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# ON A CLASS OF CAUCHY EXPONENTIAL SERIES

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This paper was received before the synoptic introduction became a requirement.

1. Introduction. Let Q(z) be a meromorphic function with poles  $z_1, z_2, z_3, \cdots$ , the notation being so chosen that  $|z_1| \leq |z_2| \leq |z_3| \leq \cdots$ . If  $f \in L(0, 1)$ , define

$$c_{
u}e^{z_{
u}x}=\operatorname{res}_{z_{
u}}Q(z)\int_{0}^{1}f(t)e^{z(x-t)}dt$$
 .

Then, the series  $\Sigma c_{\nu}e^{z_{\nu}x}$  is called the Cauchy Exponential Series (CES) of f with respect to Q(z). If  $z_{\nu}$  is of multiplicity m, then  $c_{\nu}$  is a polynomial in x of degree at most m-1; if the poles are all simple, with residue  $\lambda_{\nu}$  at  $z_{\nu}$ , we may write

$$(1) c_{\nu} = \lambda_{\nu} \int_{0}^{1} f(t)e^{-z_{\nu}t}dt$$

and  $\{c_{\nu}\}\$ , independent of x, are called the CE constants.

Let  $C_p$ :  $|z| = r_p$  be an expanding sequence of contours, none of which passes through a pole of Q(z). Suppose  $C_p$  contains  $n_p$  poles of Q(z). Then,

$$egin{align} \sum\limits_{
u=1}^{n_p} c_
u e^{z_
u x} &= rac{1}{2\pi i} \int_{\sigma_p} Q(z) dz \int_0^1 f(t) e^{z(x-t)} dt \;, \ &= I_p \;, \quad ext{say} \;. \end{gathered}$$

Denote by  $C_p^+$ ,  $C_p^-$  the parts of  $C_p$  lying in the right, left half-planes respectively. If Q(z) is approximately unity on  $C_p^+$ , and is small on  $C_p^-$ , in the sense that

(2) 
$$\int_{o^{+}_{-}} (Q(z) - 1) dz \int_{0}^{1} f(t) e^{z(x-t)} dt = o(1)$$

(3) 
$$\int_{\sigma_{n}^{-}} Q(z)dz \int_{0}^{1} f(t)e^{z(x-t)}dt = o(1)$$

as  $p \to \infty$ , uniformly for  $x \in [0, 1]$ , then

$$egin{align} I_p &= rac{1}{2\pi i} \int_{\sigma_p^+} dz \int_0^1 f(t) e^{z(x-t)} dt \, + \, o(1) \ &= rac{1}{\pi} \int_0^1 rac{f(t) \sin r_p(x-t)}{x-t} dt \, + \, o(1) \end{array}$$

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uniformly in [0,1], and so the sums  $I_p$  behave somewhat like the partial sums of a Fourier series (F.s.). Indeed, when

$$Q(z) = e^z/e^z - 1$$

the CES is the F.s. of f.

In this paper, we shall suppose that

$$Q(z) = \frac{e^z a(z)}{e^z a(z) + b(z)} = \frac{e^z a(z)}{G(z)}$$

where a(z), b(z) are relatively prime polynomials of degree n, and that all the poles are simple. This case was investigated first by Fullerton ([1], 1-34), using a less convenient notation.

The large zeros of G(z) approximate to those of  $e^z - c$ , where

$$c = -\lim_{|z| \to \infty} b(z)/a(z)$$

i.e. to the points  $\{\zeta+2\pi pi\}$ ,  $\zeta$  being the principal value of  $\log c$ . Hence there is a  $\delta$ ,  $0<\delta<2\pi$ , such that if  $r_p=2p\pi+\delta$ , each point of  $C_p$  is at a distance greater than a positive constant from the zeros of G(z) and of  $e^z-c$ . This enables us to prove

THEOREM 1. Let  $f \in L(0, 1)$ . Then, as  $p \to \infty$ ,

$$\sum_{\nu=1}^{n_p} c_{\nu} e^{z_{\nu} x} - e^{\zeta x} s_p(x) \to 0$$

uniformly for  $x \in [0, 1]$ , where  $s_p(x)$  is the pth partial sum of the F.s. of  $f(t)e^{-\zeta t}$ .

We next show that there are n relations connecting the CE constants.

