

Pacific Journal of Mathematics

**A GENERALIZATION OF THE COSET DECOMPOSITION OF A
FINITE GROUP**

BASIL GORDON

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Let G be a finite group, and suppose that G is partitioned into disjoint subsets: $G = \bigcup_{i=1}^h A_i$. If the A_i are the left (or right) cosets of a subgroup $H \subseteq G$, then the products xy , where $x \in A_i$ and $y \in A_j$, represent all elements of any A_k the same number of times. It turns out that certain other decompositions of G of interest in algebra enjoy this same property; we will call such a partition π an α -partition.

In this paper all α -partitions are determined in the case G is a cyclic group of prime order p ; they arise by choosing a divisor d of $p-1$, and letting the A_i be the sets on which the d 'th power residue symbol $(x/p)_d$ has a fixed value. It is shown that if an α -partition is invariant under the inner automorphisms of G (strongly normal) then it is also invariant under the antiautomorphism $x \rightarrow x^{-1}$. For such α -partitions (called weakly normal) it is shown that the set A_i containing the identity element is a group. An example shows that this need not hold for nonnormal partitions.

1. For any α -partition π , let N_{ijk} denote the number of times each element of A_k is represented among the products xy , $x \in A_i$, $y \in A_j$. Then if $\mathfrak{A}(G)$ is the group algebra of G over a field K , and if we put

$$(1) \quad a_i = \sum_{x \in A_i} x,$$

it is clear that $a_i a_j = \sum_{k=1}^h N_{ijk} a_k$. Therefore the vector space spanned over K by a_1, \dots, a_h is a subalgebra \mathfrak{A}_π of $\mathfrak{A}(G)$, with structure constants N_{ijk} . Conversely, if $\pi: G = \bigcup_{i=1}^h A_i$ is any partition of G into disjoint subsets, and if the elements a_i defined by (1) span a subalgebra of $\mathfrak{A}(G)$, then π is an α -partition.

In the case where π is the decomposition of G into the cosets of a normal subgroup H whose order m is not divisible by the characteristic of K , the algebra \mathfrak{A}_π is the group algebra $\mathfrak{A}(G/H)$ of the factor group G/H . For then the elements a_i/m form a group isomorphic to G/H , and are a basis of \mathfrak{A}_π .

In this paper some of the elementary properties of α -partitions are developed. I plan in a second paper to discuss in more detail the structure of the algebras \mathfrak{A}_π and their application to the representation of G by matrices.

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2. **Normal partitions.** Since the α -partitions are a generalization of the coset decomposition of G with respect to a subgroup H , it is natural to begin the study of them by asking which α -partitions should be called normal. Several different definitions of normality are possible, and two of them will be considered here. Note first that if π is an α -partition, and σ is an automorphism or anti-automorphism of G , then the partition π^σ obtained by applying σ to the sets of π , is again an α -partition. If $\pi = \pi^\sigma$, we will say that π is *invariant under σ* . This means that the sets of π are permuted among themselves by σ . If σ has the stronger property of mapping each set of π onto itself, π is called *setwise invariant under σ* .

An α -partition π is called *weakly normal* if it is invariant under the anti-automorphism $\sigma: x \rightarrow x^{-1}$. On the other hand π is called *strongly normal* if it is invariant under all inner automorphisms $\tau: x \rightarrow t^{-1}xt$. It is easily seen that in the case where π is the left coset decomposition of G with respect to a subgroup H , either type of normality of π is equivalent to normality of H . The following theorem explains the choice of terminology.

