# CHEBYSHEV APPROXIMATION TO ZERO

James McLean Sloss

# CHEBYSHEV APPROXIMATION TO ZERO

## James M. Sloss

In this paper we shall be concerned with the questions of existence, uniqueness and constructability of those polynomials in $k + 1$ variables $(x_1, x_2, \cdots, x_k, y)$ of degree not greater than $n_s$ in $x_s$ and $m$ in $y$ which best approximate zero on $I_1 \times I_2 \times \cdots \times I_{k+1}, I_s = [-1, 1]$, in the Chebyshev sense.

It is a classic result that among all monic polynomials of degree not greater than $n$ there is a unique polynomial whose maximum over the interval $[-1, 1]$ is less than the maximum over $[-1, 1]$ of any other polynomial of the same type and moreover it is given by $\tilde{T}_n(x) = 2^{1-n} \cos [n \text{ are } \cos x]$, the normalized Chebyshev polynomial.

Our method of attack will be to prove a generalization of an inequality for monic polynomials in one variable concerning the lower bound of the maximum viz. $\max_{-1 \leq x \leq 1} | P_n(x) | \geq 2^{1-n}$ where $P_n(x)$ is a monic polynomial of degree not greater than $n$. The theorem will show that the only hope for uniqueness is to normalize our class of polynomials. This is done in a very natural way viz. by considering only polynomials, if they exist, of the form:

$$(0.1) \qquad P(x_1, x_2, \cdots, x_k, y) = A_m(x_1, \cdots, x_k)y^m$$
$$+ A_{m-1}(\cdots)y^{m-1} + \cdots + A_0(\cdots)$$

for which $A_m(x_1, x_2, \cdots, x_k)$ is the best polynomial approximation to zero on $I_1 \times I_2 \times \cdots \times I_k$. Thus if $k = 1$, we consider only polynomials of the form:

$$(0.2) \qquad P(x_1, y) = \tilde{T}_n(x_1)y^m + A_{m-1}(x_1)y^{m-1} + \cdots + A_0(x_1) .$$

We find in the case of (0.2) that there is a unique best polynomial approximation and it is given by $\tilde{T}_n(x_1)\tilde{T}_m(y)$. Thus we can consider the question of existence, uniqueness and constructability of a polynomial of the form:

$$(0.3) \qquad P(x_1, x_2, y) = \tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2)y^m$$
$$+ A_{m-1}(x_1, x_2)y^{m-1} + \cdots + A_0(x_1, x_2)$$

that best approximates zero. We find in this case there is a unique best polynomial approximation and it is given by $\tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2)\tilde{T}_m(y)$. Continuing in this way we shall show that the question is meaning-

ful in general and that there is a unique best polynomial approxima-
tion to zero of the form (0.1) given by $\tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2) \cdots \tilde{T}_{n_k}(x_k)\tilde{T}_m(y)$.

The uniqueness and constructability are the most surprising results,
since as Buck [1] has shown, $F(x, y) = xy$ has amongst those polyno-
mials of the form

$$p(x, y) = a_0 + a_1(x + y) + a_2(x^2 + y^2)$$

infinitely many polynomials of best approximation which are given by:

$$\alpha f_1 + \beta f_2, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta = 1$$

where

$$f_1(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{4},$$

$$f_2(x, y) = x + y - \frac{1}{2}(x^2 + y^2) - \frac{1}{4}.$$

We shall finally normalize the polynomials in a different way and
show by construction, the existence of a polynomial, of best approxi-
mation in this class. However in this case the question of uniqueness
remains open.

1.  NOTATION.  Let $n_1, n_2, \cdots, n_k$ be positive fixed integers. Let
$\sigma$ be the finite set of vectors $\{(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})\}$, where $j_1, j_2, \cdots, j_k$
are integers with $0 \leq j_1 \leq n_1, 0 \leq j_2 \leq n_2, \cdots, 0 \leq j_k \leq n_k$; and where
also $-1 \leq x_{1j_1} \leq 1, -1 \leq x_{2j_2} \leq 1, \cdots, -1 \leq x_{kj_k} \leq 1$ and no two of
the $x_{1j_1}$ are the same, no two of the $x_{2j_2}$ are the same, $\cdots$, no two
of the $x_{kj_k}$ are the same. Let $Q(x, y) = Q(x_1, x_2, \cdots, x_k, y)$ be any
polynomial in $x_1, x_2, \cdots, x_k$ and $y$ of degree $\leq n_1 + n_2 + \cdots + n_k + m - 1$
where $Q$ is of degree $\leq n_s$ in $x_s$, $s = 1, 2, \cdots, k$ and of degree $\leq m$ in
$y$. Let $\pi$ be the set of all such polynomials. Thus if $Q(x, y)$ is in $\pi$