THEOREM 2. Let  $f \in L(0,1)$ . If  $c_{\nu}$  is defined by (1), for  $\nu=1,2,\cdots$ , then

$$\sum_{\nu=1}^{\infty} \frac{c_{\nu} z_{\nu}^{r}}{\lambda_{\nu} F'(z_{\nu})} = 0$$

$$(r = 0, 1, \dots, n - 1), where F(z) = e^{-z}G(z).$$

This naturally leads to the following question: if a sequence of numbers  $\{\beta_{\nu}\}$  satisfies  $\sum_{\nu=1}^{\infty} c_{\nu}\beta_{\nu} = 0$ , what is the nature of the  $\beta_{\nu}$ ? The answer is given by

THEOREM 3. Let  $\{\beta_{\nu}\}$  be a sequence of numbers such that  $\sum_{\nu=1}^{\infty} c_{\nu}\beta_{\nu} = 0$  for every CES  $\Sigma c_{\nu}e^{z_{\nu}x}$ . Then, there are constants

 $\alpha_0, \cdots, \alpha_{n-1}$  such that

$$eta_{
u} = \sum\limits_{r=0}^{n-1} rac{lpha_r z_{
u}^r}{\lambda_{
u} F'(z_{
u})}$$
 .

Because of the relations (6), we cannot expect that, given a sequence  $\{c_{\nu}\}$  with  $\sum_{\nu=1}^{\infty}|c_{\nu}|^2<\infty$ , there is a function  $f\in L^2(0,1)$  such that (1) is true for each  $\nu$ . However, we can prove

THEOREM 4. If  $\{c_{\nu}\}$ ,  $\nu > n$ , is a sequence with  $\sum_{\nu > n} |c_{\nu}|^2 < \infty$ , there is a function  $f \in L^2(0, 1)$  such that (1) is true for each  $\nu > n$ , and upon defining  $c_1, \dots, c_n$  by (1), such that  $\sum_{\nu=1}^{\infty} c_{\nu}e^{z_{\nu}x}$  converges in mean to f.

Alternatively, we can alter every  $c_{\gamma}$  and so obtain a Riesz-Fischer analogue. We have

Theorem 5. Let  $\{c_{\nu}\}$  be a sequence with  $\sum_{\nu=1}^{\infty}|c_{\nu}|^2<\infty$ . Then, there are constants  $\gamma_0,\cdots,\gamma_{n-1}$  such that if

$$d_{
u}=c_{
u}+\sum\limits_{r=0}^{n-1}rac{\gamma_{r}z_{
u}^{r}}{G'(z_{
u})}$$
 ,

the numbers  $d_{\gamma}$  are the CE constants of a function  $f \in L^2(0,1)$ .

We next investigate the problem of the uniqueness of CES. We prove

Theorem 6. If  $\sum_{\nu=1}^{\infty} d_{\nu}e^{s_{\nu}x} = f(x)$  almost everywhere in [0, 1], then there are constants  $\sigma_0, \dots, \sigma_{n-1}$  such that

(7) 
$$d_{\nu} = \lambda_{\nu} \int_{0}^{1} f(t) e^{-z_{\nu}t} dt + \sum_{r=0}^{n-1} \frac{\sigma_{r} z_{\nu}^{r}}{G'(z_{\nu})}$$

Finally, the question arises whether it is possible to generalise the function Q(z) given by (4), so that the CES of f is uniformly equiconvergent with a F.s. The functions

$$P(z) = rac{e^z lpha(z) + eta(z)}{e^z a(z) + b(z)}$$

where  $\alpha(z)$ ,  $\beta(z)$  are polynomials of degree n, are obvious generalisations. As  $\operatorname{Re} z \to \infty$ , P(z) tends to a number  $\omega_1 \neq 0$ ; as  $\operatorname{Re} z \to -\infty$ , to  $\omega_2 \neq 0$ . Suppose  $\omega_1 \neq \omega_2$ , and define

$$Q_1(z) = \frac{1}{\omega_1 - \omega_2} \{P(z) - \omega_2\}$$
;

then  $Q_1(z)$  satisfies (2), (3). If the CES of f with respect to  $Q_1(z)$  is uniformly equiconvergent in [0,1] with  $e^{\xi x}$  multiplied by the F.s. of  $f(t)e^{-\xi t}$ , for each  $f \in L(0,1)$ , then

$$\alpha(z) = \omega_1 a(z)$$
 and  $\beta(z) = \omega_2 b(z)$ ,

so that  $P(z) = (\omega_1 - \omega_2)Q(z) + \omega_2$ . We omit the proof.