THEOREM 1. *If π is strongly normal, then it is also weakly normal.*

Proof. Let π be strongly normal, let A_i be any set of π , and let x be any element of A_i . Suppose $x^{-1} \in A_j$. If n is the order of G , there exists a prime p such that $p > n$, $p \equiv -1 \pmod{n}$, by Dirichlet's theorem on primes in an arithmetic progression. Let H_i be the group generated by the elements of A_i , and denote its order by m_i . Consider the set S of all ordered $(p+1)$ -tuples $(t, x_1, x_2, \dots, x_p)$ with $t \in H_i$, all $x_v \in A_i$, and such that $t^{-1}x^{-1}t = x_1x_2 \cdots x_p$. The mapping $\theta: (t, x_1, \dots, x_p) \rightarrow (tx_1, x_2, \dots, x_p, x_1)$ maps S onto itself, and so S is decomposed into orbits by the cyclic group of mappings generated by θ . Clearly the cardinality of the orbit of (t, x_1, \dots, x_p) is a multiple of p unless $x_1 = x_2 = \cdots = x_p$. In this case we have $t^{-1}x^{-1}t = x_1^p = x_1^{-1}$, or equivalently $t^{-1}xt = x_1$. Therefore the number of such $(p+1)$ -tuples is equal to the number of elements $t \in H_i$ such that $t^{-1}A_it = A_i$. But every element $t \in H_i$ has this property. Indeed, if $t \in A_i$ then $t^{-1}tt = t$, so that the assumed strong normality of π implies $t^{-1}A_it = A_i$; the same is then of course true for all $t \in H_i$.

From this we see that if N is the cardinality of S , then $N \equiv m_i \pmod{p}$. On the other hand it is immediately seen from the definition of a strongly normal α -partition that if y is any element of A_j , then the number of ordered $(p+1)$ -tuples (t, x_1, \dots, x_p) , $t \in H_i$, $x_v \in A_i$ such that $t^{-1}yt = x_1x_2 \cdots x_p$ is also N . Since these $(p+1)$ -tuples can be

divided into orbits as above, we see that there are exactly m_i solutions of the equation $t^{-1}yt = x_i^p = x_i^{-1}$, where $t \in H_i$, $x_i \in A_i$ (here we use the fact that $m_i \leq n < p$). Hence all $t \in H_i$, give rise to solutions of this equation. Taking $t = e$ we get $y = x_i^{-1}$, so that the inverse of any element of A_j is in A_i . Since the roles of A_i and A_j can be interchanged, we have $A_j = \{z^{-1} \mid z \in A_i\}$, and the proof is complete.

In general weak normality does not imply strong normality. This can be seen by considering the example where A_1 is a nonnormal subgroup of G and $A_2 = G - A_1$.

3. Weakly normal partitions. In this section we obtain a characteristic property of weakly normal α -partitions which is useful in the further development of the theory. Let $\pi : G = \cup_{i=1}^h A_i$ be any decomposition of G into disjoint sets (not necessarily an α -partition). Suppose that for any $x \in A_i$, the cardinality of the $xA_j \cap A_k$ depends only on i, j, k (that is, does not depend on the particular x chosen from A_i) and for any $y \in A_j$, the cardinality of $A_i y \cap A_k$ depends only on i, j, k . We will use the tentative term β -partition to describe such π 's, and will prove that they are precisely the weakly normal α -partitions. Half of this can be proved at once.

THEOREM 2. *Every weakly normal α -partition is a β -partition.*

Proof. Suppose $x \in A_i$, and form the set $xA_j \cap A_k$. The cardinality of this set is the number of solutions of the equation $xy = z$, where $y \in A_j$, $z \in A_k$. Since this equation is equivalent to $x = zy^{-1}$, and since $\{y^{-1} \mid y \in A_j\} = A_j'$ for some j' , the number of solutions is $N_{kj' i}$, which depends only on i, j, k . In the same way we see that the cardinality of $A_i y \cap A_k$, where $y \in A_j$, depends only on i, j, k , and the proof is complete.

The proof that every β -partition is a weakly normal α -partition is somewhat more complicated, and we need two lemmas. For any β -partition, let Q_{ijk} denote the cardinality of $A_i y \cap A_k$, where $y \in A_j$.

