Pacific Journal of Mathematics

AN INEQUALITY FOR THE NUMBER OF ELEMENTS IN A SUM OF TWO SETS OF LATTICE POINTS

BETTY KVARDA

Vol. 15, No. 2 October 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

AN INEQUALITY FOR THE NUMBER OF ELEMENTS IN A SUM OF TWO SETS OF LATTICE POINTS

BETTY KVARDA

For a fixed positive integer n, let Q be the set of all n-dimensional lattice points $(x_1, \cdots, x_n)$ with each x_i a nonnegative integer and at least one x_i positive. A finite nonempty subset R of Q is called a *fundamental set* if for every $(r_1, \cdots, r_n)$ in R, all vectors $(x_1, \cdots, x_n)$ of Q with $x_i \leqq r_i$, $i=1, \cdots, n$, are also in R. If A is any subset of Q and R is any fundamental set, let $A(R)$ denote the number of vectors in $A \cap R$. Finally, if A is any proper subset of Q, let the *density* of A be the quantity

$$\alpha = \text{glb} \frac{A(R)}{Q(R) + 1},$$

taken over all fundamental sets R for which $A(R) < Q(R)$. Then the theorem proved in this paper can be stated as follows.

THEOREM. Let A and B be subsets of Q, let C be the set of all vectors of the form a, b, or $a + b$ where $a \in A$ and $b \in B$, let α be the density of A, and let R be any fundamental set such that (1) there exists at least one vector in R which is not in C, and (2) for each b in $B \cap R$ (if any) there exists g in R but not in C such that $g - b$ is in Q. Then

$$C(R) \geqq \alpha[Q(R) + 1] + B(R) .$$

It will be seen that for $n = 1$ this theorem implies a result of H. B. Mann [2].

Let A and B be sets of positive integers, and for any positive integer x denote by $A(x)$ the number of integers in A which are not greater than x. Let the *modified density* (or *Erdös density*) of A be the quantity

$$\alpha = \text{glb}_{x \geqq k} \frac{A(x)}{x + 1}$$

where k is the smallest positive integer not in A. If $C = A + B$ is the set of all integers of the form a, b, or $a + b$, where a is in A and b is in B, and if x is a positive integer not in C, then Mann has shown [2] that

$$C(x) \geq \alpha x + B(x) .$$

Received February 17, 1964. The material in this paper is based upon a portion of the author's Ph. D. thesis, written under the direction of Dr. Robert D. Stalley at Oregon State University.

(Actually, Mann's work is sufficient to establish $C(x) \geqq \alpha(x+1) + B(x)$.) We will show that this theorem, with somewhat weaker hypotheses, can be extended to certain sets of n-dimensional lattice points.

Let Q be the set of all lattice points $\boldsymbol{x} = (x_1, \cdots, x_n)$ for which each component is a nonnegative integer and at least one component is positive. Define the sum of subsets of Q in the same manner as was done for sets of positive integers, addition of lattice points being done componentwise, and for any subsets A and B of Q let $A - B$ denote the set of all elements of A which are not in B. If A and S are subsets of Q and S is finite let $A(S)$ be the number of elements in $A \cap S$. Let $\boldsymbol{\omega}_i$ be that element of Q for which the ith component is 1 and the others are 0.

DEFINITION 1. A finite nonempty subset R of Q will be called a *fundamental set* if whenever $\boldsymbol{r} = (r_1, \cdots, r_n)$ is in R then all vectors $\boldsymbol{x} = (x_1, \cdots, x_n)$ of Q such that $x_i \leqq r_i$, $i = 1, \cdots, n$, are also in R.

DEFINITION 2. Let A be any proper subset of Q. Then the *density* of A is the quantity

$$\alpha = \operatorname{glb} \frac{A(R)}{Q(R)+1},$$

taken over all fundamental sets R for which $A(R) < Q(R)$.