2. Proof of Theorem 1. In (4), write

$$Q(z) = \frac{e^z}{e^z - c} + R(z) ;$$

then

$$R(z) = \frac{-e^z\{ca(z) + b(z)\}}{(e^z - c)G(z)}.$$

By the choice of  $C_p$ , there is a positive constant A such that, on  $C_p$ ,

$$|e^z-c|>A\max\left(|e^z|,1
ight) \ |G(z)|>A\max\left(|e^zz^n|,|z^n|
ight)$$
 .

Further, by (5),

$$ca(z) + b(z) = O(|z^{n-1}|)$$

as  $|z| \to \infty$ . Hence,

$$egin{aligned} \int_{\sigma_p^+} R(z) dz \int_0^1 f(t) e^{z(x-t)} dt &= O\Bigl(\int_{\sigma_p^+} \Bigl| rac{e^{z(x-1)}}{z} dz \int_0^1 f(t) e^{-zt} dt \, \Bigr| \Bigr) \ &= o\Bigl(\int_{\sigma_p^+} \Bigl| rac{e^{z(x-1)}}{z} dz \Bigr| \Bigr) \ &= o(1) \end{aligned}$$

as  $p \to \infty$ , uniformly for  $x \le 1$ . Similarly,

$$\int_{\sigma_{\overline{p}}^{-}} R(z)dz \int_{0}^{1} f(t)e^{z(z-t)}dt = O\left(\int_{\sigma_{\overline{p}}^{-}} \left| \frac{e^{zz}}{z} dz \int_{0}^{1} f(t)e^{z(1-t)}dt \right| \right)$$

$$= o\left(\int_{\sigma_{\overline{p}}^{-}} \left| \frac{e^{zz}}{z} dz \right| \right)$$

$$= o(1)$$

as  $p \to \infty$ , uniformly for  $x \ge 0$ .

Since, for large p, the number of zeros of  $e^z-c$  inside  $C_p$  differs from 2p+1 by at most 1 and

$$\int_{0}^{1} f(t)e^{(\zeta+2p\pi_{i})(x-t)}dt = o(1)$$
 ,

it follows that

$$egin{align} \sum_{
u=1}^{n_p} c_
u e^{z
u} x &= rac{1}{2\pi i} \int_{\sigma_p} rac{e^z}{e^z-c} dz \int_0^1 f(t) e^{z(x-t)} dt + o(1) \ &= \sum_{
u=-p}^p \int_0^1 f(t) e^{(\zeta+2\pi i 
u)(x-t)} dt + o(1) \ &= e^{\zeta x} s_p(x) + o(1) \end{aligned}$$

as  $p \to \infty$ , uniformly in [0,1], and this completes the proof.

3. The proof of Theorem 2 will depend upon

LEMMA 1. For 
$$r=0,1,\cdots,n-1,$$
 
$$\int_{C_n} \frac{z^r e^{-zt}}{F(z)} dz = o(1)$$

as 
$$p o \infty$$
, boundedly for  $0 < t < 1$ .

*Proof.* Define  $C_p^+$ ,  $C_p^-$  as in § 1; then, for  $r=0,1,\cdots,n-1$ ,

$$egin{aligned} \int_{\sigma_p^+} rac{z^r e^{-zt}}{F(z)} dz &= O\Bigl(\int_{\sigma_p^+} |\, z^{r-n} e^{-zt} dz\,|\Bigr) \ &= O\Bigl(\int_{-\pi/2}^{\pi/2} \exp{(-t
ho\cos{ heta})} d heta\Bigr) \quad (
ho = r_p) \ &= O\Bigl(\int_0^{\pi/2} \exp{(-t
ho\sin{ heta})} d heta\Bigr) \ &= O\Bigl(\int_0^{\pi/2} \exp{\left(-rac{2t
ho heta}{\sigma}\right)} d heta\Bigr) \end{aligned}$$

which is o(1) as  $p \to \infty$ , boundedly for t > 0. Similarly,

$$\int_{\sigma_{p}^{-}} \frac{z^{r} e^{-zt}}{F(z)} dz = o(1)$$

boundedly for t < 1. Hence the result.

4. Proof of Theorem 2. Since the zeros of F(z) are simple,

$$\operatorname{res}_{z_{\mathcal{V}}}rac{z^{r}e^{-zt}}{F(z)}=rac{z_{\mathcal{V}}^{r}e^{-z_{\mathcal{V}}t}}{F'(z_{\mathcal{V}})}\;;$$

hence, by Lemma 1, for  $r = 0, 1, \dots, n-1$ ,

$$\sum_{\nu=1}^{n} rac{z_{
u}^{r} e^{-z_{
u} t}}{F'(z_{
u})} = o(1)$$

as  $p o \infty$ , boundedly for 0 < t < 1. By the choice of  $C_p$ ,  $n_{p+1} - n_p = 2$ 

for large p, and so, since the terms are o(1) as  $\nu \to \infty$ , we may replace  $n_p$  by p in the above summation. If we multiply by f(t) and integrate over [0,1], we have (6).