LEMMA 1. *Suppose that the identity element e of G is in the set A_1 of a β -partition. Then A_1 is a group. Each A_i is a union of right cosets $A_1 t$, $t \in G$, and also a union of left cosets tA_1 , $t \in G$.*

Proof. Since $eA_1 = A_1$, we must have $xA_1 = A_1$ for any $x \in A_1$, which proves that A_1 is a subgroup of G . For any other set A_i we have $eA_i = A_i$, and therefore $xA_i = A_i$ for all $x \in A_1$. Hence whenever A_i contains an element t , it also contains the right coset $A_1 t$. By the same reasoning A_i contains the left coset tA_1 , which completes the proof.

LEMMA 2. *Let A_i be any set of a β -partition π . Then $\{x^{-1} \mid x \in A_i\}$ is also a set of π .*

Proof. Choose a fixed element $y \in A_i$, and let C be the set of π to which y^{-1} belongs (of course C may coincide with A_i). Then the complex yC contains at least one number of A_i , namely e . Hence if x is any other element of A_i , the complex xC must contain a member of A_i . Thus $xc = w$, where $c \in C$ and $w \in A_i$. Then $x^{-1} = cw^{-1}$ is in C by Lemma 1, which shows that $C \supseteq \{x^{-1} \mid x \in A_i\}$. By the same reasoning $A_i \supseteq \{z^{-1} \mid z \in C\}$, and hence $C = \{x^{-1} \mid x \in A_i\}$.

We define the mapping $i \rightarrow i'$ by putting $A_{i'} = \{x^{-1} \mid x \in A_i\}$.

THEOREM 3. *Every β -partition is a weakly normal α -partition.*

Proof. Let $\pi : G = \bigcup_{i=1}^h A_i$ be a β -partition. Fix $z \in A_k$ and consider the equation $xy = z$, where $x \in A_i$, $y \in A_j$. Since this equation is equivalent to $y = x^{-1}z$, it has $Q_{i'kj}$ solutions. Therefore every element of A_k is represented $Q_{i'kj}$ times among the products xy , $x \in A_i$, $y \in A_j$, and so π is an α -partition. It is weakly normal by Lemma 2.

In the next theorem we again let A_1 be the set of π containing e , and denote its cardinality by ν_1 .

THEOREM 4. *If π is weakly normal, and if ν_1 is not a multiple of the characteristic of K , then \mathfrak{A}_π has a two-sided identity element.*

Proof. By Lemma 1 each A_i is a union of right cosets of A_1 . Hence $xA_i = A_i$ for any $x \in A_1$. Therefore, defining the elements a_i by (1), we have $a_1a_i = \nu_1a_i$. Similarly $a_ia_1 = \nu_1a_i$, so that $\nu_1^{-1}a_1$ is a two-sided identity in \mathfrak{A}_π .

We conclude this section with some remarks and examples. Lemma 1 shows that if π is a weakly normal α -partition, then the set of π containing the identity element is a subgroup of G . If G is Abelian, then every α -partition is clearly strongly normal, and hence weakly normal by Theorem 1. Thus in this case the set containing e is always a subgroup. For non-Abelian groups this need not be so, as can be seen by considering the double coset decomposition $G = \bigcup_{i=1}^h Ha_iK$, where H and K are nonnormal subgroups of G . For example if $G = S_3$, the symmetric group on 3 letters, $H = \{e, (12)\}$, $K = \{e, (13)\}$, we obtain an α -partition into the two sets $A_1 = \{e, (12), (13), (123)\}$, $A_2 = \{(23), (132)\}$. Here A_1 is not a group.