2. Extension of Mann's result. The theorem to be proved can now be stated as follows.

THEOREM. *Let A and B be subsets of Q, let $C = A + B$, and let α be the density of A. Let R be any fundamental set such that for each $\boldsymbol{b}$ in $B \cap R$ there exists $\boldsymbol{g}$ in $R - C$ such that $\boldsymbol{g} - \boldsymbol{b}$ is in Q, and $Q(R - C) \geqq 1$. Then*

$$C(R) \geqq \alpha[Q(R) + 1] + B(R).$$

Proof. Let the elements of Q be ordered so that $(x_1, \cdots, x_n) > (y_1, \cdots, y_n)$ if $x_1 > y_1$, or if $x_1 = y_1, \cdots, x_k = y_k, x_{k+1} > y_{k+1}$. Consider a nonempty set $S = R' - R''$, where R' and R'' are fundamental sets, and let $\boldsymbol{\delta}_1 = (\delta_{11}, \cdots, \delta_{1n}), \cdots, \boldsymbol{\delta}_u = (\delta_{u1}, \cdots, \delta_{un})$ be all the vectors of S such that for each $i = 1, \cdots, n$ and for each $j = 1, \cdots, u$ we have either (1) $\boldsymbol{\delta}_j - \boldsymbol{\omega}_i$ is in R'', or (2) $\boldsymbol{\delta}_j - \boldsymbol{\omega}_i = \boldsymbol{0} = (0, \cdots, 0)$, or (3) $\delta_{ji} = 0$. There must be at least one such vector in S, for S is a nonempty finite set, and hence has a least element (in our ordering). This least element will satisfy the given conditions. Also, it is easily seen that if $(s_1, \cdots, s_n)$ is any vector in S then for at least one of the $\boldsymbol{\delta}_j$ we have $\delta_{ji} \leqq s_i$, $i = 1, \cdots, n$.

From this it follows that if for each $j = 1, \cdots, u$ we let

$$S_j = \{s = (s_1, \cdots, s_n) \mid s \in S,\ s_i \geqq \delta_{ji},\ i = 1, \cdots, n\}\ ,$$

then $S = S_1 \cup \cdots \cup S_u$. Also, let $S'_j = \{s - \boldsymbol{\delta}_j \mid s \in S_j,\ s \neq \boldsymbol{\delta}_j\}$ and let $S' = S'_1 \cup \cdots \cup S'_u$. Each S'_j, and therefore also S', is either a fundamental set or is empty.

LEMMA 1. $Q(S') + 1 \leqq Q(S)$.

Proof of Lemma 1. The lemma is obvious if $n = 1$, since then $u = 1$ also. Hence assume $n \geqq 2$. Let λ_1 be a mapping defined so that

$$S_j\lambda_1 = \{s - \delta_{j1}\omega_1 \mid s \in S_j\},\ j = 1, \cdots, u\ ,$$
$$S\lambda_1 = S_1\lambda_1 \cup \cdots \cup S_u\lambda_1\ .$$

Partition S into sets $T_{c_2 \cdots c_n}$ such that

$$T_{c_2 \cdots c_n} = \{s = (x_1, c_2, \cdots, c_n) \mid s \in S\}\ ,$$

and let

$$T_{c_2 \cdots c_n}\lambda_1 = \{s = (x_1, c_2, \cdots, c_n) \mid s \in S\lambda_1\}\ ,$$
$$k_{c_2 \cdots c_n} = \max_{1 \leqq j \leqq u} (\max \{x_1 - \delta_{j1} \mid (x_1, c_2, \cdots, c_n) \in S_j\})\ .$$

Then $Q(T_{c_2 \cdots c_n}\lambda_1) = k_{c_2 \cdots c_n} + 1$ or $Q(T_{c_2 \cdots c_n}\lambda_1) + 1 = k_{c_2 \cdots c_n} + 1$, according as $\mathbf{0} \notin T_{c_2 \ldots c_n}\lambda_1$ or $\mathbf{0} \in T_{c_2 \ldots c_n}\lambda_1$, and $k_{c_2 \ldots c_n} + 1 \leqq Q(T_{c_2 \ldots c_n})$.

Hence $Q(S\lambda_1) \leqq Q(S)$, and $Q(S\lambda_1) + 1 \leqq Q(S)$ if $\mathbf{0} \in S\lambda_1$.