## 5. We now prove

LEMMA 2. Let  $a(z) = \sum_{k=0}^{n} a_k z^k$ , and  $b(z) = \sum_{k=0}^{n} b_k z^k$ . Then

$$(\,8\,)\qquad \textstyle\sum\limits_{r=0}^{n-1}z_{\mu}^{r}\sum\limits_{k=r+1}^{n}(b_{k}\,+\,a_{k}e^{z\nu})z_{\nu}^{k-r-1}\,+\,a(z_{\mu})e^{z_{\mu}}\int_{0}^{1}e^{(z_{\boldsymbol{\mathcal{V}}}-z_{\mu})t}dt=\begin{cases} 0 & \boldsymbol{\mathcal{V}}\neq\,\mu\\ G'(z_{\mu}) & \boldsymbol{\mathcal{V}}=\mu\end{cases}$$

Proof. Write the left-hand side of (8) as

$$\mathscr{L}+\mathscr{M};$$

then,

$$\mathscr{L} = \sum_{k=1}^{n} (b_k + a_k e^{i\mathbf{y}}) \sum_{r=0}^{k-1} z_{\mu}^{r} z_{\mathbf{y}}^{k-r-1}$$
 .

If  $\nu \neq \mu$ ,

$$egin{align} \mathscr{L} &= rac{b(z_{
u}) - b(z_{\mu}) + e^{z_{
u}} \{a(z_{
u}) - a(z_{\mu})\}}{z_{
u} - z_{\mu}} \; . \ & \mathscr{M} &= a(z_{\mu}) rac{e^{z_{
u}} - e^{z_{\mu}}}{z_{
u} - z_{\mu}} \; ; \end{aligned}$$

since  $G(z_{\nu})=G(z_{\mu})=0$ , (9) is zero. If  $\nu=\mu$ , (9) is

$$egin{array}{l} \sum_{k=1}^n k(b_k + a_k e^{z_\mu}) z_\mu^{k-1} + a(z_\mu) e^{z_\mu} \ &= b'(z_\mu) + e^{z_\mu} (a'(z_\mu) + a(z_\mu)) \ &= G'(z_\mu) \; . \end{array}$$

This proves the lemma.

6. Proof of Theorem 3. We have  $\sum_{\nu=1}^{\infty} c_{\nu} \beta_{\nu} = 0$  for every sequence  $\{c_{\nu}\}$  of CE constants, i.e.

$$\sum_{
u=1}^{\infty}eta_{
u}\lambda_{
u}\int_{0}^{1}f(t)e^{-z_{
u}t}dt=0$$

for every  $f \in L(0, 1)$ . Hence, by a well-known theorem ([2], § 279),

(10) 
$$\int_{1-x}^{1} \sum_{\nu=1}^{p} \beta_{\nu} \lambda_{\nu} e^{-z_{\nu}t} dt \longrightarrow 0$$

as  $p \to \infty$ , boundedly for  $x \in [0, 1]$ . We recall (8); if we multiply by  $\beta_{\nu} \lambda_{\nu} e^{-z_{\nu}}$  and sum from  $\nu = 1$  to  $\nu = p$ , where p is greater than an

assigned integer  $\mu$ , we obtain

$$egin{align} eta_{\mu}\lambda_{\mu}e^{-z_{\mu}}G'(z_{\mu}) &= \sum\limits_{r=0}^{n-1}z_{\mu}^{r}\sum\limits_{
u=1}^{p}eta_{
u}\lambda_{
u}\sum\limits_{k=r+1}^{n}(b_{k}e^{-z_{
u}}+a_{k})z_{
u}^{k-r-1} \ &+ a(z_{\mu})e^{z_{\mu}}\int_{0}^{1}e^{-z_{\mu}t}\sum\limits_{
u=1}^{p}eta_{
u}\lambda_{
u}e^{z_{
u}(t-1)}dt \ &= \sum\limits_{r=0}^{n-1}L_{r,p}z_{\mu}^{r}+\mathscr{N}_{p} \;, \qquad ext{say.} \end{array}$$

Let

$$\phi_p(t)=\sum\limits_{
u=1}^peta_
u\lambda_
u e^{z_
u(t-1)}$$
 ,  $arPhi_p(x)=\int_0^x\phi_p(t)dt=\int_{1-x}^1\sum\limits_{
u=1}^peta_
u\lambda_
u e^{-z_
u t}dt$  .