An important class of weakly normal α -partitions can be constructed as follows. Let Γ be any group of automorphisms of G , and let the sets of π be the orbits of G under Γ , so that two elements $x_1, x_2 \in G$

are in the same set of π if and only if $x_1^\sigma = x_2$ for some $\sigma \in \Gamma$. Then if z and z^σ are two elements of A_k , to every representation $z = xy$ with $x \in A_i, y \in A_j$ corresponds the representation $z^\sigma = x^\sigma y^\sigma$ and conversely. Hence π is an α -partition. Also $x_1^\sigma = x_2$ implies $(x_1^{-1})^\sigma = x_2^\sigma$, so that if A_i is a set of π , so is $\{x^{-1} \mid x \in A_i\}$. Thus π is weakly normal. It is easily seen that π is strongly normal if and only if Γ is normalized by the group Γ_0 of inner automorphisms of G . This last situation includes the partition of G into its conjugacy classes, for then $\Gamma = \Gamma_0$.

4. The case $G = Z_p$. We next determine all α -partitions of Z_p , the cyclic group of prime order p . We use the additive notation for Z_p , so that its elements are $0, 1, \dots, p - 1$, and the group operation is addition (mod p). It is convenient in this case to call the sets of the partition A_0, \dots, A_h rather than A_1, \dots, A_h , and to let A_0 be the set containing the identity element 0 .

The only subgroups of Z_p are Z_p and $\{0\}$, and so by Lemma 1, $A_0 = Z_p$ or $A_0 = \{0\}$. The first case gives rise to a trivial α -partition, so only the second case need be considered. If ϵ is any primitive p 'th root of unity, then the mapping $x \rightarrow \epsilon^x$ maps Z_p isomorphically into the complex field, and by extension maps the group algebra $\mathfrak{A}(G)$ over the rational field Q homomorphically onto $Q(\epsilon)$. Let η_i be the image of a_i under this mapping, so that $\eta_i = \sum_{x \in A_i} \epsilon^x$.

LEMMA 3. *The η_i are algebraic integers of degree at most h .*

Proof. By (1), $\eta_i \eta_j = \sum_{k=0}^h N_{ijk} \eta_k$. Since $\eta_0 = 1 = -\eta_1 - \eta_2 - \dots - \eta_h$, this can be written in the form $\eta_i \eta_j = \sum_{k=1}^h (N_{ijk} - N_{ij0}) \eta_k$; ($1 \leq i, j \leq h$). Thus the vector (η_1, \dots, η_h) is an eigenvector of the matrix $(M_{jk}) = (N_{ijk} - N_{ij0})$ ($1 \leq j, k \leq h$) with eigenvalue η_i . Since the M_{jk} are integers, it follows that η_i is an algebraic integer of degree $\leq h$.

THEOREM 5. *Let $\bigcup_{i=0}^h A_i$ be an α -partition of Z_p with $A_0 = \{0\}$. Then*

- (i) $p \equiv 1 \pmod{h}$
- (ii) *If g is a primitive root of p , then the classes A_i can be numbered so that A_i consists of all residues x with $\text{ind}_g x \equiv i \pmod{h}$; ($i > 0$).*
- (iii) *Conversely, for any h dividing $p - 1$, the sets defined in (ii) form an α -partition of z_p .*

Proof. Let c_i be the number of elements in A_i , and suppose for the sake of the argument that $c_1 = \min_{1 \leq i \leq h} c_i$. Theorem 2 implies that