Now define mappings $\lambda_2, \cdots, \lambda_n$ such that

$$S_j\lambda_1 \cdots \lambda_{i-1}\lambda_i = \{s - \delta_{j/i}\omega_i \mid s \in S_j\lambda_1 \cdots \lambda_{i-1}\}\ ,$$

$i = 2, \cdots, n$, and obtain as above

$$Q(S\lambda_1 \cdots \lambda_i) + \theta_i \leqq Q(S\lambda_1 \cdots \lambda_{i-1}) + \theta_{i-1} \leqq Q(S)\ ,$$

where $\theta_1 = 0$ or 1 according as $\mathbf{0} \notin S\lambda_1 \cdots \lambda_i$ or $\mathbf{0} \in S\lambda_1 \cdots \lambda_i$. This establishes the lemma.

DEFINITION 3. A set S will be said to be of type I if

(1) S is a fundamental set,

(2) $Q(S - C) \geqq 1$, and

(3) for all $\boldsymbol{b}$ in $B \cap S$ (if any) and all $\boldsymbol{g}$ in $S - C$ we have $\boldsymbol{g} - \boldsymbol{b}$ contained in Q.

DEFINITION 4. A set S will be said to be of type II if

(1) there exist fundamental sets R', R'' such that $S = R' - R''$,

(2) $B(S) \geqq 1$ and $Q(S - C) \geqq 1$, and

(3) for all $\boldsymbol{b}$ in $B \cap S$ and $\boldsymbol{g}$ in $S - C$ we have $\boldsymbol{g} - \boldsymbol{b}$ contained in Q.

LEMMA 2. *If S is any set of type II then*

$$C(S) \geqq \alpha Q(S) + B(S) .$$

Proof of Lemma 2. Define the sets S'_j and S' as above. Let $\boldsymbol{b} = (b_1, \cdots, b_n)$ be the largest vector such that

(1) $\boldsymbol{b}$ is in $B \cap S$, and

(2) $b_1 + \cdots + b_n = \max\{x_1 + \cdots + x_n \mid (x_1, \cdots, x_n) \in B \cap S\}$. Likewise, let $\boldsymbol{g} = (g_1, \cdots, g_n)$ be the largest vector such that

(1) $\boldsymbol{g}$ is in $S - C$, and

(2) $g_1 + \cdots + g_n = \max\{y_1 + \cdots + y_n \mid (y_1, \cdots, y_n) \in S - C\}$.

Let $B(S) = \rho \geqq 1$, $Q(S - C) = \sigma \geqq 1$, $Q(S' - A) = \tau$. The set $\{\boldsymbol{g} - \boldsymbol{x} \mid \boldsymbol{x} \in B \cap S\}$ contains ρ elements of Q (Definition 4, part 3), none of which is in A. We show that these are in S': If $\boldsymbol{x} = (x_1, \cdots, x_n)$ is in $B \cap S$ then $\boldsymbol{x}$ is in S_j for some j such that $1 \leqq j \leqq u$. Hence $\delta_{ji} \leqq x_i \leqq g_i$ for all $i = 1, \cdots, n$, and $\boldsymbol{g}$ is in S_j. $0 \neq \boldsymbol{g} - \boldsymbol{x} = (\boldsymbol{g} - \boldsymbol{\delta}_j) - (\boldsymbol{x} - \boldsymbol{\delta}_j)$. But $\boldsymbol{g} - \boldsymbol{\delta}_j$ is in S'_j and S'_j is a fundamental set. Hence $\boldsymbol{g} - \boldsymbol{x}$ is in S'_j, therefore in S'.

Likewise, the (possibly empty) set $\{\boldsymbol{y} - \boldsymbol{b} \mid \boldsymbol{y} \in S - C,\ \boldsymbol{y} \neq \boldsymbol{g}\}$ contains $\sigma - 1$ elements, all of which are in $S' - A$. We must show that the two sets are disjoint. Hence suppose that for some $\boldsymbol{y} \neq \boldsymbol{g}$ and, therefore, $\boldsymbol{x} \neq \boldsymbol{b}$, we have

$$\boldsymbol{g} - \boldsymbol{x} = \boldsymbol{y} - \boldsymbol{b} .$$

Equating the ith components and transposing gives the n equations

$$\text{(V)} \qquad \begin{aligned} g_1 + b_1 &= y_1 + x_1 \\ g_2 + b_2 &= y_2 + x_2 \\ &\vdots \\ g_n + b_n &= y_n + x_n \end{aligned}$$

and

$$g_1 + \cdots + g_n + b_1 + \cdots + b_n = y_1 + \cdots + y_n + x_1 + \cdots + x_n .$$