$$Q(x, y) = p_m(x)y^m + p_{m-1}(x)y^{m-1} + \cdots + p_0(x)$$

where $p_m(x)$ is a polynomial in $x_1, x_2, \cdots, x_k$ of

$$\text{degree} \leq n_1 + n_2 + \cdots + n_k - 1$$

and $p_s(x), 0 \leq s \leq m - 1$, are polynomials of degree $\leq n_1 + n_2 + \cdots + n_k$
in $x_1, x_2, \cdots, x_k$. Let

$$A[p_m; \pi, \sigma] = \min_{x \text{ in } \sigma} |x_1^{n_1}x_1^{n_2} \cdots x_k^{n_k} - p_m(x_1, x_2, \cdots, x_k)|$$

which does not depend on the particular $Q$, but only on the class $\pi$
and the leading coefficient polynomial of $y$.

THEOREM 1. *If $Q(x, y)$ is any polynomial in $\pi$ and if $\sigma$ is any set of the type described above then*

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x_1, x_2, \cdots, x_k, y)| \geq A[p_m; \pi, \sigma] 2^{1-m} .$$

*Proof.* Assume not. Then there exists a $Q^*(x, y)$ in $\pi$ and a set $\sigma$ of the type described such that:

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y)| < A[p_m; \pi, \sigma] 2^{1-m}$$

consider the polynomial:

$$P(x, y) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m$$
$$- Q^*(x, y) - [x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} - p_m(x)] \tilde{T}_m(y)$$

where $p_m(x)$ is the coefficient of $y^m$ in $Q^*(x, y)$ and where

(1) $$\tilde{T}_m(y) = 2^{1-m} T_m(y) = 2^{1-m} \cos[m \text{ arc } \cos y] .$$

Then $P(x, y)$ is a polynomial of degree $\leq m - 1$ in $y$ and thus can be written:

$$P(x, y) = q_{m-1}(x) y^{m-1} + q_{m-2}(x) y^{m-2} + \cdots + q_0(x)$$

where $q_s(x)$, $0 \leq s \leq m - 1$, are polynomials in $x_1, x_2, \cdots, x_k$ of degree $\leq n_1 + n_2 + \cdots + n_k$.

Let $(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})$ belong to $\sigma$ and $y_r$ be one of the points

$$y_r = \cos \frac{r\pi}{m} , \quad 0 \leq r \leq m , \quad r = \text{integer} .$$

Then $\tilde{T}_m(y_r) = (-1)^r 2^{1-m}$ and we can show that the sign of

$$P[x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k}, y_r]$$

is the same as the sign of $-[x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} - p_m(y_{1j_1}, \cdots, x_{kj_k})]$. $\tilde{T}_m(y_r)$. To see this note that:

$$|\tilde{T}_m(y_r)| |x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} - p_m(x_{1j_1}, \cdots, x_{kj_k})|$$
$$= |x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} - p_m(x_{1j_1}, \cdots, x_{1j_k})| 2^{1-m}$$
$$\geq A[p_m; \pi, \sigma] 2^{1-m} .$$

But by the assumption

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} |x_1^{n_1} \cdots x_k^{n_k} y^m - Q^*(x, y)| < A[p_m; \pi, \sigma] 2^{1-m}$$

and thus a fortiori

$$|x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} y_r^m - Q^*(x_{1j_1}, \cdots, x_{kj_k}, y_r)| < A[p_m; \pi, \sigma]2^{1-m} .$$

If we fix $x$ in $\sigma$ then $P(x, y)$ is a polynomial of the one variable $y$ and of degree $\leqq m - 1$. And as $y$ takes on the values $y_r = \cos(\pi r/m)$, $P(x, y)$ changes sign $m + 1$ times. Thus $P(x, y)$ has $m$ zeros, which means $q_{m-1}(x) = 0, q_{m-2}(x) = 0, \cdots, q_0(x) = 0$ since $P(x, y)$ is only of degree $\leqq m - 1$.