$Q \cong Q(\eta_1) \cong Q(\varepsilon)$, where $S = [Q(\eta_1) : Q] \leq h$. But $Q(\varepsilon)$ is a normal extension of Q whose Galois group \mathfrak{G} is generated by the automorphism $\varepsilon \rightarrow \varepsilon^g$, and is cyclic of order $p - 1$. By the fundamental theorem of Galois theory, the elements of $Q(\eta_1)$ are invariant under a subgroup \mathfrak{H} of \mathfrak{G} of order $t = (p - 1)/s$. Since a cyclic group has only one subgroup of given order, \mathfrak{H} is generated by the automorphism $\varepsilon \rightarrow \varepsilon^{g^s}$. From this it follows that if ε^x is a term of η_i , then $\varepsilon^{g^s x}$ is also a term of η_i . Hence η_i contains the t distinct terms $\varepsilon^x, \varepsilon^{g^s x}, \dots, \varepsilon^{g^{(t-1)s} x}$, so that $c_i \geq t$. Hence $p - 1 = \sum_{i=1}^h c_i \geq hc_1 \geq ht \geq st = p - 1$. Equality must hold at each stage, and so $c_1 = c_2 = \dots = c_h = t$, and $h = s$. Moreover each η_i is of the form $\eta_i = \varepsilon^{x_i} + \varepsilon^{g^s x_i} + \dots + \varepsilon^{g^{(t-1)s} x_i}$, and accordingly each A_i is of the form $A_i = \{x_i, g^s x_i, \dots, g^{(t-1)s} x_i\}$. Re-numbering the A_i if necessary, this is equivalent to assertion (ii).

To prove (iii) it suffices to apply the remark made at the end of §2, taking Γ to be the group of automorphisms of G generated by the mapping $x \rightarrow \mu x$, where μ is an element of order h in the multiplicative group of non-zero residues (mod p).

The determination of the structure constants N_{ijk} of the algebras \mathfrak{A}_π of Z_p is an interesting and difficult problem. For a survey of the known results, see [1].

5. The lattice of α -partitions. If π_1 and π_2 are any two partitions of G into disjoint sets, we will say that $\pi_1 \leq \pi_2$ if every set of π_1 is contained in some set of π_2 . This clearly defines a partial ordering, and the purpose of this section is to show that the set of all α -partitions of G is a lattice under this ordering. The following theorem is the key to the proof of this fact.

THEOREM 6. *Let π_0 be a given partition of G . Then the set of α -partitions π satisfying $\pi \leq \pi_0$ has a greatest element.*

Proof. If π_0 is itself an α -partition the theorem is clearly true. So we can suppose that there are three sets A_i, A_j, A_k of π_0 such that not all elements of A_k are represented the same numbers of times among the products $xy, x \in A_i, y \in A_j$. Thus A_k can be decomposed into sets $A_{k1}, A_{k2}, \dots, A_{k\gamma} (\gamma \geq 2)$, by putting two elements $u, v \in A_k$ in the same $A_{k\gamma}$ if and only if u and v are represented the same number of times in the form xy . Call π_1 the resulting partition of G . If π is an α -partition with $\pi \leq \pi_0$, then A_i and A_j are both unions of sets of π . Therefore each $A_{k\gamma}$ is a union of sets of π , so that $\pi \leq \pi_1 < \pi_0$. If π_1 is an α -partition we are through; otherwise we can treat π_1 in the same way as π_0 , thus obtaining a partition $\pi_2 < \pi_1$ with the property that any α -partition $\pi \leq \pi_0$ is $\leq \pi_2$. Proceeding in this manner

we obtain a chain $\pi_0 > \pi_1 > \pi_2 \cdots$, which must terminate after a finite number of steps since G is finite.

THEOREM 7. *The α -partitions of G form a lattice L . The weakly and strongly normal α -partitions form sublattices L_w and L_s with $L_s \subseteq L_w \subseteq L$.*

Proof. If $\pi_1: G = \bigcup_{i=1}^k A_i$ and $\pi_2: G = \bigcup_{j=1}^k B_j$ are any two α -partitions of G , let π_0 be the partition $G = \bigcup_{i,j} A_i \cap B_j$. Clearly any α -partition π satisfying $\pi \leq \pi_1$ and $\pi \leq \pi_2$ satisfies $\pi \leq \pi_0$ and conversely. Hence by Theorem 6 there is a greatest such α -partition, which we denote by $\pi_1 \cap \pi_2$. It follows at once that any finite set π_1, \dots, π_m of α -partitions have a meet $\pi_1 \cap \dots \cap \pi_m$. Therefore any two α -partitions π_1, π_2 have a join $\pi_1 \cup \pi_2$, namely the meet of all α -partitions π such that $\pi_1 \leq \pi, \pi_2 \leq \pi$.

