can be decomposed*

*into direct sums*

$$X = X_0 \oplus X_1$$
$$Y = Y_0 \oplus Y_1 ,$$

*where* $X_1$ *and* $Y_1$ *are finite dimensional,* $X_1 \subseteqq D(A)$, $X_0 \cap D(A)$ *is dense in* $X_0$, *and* $(X_0, Y_0)$ *and* $(X_1, Y_1)$ *are invariant pairs for both* $A$ *and* $B$. *If* $A_i$ *and* $B_i$ *are the restrictions of* $A$ *and* $B$ *respectively to* $X_i$, *then* $\mu(A_0, B_0) = \infty$, *while* $A_1$ *and* $B_1$ *map* $X_1$ *onto* $Y_1$.

Furthermore, $X_1$ and $Y_1$ can be decomposed as direct sums

$$X_1 = P_1 \oplus \cdots \oplus P_p$$
$$Y_1 = Q_1 \oplus \cdots \oplus Q_p ,$$

where $A_1$ and $B_1$ map $P_j$ onto $Q_j$. Bases $\{x_j^i: 1 < i \leqq \eta(j)\}$ and $\{y_j^i: 1 \leqq i \leqq \eta(j) - 1\}$ can be chosen for $P_j$ and $Q_j$ respectively so that

$$Ax_j^{i+1} = Bx_j^i = y_j^i , \qquad 1 \leqq i \leqq \eta(j) - 1$$
$$Ax_j^1 = 0 = Bx_j^{\eta(j)} .$$

Although the decomposition is not, in general, unique, the integers $p$ and $\eta(j)$, $1 \leqq j \leqq m$, are uniquely determined by $A$ and $B$. In fact,

$$p = \alpha(A) - \sigma(A : B) .$$

*Proof.* Let $n = \alpha(A)$, and suppose that $\{z_1^1, \cdots z_n^1\}$ is a basis for $N(A)$. Since $\nu(A : B) = \infty$, $z_j^i$ can be chosen by induction so that $Az_j^i = Bz_j^{i-1}$. $\{z_j^i: 1 \leqq j \leqq n,\ 1 \leqq i \leqq m\}$ is a spanning set for $N_m$, while $\{Bz_j^i: 1 \leqq j \leqq n,\ 1 \leqq i \leqq m\}$ is a spanning set for $M_m$. Also, $\{z_i^m: 1 \leqq i \leqq n\}$ span $N_m$ modulo $N_{m-1}$.

Recall that $\sigma_m = \sigma(m) = \dim (N_m/N_{m-1})$. By induction, the order of the $z_j^i$ can be chosen so that $\{z_{n-\sigma(m)+1}^m, \cdots, z_n^m\}$ span $N_m$ modulo $N_{m-1}$. Then

$$G_m = \{z_j^i: n - \sigma(i) + 1 \leqq j \leqq n,\ 1 \leqq i \leqq m\}$$

is a basis for $N_m$.

Let $\eta(j)$ be the greatest integer $k$ such that $z_j^k \in G_k$. If $z_j^k \in G_k$ for all $k$, let $\eta(j) = \infty$. Then $1 \leqq \eta(1) \leqq \eta(2) \leqq \cdots \leqq \eta(n)$. Let $p$ be the greatest integer $k$ such that $\eta(k) < \infty$. By definition of $\sigma$, it is clear that

$$p = \alpha(A) - \sigma .$$

Suppose $1 \leqq j \leqq p$. $z_j^{\eta(j)+1}$ is linearly dependent on the set $G_{\eta(j)+1}$, and so we can write

$$z_j^{\eta(j)+1} = \sum \alpha_{ik} z_k^i ,$$

where the sum is taken over all pairs of integers $(i, k)$, with the understanding that $z_k^i = 0$ if $i \leqq 0$ and $\alpha_{ik} = 0$ if $z_k^i \notin G_{\eta(j)+1}$. For $-1 \leqq q \leqq \eta(j)$ define

$$x_j^{\eta(j)-q} = z_j^{\eta(j)-q} - \sum \alpha_{ik} z_k^{i-q-1}.$$

For $0 \leqq q \leqq \eta(j)$,

$$\begin{aligned} Bx_j^{\eta(j)-q} &= Bz_j^{\eta(j)-q} - \sum \alpha_{ik} Bz_k^{i-q-1} \\ &= Az_j^{\eta(j)-q+1} - \sum \alpha_{ik} Az_k^{i-q} \\ &= Ax_j^{\eta(j)-q+1} \end{aligned}$$

In particular, $Bx_j^{\eta(j)} = 0$.