By (10),  $\Phi_p(x) \to 0$  as  $p \to \infty$ , boundedly for  $x \in [0, 1]$ . Thus,

$$egin{align} \mathscr{N}_p &= a(z_\mu)e^{z_\mu}\int_0^1 e^{-z_\mu t}\phi_p(t)dt \ &= a(z_\mu)e^{z_\mu}\Big\{arPhi_p(1)e^{-z_\mu} + z_\mu\int_0^1 e^{-z_\mu t}arPhi_p(t)dt\Big\} \ &= o(1) \quad ext{ as } p o \infty \;. \end{aligned}$$

Hence, since  $e^{-z}G(z) = F(z)$ ,

where the numbers  $\{L_{r,p}\}$  are independent of  $\mu$ , and  $\varepsilon_{\mu} \to 0$  as  $p \to \infty$ . Giving  $\mu$  distinct values  $\mu_1, \dots, \mu_n$ , (11) yields a regular system of n linear equations for  $L_{0,p}, \dots, L_{n-1,p}$ . The solution is

$$L_{r,p} = rac{\sum\limits_{i=1}^{n} \{eta_{\mu_{i}} \lambda_{\mu_{i}} F'(z_{\mu_{i}}) \, + \, arepsilon_{\mu_{i}} \} arDelta_{i}^{(r)}}{\det{(z_{\mu_{i}}^{i-1})}}$$

where  $\Delta_i^{(r)}$  are cofactors of elements in the (r+1)th column of the matrix  $(z_{\mu_i}^{j-1})$ ,  $(i,j=1,2,\cdots,n)$ . The only nonconstant terms in this expression for  $L_{r,p}$  are  $\varepsilon_{\mu_i}$ , which are o(1) as  $p\to\infty$ . Hence, for  $r=0,1,\cdots,n-1$ ,  $\{L_{r,p}\}$  converges, to  $\alpha_r$  say. Letting  $p\to\infty$  in (11), we have the result.

7. To prove Theorem 4, we require three lemmas.

LEMMA 3. If p > n, there are numbers  $d_1, \dots, d_n$  such that

$$e^{z_p x} + \sum_{k=1}^n d_k e^{z_k x}$$

is its own CES.

*Proof.* We shall show that there are numbers  $d_1, \dots, d_n$  such that, if

$$S(x) = e^{z_p x} + \sum_{k=1}^n d_k e^{z_k x}$$
 ,

then, for  $\mu \notin \{1, \dots, n, p\}$ ,

(12) 
$$\int_0^1 S(x)e^{-z_{\mu}x}dx = 0.$$

Since the functions  $e^{z_1x}$ , ...,  $e^{z_nx}$ ,  $e^{z_px}$  are linearly independent, and by Theorem 1, the CES of S(x) converges everywhere in (0,1) to S(x), it will then follow that S(x) is its own CES.

For  $\mu \neq k$ ,

$$egin{aligned} \int_0^1 e^{(z_k-z_\mu)x} dx &= rac{e^{-z_\mu}}{z_k-z_\mu} \{e^{z_k}-e^s_\mu\} \ &= rac{e^{-z_\mu} \{a(z_k)b(z_\mu)-a(z_\mu)b(z_k)\}}{a(z_k)a(z_\mu)(z_k-z_\mu)} \ &= rac{e^{-z_\mu}\sigma(z_k,z_\mu)}{a(z_k)a(z_\mu)} \ , \qquad ext{say.} \end{aligned}$$

Thus, if  $\mu \notin \{1, \dots, n, p\}$ , and  $d_1, \dots, d_n$  are any n numbers, the left-hand side of (12) is

$$egin{aligned} rac{e^{-z_\mu}}{a(z_\mu)} & \left\{ rac{\sigma(z_p,\,z_\mu)}{a(z_p)} + \sum\limits_{k=1}^n rac{d_k \sigma(z_k,\,z_\mu)}{a(z_k)} 
ight\} \ &= rac{e^{-z_\mu}}{a(z_\mu) a(z_p)} & \left\{ \sigma(z_p,\,z_\mu) + \sum\limits_{k=1}^n \delta_k \sigma(z_k,\,z_\mu) 
ight\} \ &= I_\mu \qquad ext{say, where } \delta_k = rac{a(z_p) d_k}{a(z_k)}. \end{aligned}$$

The symmetric polynomial

$$\sigma(x, y) = \frac{a(x)b(y) - a(y)b(x)}{x - y}$$

can be expressed in the form

$$\sum_{r=0}^{n-1} P_r(x) y^r$$

where  $P_r(x)$  is a polynomial in x of degree at most n-1. Then,

$$I_{\mu} = rac{e^{-z_{\mu}}}{a(z_{\mu})a(z_p)} \sum\limits_{r=0}^{n-1} z_{\mu}^r \Bigl\{ P_r(z_p) + \sum\limits_{k=1}^n \delta_k P_r(z_k) \Bigr\}$$
 .