To prove the second part of the theorem, suppose that π_1 and π_2 are both invariant under a group Σ of automorphisms and antiautomorphisms of G . Then for any $\sigma \in \Sigma$ we have $(\pi_1 \cap \pi_2)^\sigma \leq \pi_1^\sigma = \pi_1$ and similarly $(\pi_1 \cap \pi_2)^\sigma \leq \pi_2$. Therefore $(\pi_1 \cap \pi_2)^\sigma \leq \pi_1 \cap \pi_2$, and reasoning in the same way with σ^{-1} , we see that $(\pi_1 \cap \pi_2)^\sigma = \pi_1 \cap \pi_2$. This shows that $\pi_1 \cap \pi_2$ is invariant under Σ , and the same is of course true of $\pi_1 \cup \pi_2$.

The lattice of α -partitions of G conveys more information about G than its lattice of subgroups. A fuller account of this will be given elsewhere.

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Pacific Journal of Mathematics

Vol. 15, No. 2

October, 1965

Patrick Robert Ahern, <i>On the generalized F. and M. Riesz theorem</i>	373
A. A. Albert, <i>On exceptional Jordan division algebras</i>	377
J. A. Anderson and G. H. Fullerton, <i>On a class of Cauchy exponential series</i>	405
Allan Clark, <i>Hopf algebras over Dedekind domains and torsion in H-spaces</i>	419
John Dauns and D. V. Widder, <i>Convolution transforms whose inversion functions have complex roots</i>	427
Ronald George Douglas, <i>Contractive projections on an L_1 space</i>	443
Robert E. Edwards, <i>Changing signs of Fourier coefficients</i>	463
Ramesh Anand Gangolli, <i>Sample functions of certain differential processes on symmetric spaces</i>	477
Robert William Gilmer, Jr., <i>Some containment relations between classes of ideals of a commutative ring</i>	497
Basil Gordon, <i>A generalization of the coset decomposition of a finite group</i>	503
Teruo Ikebe, <i>On the phase-shift formula for the scattering operator</i>	511
Makoto Ishida, <i>On algebraic homogeneous spaces</i>	525
Donald William Kahn, <i>Maps which induce the zero map on homotopy</i>	537
Frank James Kosier, <i>Certain algebras of degree one</i>	541
Betty Kvarda, <i>An inequality for the number of elements in a sum of two sets of lattice points</i>	545
Jonah Mann and Donald J. Newman, <i>The generalized Gibbs phenomenon for regular Hausdorff means</i>	551
Charles Alan McCarthy, <i>The nilpotent part of a spectral operator. II</i>	557
Donald Steven Passman, <i>Isomorphic groups and group rings</i>	561
R. N. Pederson, <i>Laplace's method for two parameters</i>	585
Tom Stephen Pitcher, <i>A more general property than domination for sets of probability measures</i>	597
Arthur Argyle Sagle, <i>Remarks on simple extended Lie algebras</i>	613
Arthur Argyle Sagle, <i>On simple extended Lie algebras over fields of characteristic zero</i>	621
Tôru Saitô, <i>Proper ordered inverse semigroups</i>	649
Oved Shisha, <i>Monotone approximation</i>	667
Indranand Sinha, <i>Reduction of sets of matrices to a triangular form</i>	673
Raymond Earl Smithson, <i>Some general properties of multi-valued functions</i>	681
John Stuepnel, <i>Euclidean fiberings of solvmanifolds</i>	705
Richard Steven Varga, <i>Minimal Gerschgorin sets</i>	719
James Juei-Chin Yeh, <i>Convolution in Fourier-Wiener transform</i>	731