Since the sum for $x_j^{\eta(j)-q}$ involves $z_j^{\eta(j)-q}$ only in the first term, the $z^{\eta}_{j(j)-q}$ may be replaced by the $x_j^{\eta(j)-q}$, $0 \leqq q \leqq \eta(j)$, to obtain another basis for $N_{\eta(j)+1}$. Repeating this process for $1 \leqq j \leqq p$, and making other appropriate replacements, we arrive at vectors $x_j^i$ such that.

(1) $$x_1^1, \cdots, x_1^n \text{ are a basis for } N(A)$$

(2) $$Bx_j^i = Ax_j^{i+1} , \qquad 1 \leqq i \leqq \eta(j)$$

(3) $$Bx_j^{\eta(j)} = 0 , \qquad 1 \leqq j \leqq p .$$

For convenience, it is assumed that

(4) $$x_j^i = 0 \qquad \text{if } i > \eta(j) .$$

If $1 \leqq j \leqq p$, let $P_j$ be the subspace of $X$ with basis $\{x_j^1, \cdots, x_j^{\eta(j)}\}$. Let $Q_j$ be the subspace of $Y$ with basis $\{y_j^1, \cdots, y_j^{\eta(j)-1}\}$, where $y_j^i = Bx_j^i = Ax_j^{i+1}$. Let $X_1 = P_1 \oplus \cdots \oplus P_p$ and $Y_1 = Q_1 \oplus \cdots \oplus Q_p$. Then $X_1$ and $Y_1$ satisfy all the conclusions of the theorem. To conclude the proof, it suffices to produce complementary subspaces to $X_1$ and $Y_1$ which also form an invariant pair.

We will construct functionals

$$\{g_j^i \colon 1 \leqq i \leqq \eta(j) , \qquad 1 \leqq j \leqq p\} \text{ on } X \text{ and}$$
$$\{f_j^i \colon 1 \leqq i \leqq \eta(j) - 1 , \quad 1 \leqq j \leqq p\} \text{ on } Y \text{ such that}$$

the $f_j^i$ are in the domain of $A^*$ and

(5) $$g_j^{i+1} = A^* f_j^i , \qquad 1 \leqq i \leqq \eta(j) - 1$$

(6) $$g_j^i = B^* f_j^i , \qquad 1 \leqq i \leqq \eta(j) - 1$$

(7) $$f_j^i(y_k^q) = \delta_{iq}\delta_{jk} , \qquad \begin{array}{l} 1 \leqq j,\ k \leqq n \\ 1 \leqq q \leqq i \end{array}$$

$$(8) \qquad g_j^i(x_k^q) = \delta_{iq}\delta_{jk}\,, \qquad \begin{array}{l} 1 \leqq j,\ k \leqq n \\ 1 \leqq q \leqq i\,. \end{array}$$

Let $g_j^{\eta(j)}$ be any functional on $X$ which satisfies (8). The other $g_j^i$ will be chosen by induction.

Suppose that $f_k^q$ and $g_k^q$ are chosen, for $q > i \geqq 1$, to satisfy (5) through (8). By (8), $g_k^{i+1}$ is orthogonal to $N(A)$, and so $g_k^{i+1}$ is in the closure of $R(A^*)$. Since $R(A)$ is closed, $R(A^*)$ is closed, and there is an $f_k^i \in D(A^*)$ for which $A^*f_k^i = g_k^{i+1}$. Let $g_k^i = B^*f_k^i$. Then (5) and (6) hold by definition.