Since $x$ was an arbitrary point of $\sigma$, then

$$q_s[x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k}] = 0 , \quad 0 \leqq s \leqq m - 1$$

where $0 \leqq j_1 \leqq n_1, 0 \leqq j_2 \leqq n_2, \cdots, 0 \leqq j_k \leqq n_k$. But $q_s(x)$ is a polynomial of degree $\leqq n_1$ in $x_1$, of degree $\leqq n_2$ in $x_2$, $\cdots$, of degree $\leqq n_k$ in $x_k$ and thus

$$q_s[x_1, x_2, \cdots, x_k] \equiv 0 , \quad 0 \leqq s \leqq m - 1 .$$

From which we see $P(x, y) \equiv 0$ and thus:

$$x_1^{n_1} \cdots x_k^{n_k} y^m - Q^*(x, y) \equiv [x_1^{n_1} \cdots x_k^{n_k} - p_m(x)]\tilde{T}_m(y) .$$

But clearly:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |x_1^{n_1} \cdots x_k^{n_k} - p_m(x)| \, |\tilde{T}(y)| \geqq A[p_m; \pi, \sigma]2^{1-m}$$

which is a contradiction and thus the theorem is proved.

Let us now consider the subset of polynomials $\pi_0$ of $\pi$ for which $Q(x, y)$ belongs to $\pi$ and $p_m(x) = 0$. Then by the above theorem, a lower bound for the maximum is

$$A[0; \pi, \sigma] = \min_{x \text{ in } \sigma} |x_1^{n_1} \cdots x_k^{n_k}| < 1$$

which clearly depends on the set $\sigma$. We shall next show that for this subset $\pi_0$, we get a lower bound for the maximum that is independent of $\sigma$ and moreover the lower bound is larger than $A[0; \pi, \sigma]$ for all $\sigma$, namely it is unity. In the third theorem we shall show that unity is the best possible lower bound i.e. there is a polynomial in $\pi_0$ for which the maximum is $2^{1-m}$.

THEOREM 2. *Let $Q(x, y)$ be any polynomial in $\pi_0$, then*

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x_1, x_2, \cdots, x_k, y| \geqq 2^{1-m} .$$

*Proof.* By contradiction. Assume there exists a $Q(x_1, \cdots, x_k, y)$ in $\pi_0$ such that:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y)| < 2^{1-m} .$$

Then there exist $\delta_s$'s, $1 \leqq s \leqq k$, $1 > \delta_s > 0$ such that:

$$\max_{\substack{-1 \leqq x_s \leqq 1 \\ -1 \leqq y \leqq 1}} |x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y)| < 2^{1-m} \prod_{s=1}^{k} \delta_s^{n_s} .$$

Let $\tilde{T}_m(y)$ be given by (1) and consider the polynomial

$$P(x_1, \cdots, x_k, y) \equiv x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y) - x_1^{n_1} \cdots x_k^{n_k} \tilde{T}_m(y) .$$

$P(x_1, \cdots, x_k, y)$ is a polynomial of degree $\leqq m - 1$ in $y$ and of degree $\leqq n_s$ in $x_s$ $1 \leqq s \leqq k$.

Let $\sigma^* = \{(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})\}$ where $j_1, \cdots, j_k$ are integers with

$$0 \leqq j_1 \leqq n_1 + 1, 0 \leqq j_2 \leqq n_2 + 1, \cdots, 0 \leqq j_k \leqq n_k + 1;$$
$$\delta_1 < x_{1j_1} \leqq 1, \delta_2 < x_{2j_2} \leqq 1, \cdots, \delta_k < x_{kj_k} \leqq 1$$

and the $x_{1j_1}$ are distinct, $\cdots$, the $x_{kj_k}$ are distinct.