This is zero for each  $\mu \notin \{1, \dots, n, p\}$  if

$$P_r(z_p)+\sum\limits_{k=1}^n\delta_kP_r(z_k)=0$$
  $(r=0,1,\,\cdots,\,n-1)$  ,

which happens if

$$z_p^r + \sum\limits_{k=1}^n \delta_k z_k^{\! eta} = 0$$
  $(r=0,1,\,\cdots,\,n-1)$  .

Since this system of n linear equations for the unknowns  $\delta_1, \dots, \delta_n$  is regular, the lemma follows.

COROLLARY. Given the constants  $c_{n+1}, \dots, c_p$  of Theorem 4, there are numbers  $c_1^{(p)}, \dots, c_n^{(p)}$  such that

$$T_p(x) = \sum_{k=1}^n c_k^{(p)} e^{z_k x} + \sum_{\nu=n+1}^p c_{\nu} e^{z_{\nu} x}$$

is its own CES.

LEMMA 4. The numbers  $c_1^{(p)}, \dots, c_n^{(p)}$  are unique and, for  $k=1,2,\dots,n$ , the sequence  $\{c_k^{(p)}\}$  converges.

*Proof.* By Theorem 2, the numbers  $c_1^{(p)}$ ,  $\cdots$ ,  $c_n^{(p)}$  satisfy the regular system of linear equations

$$rac{c_1^{(p)}z_1^r}{\lambda_1 F'(z_1)} + \cdots + rac{c_n^{(p)}z_n^r}{\lambda_n F'(z_n)} = -\sum_{
u=n+1}^p rac{c_
u z_
u^r}{\lambda_
u F'(z_
u)}$$

 $(r=0,1,\cdots,n-1)$ , and so are determined uniquely. Since  $\sum_{\nu>n}|c_{\nu}|^2<\infty$ , and

$$|\lambda_{\nu}F'(z_{\nu})| > K|z_{\nu}^n|$$

where K is a constant,

$$\sum_{\nu=n+1}^p \frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)}$$

converges, for  $r=0,1,\cdots,n-1$ . Hence, by an argument used in the proof of Theorem 3,  $\{c_k^{(n)}\}$  converges, for  $k=1,2,\cdots,n$ .

LEMMA 5. There is a positive constant A such that if  $\{a_{\nu}\}$  is any finite set of numbers, then

$$\int_0^1 | \, \Sigma a_{
u} e^{z_{
u} x} \, |^2 \, dx \leq A \Sigma \, | \, a_{
u} \, |^2 \, .$$

This may be proved by an argument similar to that of Lemma 3 of [3].

8. Proof of Theorem 4. Let p, q be integers such that q>p>n. Then,

$$T_{q}(x)-T_{p}(x)=\sum\limits_{k=1}^{n}(c_{k}^{(q)}-c_{k}^{(p)})e^{z_{k}x}+\sum\limits_{
u=p+1}^{q}c_{
u}e^{z_{
u}x}$$
 .

By Lemma 5, there is a constant A > 0 such that

$$\int_0^1 |T_q(x) - T_p(x)|^2 \, dx \le A \Bigl\{ \sum_{k=1}^n |c_k^{(q)} - c_k^{(p)}|^2 + \sum_{
u = p+1}^q |c_
u|^2 \Bigr\}$$
 .

Hence, by Lemma 4,  $\{T_p(x)\}$  converges in mean to a function  $f \in L^2(0, 1)$ . Let  $\nu > n$ . Since  $T_p(x)$  is its own CES,

$$c_{\scriptscriptstyle 
u} = \lambda_{\scriptscriptstyle 
u} \int_{\scriptscriptstyle 0}^{\scriptscriptstyle 1} T_{\scriptscriptstyle p}(x) e^{-z_{\scriptscriptstyle 
u} x} dx \qquad (p \geqq 
u).$$

Hence,

$$egin{align} c_{
u} &= \lambda_{
u} \lim_{p o \infty} \int_0^1 T_p(x) e^{-z_{
u} x} \, dx \ &= \lambda_{
u} \int_0^1 f(x) e^{-z_{
u} x} dx \; . \end{align}$$

Define  $c_1, \dots, c_n$  by this formula; then,

$$c_k = \lim_{p o\infty} c_k^{(p)}$$
  $(k=1,2,\cdots,n)$  ,

and  $\sum_{\nu=1}^{\infty} c_{\nu} e^{z_{\nu}x}$  converges in mean to f. This completes the proof.