To verify (7), we have for $q \leqq i$,

$$\begin{aligned} f_j^i(y_k^q) &= f_j^i(Ax_k^{q+1}) \\ &= (A^*f_j^i)(x_k^{q+1}) \\ &= g_j^{i+1}(x_k^{q+1}) = \delta_{iq}\delta_{jk}\,. \end{aligned}$$

(8) is an immediate consequence of (7).

$$\begin{aligned} \text{Let } X_0 &= \cap\,\{N(g_j^i)\colon 1 \leqq i \leqq \eta(j)\,, && 1 \leqq j \leqq p\} \\ Y_0 &= \cap\,\{N(f_j^i)\colon 1 \leqq i \leqq \eta(j) - 1\,, && 1 \leqq j \leqq p\}\,. \end{aligned}$$

From (7) and (8), it is clear that $X_0 \cap X_1 = \{0\}$ and $Y_0 \cap Y_1 = \{0\}$. Since the codimension of $X_0$ in $X$ is no greater than the number of functionals $g_j^i$ defining it, and since this number is the dimension of $X_1$, we must have $X = X_0 \oplus X_1$. Similarly, $Y = Y_0 \oplus Y_1$.

Suppose $x \in D(A) \cap X_0$. Then $f_j^i(Ax) = (A^*f_j^i)(x) = g_j^{i+1}(x) = 0$, and so $Ax \in Y_0$. Similarly, $Bx \in Y_0$, and $(X_0, Y_0)$ is an invariant pair for both $A$ and $B$.

Since $(X_0, Y_0)$ and $(X_1, Y_1)$ are invariant pairs, $N_k(A:B) \cap X_0 = N_k(A_0:B_0)$. For $k$ sufficiently large, $X_1 \subseteqq N_k(A:B)$, and so

$$\begin{aligned} \dim\{N_{k+1}(A_0:B_0)/N_k(A_0:B_0)\} &= \dim\{N_{k+1}(A:B)/N_k(A:B)\} \\ &= \sigma \\ &= \alpha(A) - p \\ &= \alpha(A_0)\,. \end{aligned}$$

This can occur only if $\dim N_k(A_0:B_0) = k\alpha(A_0)$ for all integers $k$. Hence $\mu(A_0:B_0) = \infty$.

3. Let $(P, Q)$ be an invariant pair of finite dimensional subspaces for $A$ and $B$. $(P, Q)$ is an *irreducible invariant pair of type* $\nu$ if there are bases $\{x_i\}_{i=1}^n$ for $P$ and $\{y_i\}_{i=1}^n$ for $Q$ such that $Bx_i = y_i$, $Ax_1 = 0$, and $Ax_i = y_{i-1}$, $2 \leqq i \leqq n$.

$(P, Q)$ is an *irreducible invariant pair of type* $\mu$ if there are bases $\{x_i\}_{i=1}^n$ for $P$ and $\{y_i\}_{i=1}^{n-1}$ for $Q$ such that

$$Ax_1 = 0 = Bx_n$$
$$Ax_{i+1} = y_i = Bx_i\ , \qquad 1 \leqq i \leqq n-1\ .$$

$(P, Q)$ is an *irreducible invariant pair of type* $\mu^*$ if there are bases $\{x_i\}_{i=1}^{n-1}$ for $P$ and $\{y_i\}_{i=1}^{n}$ for $Q$ such that

$$Bx_i = y_i\ , \qquad 1 \leqq i \leqq n-1$$
$$Ax_i = y_{i+1}\ , \qquad 1 \leqq i \leqq n-1\ .$$

$(P, Q)$ is an *invariant pair of type* $\nu$ if $P = P_1 \oplus \cdots \oplus P_k$ and $Q = Q_1 \oplus \cdots \oplus Q_k$, where $(P_j, Q_j)$ is an irreducible invariant pair of type $\nu$, $1 \leqq j \leqq k$. *Invariant pairs of type* $\mu$ or *type* $\mu^*$ are defined similarly.

It is straightforward to verify that if $(P, Q)$ is an (irreducible) invariant pair of type $\mu(A:B)$ (resp. $\mu^*(A:B)$), then $(P, Q)$ is an (irreducible) invariant pair of type $\mu(A-\lambda B:B)$ (resp. $\mu^*(A-\lambda B:B)$), for all complex numbers $\lambda$. If $(P, Q)$ is an invariant pair of type $\mu$, then $\nu(A \mid P, B \mid P) = \infty$ and $\mu((A \mid P)^*, (B \mid P)^*) = \infty$. If $(P, Q)$ is of type $\mu^*$, then $\nu(A \mid P, B \mid P) = \infty$ and $\mu(A \mid P, B \mid P) = \infty$.