Note that for $x$ in $\sigma^*$, the sign of $P(x_{1j_1}, \cdots, x_{kj_k}, y)$ is the same as the sign of $-x_{1j_1}^{n_1} \cdots x_{kn_k}^{n_k} \tilde{T}_m(y_r)$ for $y_r = \cos (r\pi/m)$, $r = 0, 1, \cdots, m$. This follows from the fact that:

$$|x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} y_r^m - Q(x_1, \cdots, x_k, y_r)| < 2^{1-m} \prod_{s=1}^{k} \delta_s^{n_s}$$

and the fact that:

$$|x_{1j_1}^{n_1} \cdots x_{kj_k}^{n_k} \tilde{T}_m(y_r)| = 2^{1-m} \prod_{s=1}^{k} x_{sj_s}^{n_s} > 2^{1-m} \prod_{s=1}^{k} \delta_s^{n_s} .$$

Thus we conclude that $P(x_{1j_1}, \cdots, x_{kj_k}, y)$ has $m + 1$ sign changes for $(x_{1j_1}, \cdots, x_{kj_k})$ in $\sigma^*$. Let us write

$$P(x, y) = p_{m-1}(x)y^{m-1} + p_{m-2}(x)y^{m-2} + \cdots + p_0(x)$$

where $p_s(x)$, $0 \leqq s \leqq m - 1$, are polynomials of degree $\leqq n_s$ in $x_s$, $0 \leqq s \leqq k$. For each $x$ in $\sigma^*$, $P(x, y)$ has $m + 1$ sign changes and thus $p_{m-1}(x) = 0$, $p_{m-2}(x) = 0, \cdots, p_0(x) = 0$ for each $x$ in $\sigma^*$. If for $(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})$ in $\sigma^*$, we fix all but the first component, we get $n_1 + 2$ values in $\sigma^*$ for which $p_s(x) = 0$, $0 \leqq s \leqq m - 1$, but these $p_s(x)$ are of degree $\leqq n_1$ in $x_1$ and thus $p_s(x_1, x_{2j_2}, x_{3j_3}, \cdots, x_{kj_k}) = 0$ for all real $x_1$. Continuing in this way, we see that $p_s(x_1, x_2, \cdots, x_k) \equiv 0$ for all $(x_1, x_2, \cdots, x_k)$, $x_s$ real. Thus:

$$P(x_1, x_2, \cdots, x_k, y) \equiv 0$$

for all real $x_s$ and real $y$. Thus

$$x_1^{n_1} \cdots x_k^{n_k} \tilde{T}_m(y) \equiv x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y) .$$

But

$$\max_{\substack{-1 \le x_s \le 1 \\ -1 \le y \le 1}} |x_1^{n_1} \cdots x_k^{n_k} \tilde{T}_m(y)| = 2^{1-m}$$

which gives a contradiction and the theorem is proved.

2. **Normalization of competing polynomials and construction of the best polynomial.** We shall now consider a subset $\pi(\beta)$ of the set of polynomials $\pi$. We shall then answer the question of existence, uniqueness and constructability of the best polynomial approximation in the maximum norm to zero within this class $\pi(\beta)$ on the cube

$$-1 \le x_1 \le 1, \cdots, -1 \le x_k \le 1, -1 \le y \le 1 \, .$$

It is apparent from Theorem 1, that if we want uniqueness independent of $\sigma$, it is necessary to consider some subset of $\pi$.

DEFINITION.   A polynomial

$$Q(x, y) = p_m(x_1, x_2, \cdots, x_k)y^m$$
$$+ p_{m-1}(x_1, x_2, \cdots, x_k)y^{m-1} + \cdots + p_0(x_1, x_2, \cdots, x_k)$$

which is in $\pi$ and for which

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} - p_m(x_1, x_2 \cdots x_k) = \tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2) \cdots \tilde{T}_{n_k}(x_k)$$

is said to be in $\pi(\beta)$.

LEMMA.   *Let $q(y)$ be a polynomial in $y$, let $y_0 > y_1 > \cdots > y_m$ be any set of real numbers for which*

$$q(y_0) \le 0, \, q(y_1) \ge 0, \, q(y_2) \le 0, \cdots (-1)^m q(y_m) \le 0 \, .$$

*Then $q(y)$ has $m$ zeros including multiplicities on $[y_0, y_m]$.*

*Proof.* (by induction): For $m = 1$ obvious. Assume theorem to be true for $m \le k$. Let $y_0 > y_1 > y_2 > \cdots > y_{k+1}$ be any set of real numbers such that

$$q(y_0) \le 0, \, q(y_1) \ge 0, \cdots (-1)^k q(y_k) \le 0, \, (-1)^{k+1} q(y_{k+1}) \le 0 \, .$$

*Case 1.* $q(y_s) \ne 0$ for some $1 \le s \le k$. Then by the induction hypothesis $q(y)$ has $s$ zeros on $[y_0, y_s]$ and has $k + 1 - s$ zeros on $[y_s, y_{k+1}]$. But $q(y_s) \ne 0$ thus $q(y)$ has $s$ zeros on $y_0 \le y \le y_s$ and thus $q(y)$ has $s + (k + 1 - s) = k + 1$ zeros on $[y_0, y_{k+1}]$.