9. Proof of Theorem 5. If we multiply (8) by  $c_{\nu}$  and sum from  $\nu=1$  to  $\nu=p$ , where p is greater than an assigned integer  $\mu$ , we obtain

(13) 
$$c_{\mu}G'(z_{\mu}) = \sum_{r=0}^{n-1} z_{\mu}^{r} \sum_{\nu=1}^{p} c_{\nu} \sum_{k=z+1}^{n} (a_{k}e^{r_{\nu}} + b_{k})z_{\nu}^{k-r-1} + a(z_{\mu})e^{z_{\mu}} \int_{0}^{1} e^{-z_{\mu}t} \sum_{\nu=1}^{p} c_{\nu}e^{z_{\nu}t}dt$$
$$= \mathscr{L}_{p} + \mathscr{M}_{p}, \quad \text{say.}$$

Since  $\sum_{\nu=1}^{\infty} |c_{\nu}|^2 < \infty$ ,  $\sum_{\nu=1}^{\infty} c_{\nu} e^{z_{\nu}t}$  converges in mean to a function  $f \in L^2(0, 1)$ . Hence,

$$\mathscr{M}_p \longrightarrow d_\mu G'(z_\mu) \qquad \text{as } p \longrightarrow \infty$$

where

$$d_{\mu}=\lambda_{\mu}\int_{0}^{1}f(t)e^{-z_{\mu}t}dt$$
 ,

Next,

(14) 
$$\mathscr{L}_{p} = \sum_{r=0}^{n-1} \delta_{r} z_{\mu}^{r} - \sum_{r=0}^{n-1} z_{\mu}^{r} \sum_{\nu=2}^{p} c_{\nu} \sum_{k=0}^{z} (a_{k} e^{r_{\nu}} + b_{k}) z_{\nu}^{k-r-1}$$

where

$$\delta_r = c_1 \sum_{k=r+1}^n (a_k e^{r_1} + b_k) z_1^{k-r-1}$$
 .

Since

$$\sum_{k=0}^{r} (a_k e^{z_y} + b_k) z_y^{k-r-1} = O(\nu^{-1})$$

the summation over  $\nu$  in (14) converges, as  $p \to \infty$ , to  $\eta_r$  say. The result now follows upon writing

$$\eta_r + \delta_r = \gamma_r$$
.

10. Before establishing the uniqueness theorem, we prove two lemmas.

LEMMA 6. If  $\sum_{\nu=1}^{\infty} d_{\nu}e^{z_{\nu}x} = f(x)$  almost everywhere in [0, 1], and  $d_{\nu} = O(\nu^{-2})$ , there are constants  $\sigma_0, \dots, \sigma_{n-1}$  such that (7) is satisfied for  $\nu = 1, 2, \dots$ .

*Proof.* We have (13), with  $c_{\nu}$  replaced by  $d_{\nu}$ . We may write this as

$$d_{\mu}G'(z_{\mu}) = \sum_{r=0}^{n-1} M_{r,p} z_{\mu}^{r} + \lambda_{\mu}G'(z_{\mu}) \int_{0}^{1} e^{-z_{\mu}t} \Big\{ f(t) - \sum_{\nu=p+1}^{\infty} d_{\nu}e^{z_{\nu}t} \Big\} dt$$
 .

Since

$$egin{align} \int_0^1 e^{-s_{\mu^t}} \sum_{
u=p+1}^\infty d_
u e^{s_
u^t} dt &= O\Bigl(\sum_{
u=p+1}^\infty \mid d_
u \mid\Bigr) \ &= o(1) \quad ext{as} \ \ p 
ightarrow \infty \ , \end{aligned}$$

and  $\{M_{r,p}\}$  converges, to  $\sigma_r$  say, for  $r=0,1,\cdots,n-1$ , we obtain (7).

LEMMA 7. If the series  $\sum_{\nu=2}^{\infty} b_{\nu}$  is convergent, then

$$\textstyle\sum\limits_{\nu=2}^{\infty}b_{\nu}\left(\frac{\sinh z_{\nu}\,h}{z_{\nu}h}\right)^{2}\longrightarrow\sum\limits_{\nu=2}^{\infty}\,b_{\nu}$$

as  $h \downarrow 0$ .