THEOREM 2. *Suppose A and B satisfy the standing hypothesis. Then there exist decompositions*

$$X = X_0 \oplus X_1 \oplus X_2$$
$$Y = Y_0 \oplus Y_1 \oplus Y_2$$

*Where* $(X_0, Y_0)$ *is an invariant pair,* $(X_1, Y_1)$ *is an invariant pair of type* $\mu$, *and* $(X_2, Y_2)$ *is an invariant pair of type* $\nu$. *If* $A_0$ *and* $B_0$ *are the restrictions of A and B respectively to* $X_0$, *then* $\nu(A_0, B_0) = \infty$ *and* $\mu(A_0, B_0) = \infty$.

*Proof.* Theorem 2 follows from Theorem 1 and Kato's Theorem 4 [1], after it is noted that the latter theorem, although stated only for bounded operators $B$, is valid under the less restrictive assumption that $D(B^*) \supseteqq D(A^*)$.

THEOREM 3. *In addition to the standing hypotheses, suppose that A is a Fredholm operator. Then there exist decompositions*

$$X = X_0 \oplus X_1 \oplus X_2 \oplus X_3$$
$$Y = Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3\ ,$$

*where each* $(X_i, Y_i)$ *is an invariant pair,* $(X_1, Y_1)$ *is of type* $\mu$, $(X_2, Y_2)$ *is of type* $\nu$, *and* $(X_3, Y_3)$ *is of type* $\mu^*$. *If* $A_0$ *and* $B_0$ *are the restrictions of A and B to* $X_0$, *then* $\nu(A_0 : B_0) = \infty$, $\mu(A_0 : B_0) = \infty$, $\mu(A_0^* : B_0^*) = \infty$, and $\nu(A_0^* : B_0^*) = \infty$.

If $X^* = X_0^* \oplus X_1^* \oplus X_2^* \oplus X_3^*$ and $Y^* = Y_0^* \oplus Y_1^* \oplus Y_2^* \oplus Y_3^*$ are the corresponding decompositions of the adjoint spaces, then $(Y_1^*, X_1^*)$ is an invariant pair of type $\mu_*(A^* : B^*)$, $(Y_2^*, X_2^*)$ is an invariant pair of type $\nu(A^* : B^*)$, and $(Y_3^*, X_3^*)$ is an invariant pair of type $\mu(A^* : B^*)$.

*Proof.* In view of Theorem 2, we may assume that $\mu(A : B) = \infty$ and $\nu(A : B) = \infty$. Then $\nu(A^* : B^*) = \infty$, and we can proceed to decompose $X^*$ and $Y^*$, as in the proof of Theorem 1. The only difficulty encountered is to produce vectors $x_j^i$ to span $X_3$ which actually lie in $D(A)$. An induction argument similar to that used in Theorem 1 to produce the $f_j^i$ and $g_j^i$ can also be employed in this case.

4. Let $\Phi^+(A : B)$ be the set of complex numbers $\lambda$ such that $A - \lambda B$ is a closed operator from $D(A)$ to $Y$, and such that $R(A - \lambda B)$ is closed and $\alpha(A - \lambda B) < \infty$. $\Phi^+(A : B)$ is an open subset of the complex plane which, by assumption, contains the point $\lambda = 0$.

For all $\lambda \in \Phi^+(A : B)$, Theorems 1 and 2 are applicable to the operators $A - \lambda B$ and $B$. Also, for $\lambda \in \Phi^+(A : B)$ we define

$$\begin{aligned} \sigma_k(\lambda) &= \sigma_k(A - \lambda B : B) \\ \pi_k(\lambda) &= \pi_k(A - \lambda B : B) \\ \rho_k(\lambda) &= \rho_k(A - \lambda B : B) \\ \sigma(\lambda) &= \sigma(A - \lambda B : B) . \end{aligned}$$

THEOREM 4. *Let $A$ and $B$ satisfy the standing hypotheses. There exists a decomposition*

$$\begin{aligned} X &= X_0 \oplus X_1 \\ Y &= Y_0 \oplus Y_1 \end{aligned}$$

*such that $(X_0, Y_0)$ is an invariant pair, and $(X_1, Y_1)$ is an invariant pair of type $\mu(A - \lambda B : B)$ for all complex numbers $\lambda$. If $A_0$ and $B_0$ are the restrictions of $A$ and $B$ to $X_0$, then $\mu(A_0 - \lambda B_0 : B_0) = \infty$ for all $\lambda \in \Phi^+(A : B)$ satisfying $\nu(A - \lambda B : B) = \infty$.*