*Case 2.* $q(y_0) < 0$. Then unless $q(y_s) = 0$ for $1 \le s \le k$ we are in Case 1 and we are finished. Therefore, assume $q(y_s) = 0, 1 \le s \le k$.

We may as well assume $q(y) < 0$ on $(y_0, y_1)$ since if not then $q(y)$ has a zero there because $q(y_0) < 0$, and we are finished. Also, we may as well assume $q(y) > 0$ on $(y_1, y_2)$ since if not and $q(y)$ has no zeros on $(y_1, y_2)$ (if does have a zero then we are finished) then since $q(y_0) < 0$ and $q(y_1) = 0$, we must have that $q(y)$ has 2 zeros in $(y_0, y_2)$, continuing in this way we see that we may as well assume that $(-1)^s q(y) < 0$ on $(y_s, y_{s+1})$ for $0 \leq s \leq k$. In particular $(-1)^k q(y) < 0$ for $y$ on $(y_k, y_{k+1})$. But by assumption $(-1)^{k+1} q(y_{k+1}) \leq 0$. Thus by the continuity of $q(y)$, we have $q(y_{k+1}) = 0$ and $q(y_s) = 0$ for $1 \leq s \leq k+1$ i.e. $q(y)$ has $k+1$ zeros on $[y_0, y_{k+1}]$.

*Case 3.* $q(y_0) = 0$ proof is obvious making use of Case 1.

THEOREM 3. There exists a unique $Q^*(x, y)$ in $\pi(\beta)$ such that

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y)|$$

is a minimum. Moreover:

$$Q^*(x, y) = -\widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2) \cdots \widetilde{T}_{n_k}(x_k)\widetilde{T}_m(y) + x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m .$$

*Proof. Existence by construction.* Let the $\sigma$ of Theorem 1 be the special set of vectors

$$\sigma(\beta) = \{(x_{1j_1}, x_{2j_2}, \cdots, x_{kj_k})\}$$

where

$$x_{1j_1} = \cos(j_1\pi/n_1),\ x_{2j_2},\ \cdots,\ x_{kj_k} = \cos(j_k\pi/n_k)$$

$$0 \leq j_1 \leq n_1,\ 0 \leq j_2 \leq n_2,\ \cdots,\ 0 \leq j_k \leq n_k .$$

Then

$$A[p_m, \pi(\beta), \sigma(\beta)] = \min_{x \text{ in } \sigma(\beta)} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} - p_m(x_1, x_2, \cdots, x_k)|$$

$$= \min_{x \text{ in } \sigma(\beta)} |\widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_1) \cdots \widetilde{T}_{n_k}(x_k)|$$

$$= 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} .$$

Thus by Theorem 1

$$\max_{\substack{-1 \leq x_j \leq 1 \\ -1 \leq y \leq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q(x, y)| \geq 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} 2^{1-m} .$$

But the polynomial

$$Q^*(x, y) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - \widetilde{T}_{n_1}(x_1)\widetilde{T}_{n_2}(x_2) \cdots \widetilde{T}_{n_k}(x_k)\widetilde{T}_m(y)$$

clearly belongs to $\pi(\beta)$ and

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} |x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y)| = 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} 2^{1-m}.$$

Thus $Q^*(x, y)$ is a best approximation from the set $\pi(\beta)$

*Uniqueness.* Let $Q^*(x, y)$ in $\pi(\beta)$ be a polynomial of best approximation and let

$$\begin{aligned}
P(x, y) &= x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} y^m - Q^*(x, y) - \tilde{T}_{n_1}(x_i) \cdots \tilde{T}_{n_k}(x_k) \tilde{T}_m(y) \\
&= [x_1^{n_1} x_2^{n_1} \cdots x_k^{n_k} - p_m(x)] y^m - p_{m-1}(x) y^{m-1} - \cdots p_0(x) \\
&\quad - \tilde{T}_{n_1}(x_1) \tilde{T}_{n_2}(x_2) \cdots \tilde{T}_{n_k}(x_k) \tilde{T}_m(y) \\
&= q_{m-1}(x) y^{m-1} + q_{m-2}(x) y^{m-2} + \cdots + q_0(x)
\end{aligned}$$

where $q_{m-1}(x), \cdots, q_0(x)$ are polynomials of degree $\leq n_s$ in $x_s$ $0 \leq s \leq k$ since $Q^*(x, y)$ is in $\pi(\beta)$.