*Proof.* By a classical result, it is sufficient to show that

(i) 
$$\left(\frac{\sinh z_{\nu}h}{z_{\nu}h}\right)^2 \rightarrow 1$$
 as  $h \downarrow 0$ , for  $\nu = 2, 3, \cdots$ 

(ii) 
$$\sum_{\nu=2}^{\infty} \left| \left( \frac{\sinh z_{\nu+1}h}{z_{\nu+1}h} \right)^2 - \left( \frac{\sinh z_{\nu}h}{z_{\nu}h} \right)^2 \right|$$

is bounded as  $h \downarrow 0$ . It is evident that (i) is satisfied; (ii) may be established by the method of Theorem 1 of [4].

11. Proof of Theorem 6. The hypothesis of convergence implies that  $d_{\nu} = o(1)$ . If we define

$$\Psi(x) = \sum_{\nu=2}^{\infty} \frac{d_{\nu} e^{z_{\nu} x}}{z_{\nu}^2}$$

this series is uniformly and absolutely convergent, in [0, 1]. Now

$$rac{\varPsi(x+2h)+\varPsi(x-2h)-2\varPsi(x)}{4h^2}=\sum\limits_{
u=2}^{\infty}d_{
u}e^{z_{
u}x}\Big(rac{\sinh z_{
u}h}{z_{
u}h}\Big)^2$$

and hence, by Lemma 7, the second generalised derivative of  $\Psi(x)$  equals  $f(x) - d_1 e^{z_1 x}$  almost everywhere in [0, 1]. It follows that

$$\Psi(x) = \int_{0}^{x} dt \int_{0}^{t} (f(u) - d_{1}e^{z_{1}u})du + lx + m$$

where l, m are constants. Since

$$d_{
u}/z_{
u}^{_2} = o(
u^{_{-2}})$$
 ,

we may apply Lemma 6 to the series (15). Thus, there are constants  $\alpha_0, \dots, \alpha_{n-1}$  such that

(16) 
$$\frac{d_{\nu}}{z_{\nu}^2} = \lambda_{\nu} \int_0^1 \Psi(t) e^{-z_{\nu}t} dt + \sum_{z=0}^{n-1} \frac{\alpha_z z_{\nu}^r}{G'(z_{\nu})}$$

for  $\nu = 2, 3, \cdots$ .

If we integrate by parts twice, we can write (16) in the form

$$d_{
u}=\lambda_{
u}\int_0^1 f(t)e^{-z_{
u}t}dt+\sum_{r=0}^{n+1}rac{\sigma_rz_{
u}^r}{G'(z_{
u})}$$
 ,

where  $\sigma_0, \dots, \sigma_{n+1}$  are constants. Since  $G'(z_{\nu}) \sim -b_n z_{\nu}^n$ 

$$d_
u = o(1)$$
 and  $\lambda_
u \int_0^1 f(t) e^{-z_
u t} dt = o(1)$ ,

we have

$$\sigma_n = \sigma_{n+1} = 0 ,$$

and for  $\nu = 2, 3, \dots$ , we have (7). Finally, by Theorem 1 and Lemma 1,

$$\sum_{\nu=1}^{\infty} \left\{ \lambda_{\nu} \int_{0}^{1} f(t) e^{-z_{\nu} t} dt + \sum_{r=0}^{n-1} \frac{\sigma_{r} z_{\nu}^{r}}{G'(z_{\nu})} \right\} e^{z_{\nu} x}$$

is summable (C, 1) almost everywhere in [0, 1] to

$$f(x) - \left\{ \lambda_1 \int_0^1 f(t) e^{-z_1 t} dt + \sum_{r=0}^{n-1} \frac{\sigma_r z_1^r}{G'(z_1)} \right\} e^{z_1 x}$$

so that we have (7) for  $\nu = 1$ , and the proof is complete.

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### REFERENCES

- 1. G. H. Fullerton, Expansions of a function in a series of exponentials, Ph. D. Thesis (Belfast, 1959)
- 2. E. W. Hobson, The theory of functions of a real variable, vol. II, (Cambridge, 1926)
- 3. S. Verblunsky, On an expansion in exponential series (II), Quart, J. Math. (Oxford) (2), 10 (1959), 99-109.
- 4. ——, A uniqueness theorem for the exponential series of Herglotz, Proc. Camb. Phil. Soc. 56 (1960), 220-232.

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