*Proof.* The points $\lambda \in \Phi^+(A : B)$ for which $\nu(A - \lambda B : B) < \infty$ form a discrete subset of $\Phi^+(A : B)$, and so there is a $\lambda' \in \Phi^+$ such that $\nu(A - \lambda' B : B) = \infty$. Let $X = X_0 \oplus X_1$ be the decomposition of Theorem 1 with respect to $A - \lambda' B$ and $B$. Then $(X_1, Y_1)$ is an invariant pair of type $\mu(A - \lambda B : B)$ for all complex numbers $\lambda$, as remarked earlier.

If $\lambda \in \Phi^+(A : B)$ and $\nu(A - \lambda B : B) = \infty$, then $X_0$ and $Y_0$ cannot be decomposed further as in Theorem 1, for such a decomposition would violate the fact that $\mu(A_0 - \lambda' B_0 : B) = \infty$. Hence $\nu(A - \lambda B : B) =$

$\infty$ implies $\mu(A_0 - \lambda B_0 : B_0) = \infty$.

Let $D$ be the subset of $\Phi^+(A:B)$ of complex numbers $\lambda$ for which $\nu(A - \lambda B : B) < \infty$. $D$ is a discrete subset of $\Phi^+(A:B)$ with no limit points in $\Phi^+(A:B)$(cf [1]).

THEOREM 5. *$\mu(A - \lambda B : B)$ is a constant, either finite or infinite, for $\lambda \in \Phi^+(A:B) - D$.*

*Proof.* In view of Theorem 4, it suffices to prove the theorem when $A$ and $B$ are operators in an invariant pair of type $\mu$. For this, it suffices to look at an irreducible invariant pair of type $\mu$. This case is easy to verify.

THEOREM 6. *$\sigma(\lambda)$ is constant on each component of $\Phi^+(A:B)$.*

*Proof.* It suffices to show that $\sigma(\lambda)$ is constant in a neighborhood of an arbitrary point $\lambda' \in \Phi^+(A:B)$. Let $X = X_0 \oplus X_1 \oplus X_2$ and $Y = Y_0 \oplus Y_1 \oplus Y_2$ be the decomposition of Theorem 2 with respect to $A - \lambda' B$ and $B$. Then $\nu(A_0 - \lambda B_0 : B_0) = \infty$ for $\lambda$ near $\lambda'$, and so $\sigma(\lambda) = \alpha(A_0 - \lambda B_0)$ for $\lambda$ near $\lambda'$. By Theorem 3, [2], $\alpha(A_0 - \lambda B_0) = \alpha(A_0 - \lambda' B_0)$ for $\lambda$ near $\lambda'$.

5. Let $X = X_0 \oplus X_1 \oplus X_2$ and $Y = Y_0 \oplus Y_1 \oplus Y_2$ be the decompositions of Theorem 2 with respect to $A$ and $B$. Let $\pi_k = \pi_k^0 + \pi_k^1 + \pi_k^2$ and $\rho_k = \rho_k^0 + \rho_k^1 + \rho_k^2$ be the corresponding decompositions of $\pi_k$ and $\rho_k$. Assume that $r$ is chosen small that $0 < |\lambda| < r$ implies $\lambda \in \Phi^+(A:B)$ and $\nu(A - \lambda B : B) = \infty$. Then $\pi_k^0(\lambda) = k\sigma(\lambda)$ for $|\lambda| < r$. If $k$ is sufficiently large,

$$\pi_k^1(\lambda) = \dim X_1\,, \qquad |\lambda| < r$$

$$\pi_k^2(\lambda) = \begin{cases} \dim X_2, & \lambda = 0 \\ 0\ , & 0 < |\lambda| < r\,. \end{cases}$$