Let $x^* = (x_1^*, x_2^*, \cdots, x_k^*)$ be a fixed but arbitrary element of $\sigma(\beta)$. Then we claim that $P(x^*, y)$ has $m$ zeros including multiplicities in $[-1, 1]$. To see this let $y_s = \cos(s\pi/m)$, $0 \leq s \leq m$, then since

$$|x_1^{*n_1} x_2^{*n_2} \cdots x_k^{*n_k} y^m - Q^*(x^*, y)| \leq 2^{1-n_1} 2^{1-n_2} \cdots 2^{1-n_k} 2^{1-m},$$

$$P(x^*, y_0) \leq 0, \; P(x^*, y_1) \geq 0, \; \cdots (-1)^m P(x^*, y_m) \leq 0.$$

By the lemma $P(x^*, y)$ has $m$ zeros counting multiplicities for $-1 \leq y \leq 1$.

Thus $P(x^*, y)$ has $m$ zeros but is only a polynomial of degree $m - 1$, thus $P(x^*, y) \equiv 0$. But this holds for all $x^*$ in $\sigma(\beta)$, thus $P(x, y) \equiv 0$ and the theorem is proved.

We could formulate Theorem 3 in the following way. Let $\pi(k)$, $k \geq 1$, be the set of polynomials of the form

$$Q(x, y) = p_m(x_1, \cdots, x_k) x_{k+1}^m + p_{m-1}(x) x_{k+1}^{m-1} + \cdots + p_0(x)$$

which is of degree $\leq n_s$ in $x_s$, $1 \leq s \leq k$ and for which $p_m(x_1 \cdots x_k)$ is a polynomial that best approximates zero, if such exists, on the cube $I_1 \times I_2 \times \cdots \times I_k$, $I_s = [-1, 1]$, $1 \leq s \leq k$.

*Theorem 3 alternate.* For $k = 2, 3, 4 \cdots$, the following is true:

*Statement k.* $\pi(k - 1)$ is not empty and there exists a unique $M_k(x_1, x_2, \cdots, x_k, x_{k+1})$ in $\pi(k)$ such that:

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} |M_k(x_1, x_2, \cdots, x_k, x_{k+1})|$$

is a minimum. Moreover:

$$M_k(x_1, x_2, \cdots, x_k, x_{k+1}) = \tilde{T}_{n_1}(x_1) \tilde{T}_{n_2}(x_2) \cdots \tilde{T}_{n_k}(x_k) \tilde{T}_{n_{k+1}}(x_{k_{n+1}}) \ .$$

*Proof.*   Obvious.

Finally we wish to prove:

THEOREM 4.   *There exists a monic polynomial*

$$P(x_1, \cdots, x_k, y) = x_1^{n_1} \cdots x_k^{n_k} y^m - Q(x_1, \cdots, x_k, y)$$

*where* $Q(x, y)$ *belongs to* $\pi_0$ *that best approximates zero on the cube* $I_1 \times I_2 \times \cdots \times I_{k+1}, I_s = [-1, 1]$. *The polynomial is*

$$x_1^{n_1} \cdots x_k^{n_k} \tilde{T}_m(y) \ .$$

*Proof.*   By Theorem 2

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} | P(x_1, \cdots, x_k^{n_k}, y) | \geq 2^{1-m} \ .$$

But $x_1^{n_1} \cdots x_k^{n_k} \tilde{T}_m(y)$ is a monic polynomial of the correct form with

$$\max_{\substack{-1 \leq x_s \leq 1 \\ -1 \leq y \leq 1}} | x_1^{n_1} \cdots x_k^{n_k} \tilde{T}_m(y) | = 2^{1-m} \ .$$

Thus the theorem is correct.
    The question of uniqueness in this case is an open one.

## REFERENCE

1. R. C. Buck, *Linear spaces and approximation theory, On numerical approximation*, Edited by R. E. Langer, Published by the University of Wisconsin Press, 1959.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA

# PACIFIC JOURNAL OF MATHEMATICS

# Pacific Journal of Mathematics
## Vol. 15, No. 1      September, 1965