Also, $\rho_k^0(\lambda) = k\sigma(\lambda)$ for $|\lambda| < r$. For $k$ sufficiently large,

$$\rho_k^1(\lambda) = \dim Y_1$$

$$\rho_k^2(\lambda) = \begin{cases} \dim Y_2\,, & \lambda = 0 \\ 0\ , & 0 < |\lambda| < r\,. \end{cases}$$

We define, for any $\lambda \in \Phi^+(A:B)$,

$$\pi(\lambda) = \lim_{k\to\infty} [\pi_k(\lambda) - k\sigma(\lambda)] \tag{1}$$

$$\rho(\lambda) = \lim_{k=\infty} [\rho_k(\lambda) - k\sigma(\lambda)] \tag{2}$$

$\pi(\lambda)$ and $\rho(\lambda)$ correspond to $\tau(\lambda)$ defined in [1]. From the preceding, we deduce that

$$\pi(\lambda) = \begin{cases} \dim X_1 , & 0 < |\lambda| < r \\ \dim (X_1 \oplus X_2) , & \lambda = 0 \end{cases} \tag{3}$$

$$\rho(\lambda) = \begin{cases} \dim Y_1 , & 0 < |\lambda| < r \\ \dim (Y_1 \oplus Y_2) , & \lambda = 0 . \end{cases} \tag{4}$$

From these formulae, it follows that

$$\alpha(A - \lambda B) = \sigma(\lambda) + \pi(\lambda) - \rho(\lambda) , \qquad 0 < |\lambda| < r , \tag{5}$$

for both sides of this expression are equal to

$$\alpha(A_0 - \lambda B_0) + \dim X_1 - \dim Y_1 .$$

We will assume in the remainder of the discussion that $A$ is a Fredholm operator. The set of complex numbers $\lambda$ such that $A - \lambda B$ is a Fredholm operator will be denoted by $\Phi(A : B)$. $\Phi(A : B)$ is an open subset of the complex plane, and consists of the union of those components of $\Phi^+(A : B)$ for which $R(A - \lambda B)$ is of finite codimension in $Y$, i.e., for which $\alpha(A^* - \lambda B^*) < \infty$.

The quantities $\pi_k^*(\lambda) = \pi_k(A^* - \lambda B^* : B^*)$, $\rho_k^*(\lambda)$, $\sigma^*(\lambda)$, $\pi^*(\lambda)$ and $\rho^*(\lambda)$ are then well-defined for $\lambda \in \Phi(A : B)$. The formula for the adjoint operators corresponding to (5) is

$$\alpha(A^* - \lambda B^*) = \sigma^*(\lambda) + \pi^*(\lambda) - \rho^*(\lambda) , \qquad 0 < |\lambda| < r . \tag{6}$$

Since $\alpha(A^* - \lambda B^*) = \beta(A - \lambda B)$, we have

$$\begin{aligned} \kappa(A - \lambda B) = {} & (\sigma(\lambda) - \sigma^*(\lambda)) \\ & + (\pi(\lambda) - \pi^*(\lambda)) - (\rho(\lambda) - \rho^*(\lambda)) \qquad 0 < |\lambda| < r . \end{aligned} \tag{7}$$

In view of the decomposition of Theorem 3, the jump discontinuity of $\pi^*$ at $\lambda = 0$ is equal to that of $\pi$ at $\lambda = 0$, i.e., they are both equal to $\dim X_2 = \dim Y_2$. Hence (7) holds also for $\lambda = 0$, and we arrive at the following theorem.

THEOREM 7. *For all* $\lambda \in \Phi(A : B)$,

$$\kappa(A - \lambda B) = (\sigma(\lambda) - \sigma^*(\lambda)) + (\pi(\lambda) - \pi^*(\lambda)) - (\rho(\lambda) - \rho^*(\lambda)).$$

Analogous formulae can be written down if it is assumed, further, that $B$ is a Fredholm operator. If $M(B) = \{0\}$ and $R(B)$ is dense in $Y_1$ then $\rho(\lambda) = \rho^*(\lambda) = \pi(\lambda) = \pi^*(\lambda) = 0$, and Theorem 7 reduces to

$$\kappa(A - \lambda B) = \sigma(\lambda) - \sigma^*(\lambda), \qquad \lambda \in \Phi(A : B) . \tag{8}$$

This latter formula is due to Kaniel and Schechter [1], when $X = Y$ and $B$ is the identity operator.

## Bibliography

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