Because of the way in which $\boldsymbol{g}$ and $\boldsymbol{b}$ were chosen, this implies

$$g_1 + \cdots + g_n = y_1 + \cdots + y_n \quad \text{and} \quad b_1 + \cdots + b_n = x_1 + \cdots + x_n .$$

Therefore $\boldsymbol{g} > \boldsymbol{y}$ and $\boldsymbol{b} > \boldsymbol{x}$, and at least one of the n equations of (N) must fail to hold. We now have

$$\begin{aligned} \tau &\geqq \sigma - 1 + \rho , \\ Q(S) - \sigma &\geqq Q(S) - \tau - 1 + \rho , \\ Q(S) - \sigma &\geqq Q(S') - \tau + Q(S) - Q(S') - 1 + \rho . \end{aligned}$$

We recall that $Q(S) - Q(S') - 1 \geqq 0$, and that S' is a fundamental set. Hence

$$\begin{aligned} C(S) &\geqq A(S') + Q(S) - Q(S') - 1 + B(S) \\ &\geqq \alpha[Q(S') + 1] + \alpha[Q(S) - Q(S') - 1] + B(S) \\ &= \alpha Q(S) + B(S) . \end{aligned}$$

LEMMA 3. *If S is any set of type I then*

$$C(S) \geqq \alpha[Q(S) + 1] + B(S) .$$

Proof of Lemma 3. (i) Suppose $B(S) = 0$. Then

$$C(S) = A(S) \geqq \alpha[Q(S) + 1] + B(S) .$$

(ii) Suppose $B(S) \geqq 1$. Define $\boldsymbol{b}$ and $\boldsymbol{g}$ as in the proof of Lemma 2. Let $B(S) = \rho$, $Q(S - C) = \sigma$, $Q(S - A) = \tau$. Again the two sets $\{\boldsymbol{g} - \boldsymbol{x} \mid \boldsymbol{x} \in B \cap S\}$ and $\{\boldsymbol{y} - \boldsymbol{b} \mid \boldsymbol{y} \in S - C,\ \boldsymbol{y} \neq \boldsymbol{g}\}$ give $\sigma - 1 + \rho$ elements not in A, which now will be in S. Also $\boldsymbol{g}$ is in $S - C$, hence is in $S - A$, but is in neither of the two sets above. This implies that

$$\begin{aligned} \tau &\geqq \sigma + \rho , \\ Q(S) - \sigma &\geqq Q(S) - \tau + \rho , \\ C(S) &\geqq A(S) + B(S) \geqq \alpha[Q(S) + 1] + B(S) . \end{aligned}$$

We can now return to the proof of the theorem. Let R be any fundamental set satisfying the hypotheses of the theorem. We will use induction on the number of elements in $R - C$.

(i) Let $Q(R - C) = 1$. Then R is a set of type I, and we may apply Lemma 3.

(ii) Assume the the theorem holds for any fundamental set R' satisfying the hypotheses of the theorem and such that $Q(R' - C) < k$, $k \geqq 2$, and let $Q(R - C) = k$. If $B(R) = 0$ then R is of type I, so assume $B(R) \geqq 1$.

Let $\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_k$ be the k vectors in $R - C$, $T_j = \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{g}_j$ or $\boldsymbol{g}_j - \boldsymbol{x} \in Q\}$, $j = 1, \cdots, k$. If $\boldsymbol{b} \in T_j$ for all $j = 1, \cdots, k$ and all $\boldsymbol{b}$ in $B \cap R$ then again R is of type I, so assume (by re-numbering, if necessary) that $B(R - T_1) > 0$. Let J be the maximum j such that $B(R - (T_1 \cup \cdots \cup T_j)) > 0$. Then $\boldsymbol{b} \in B$ and $\boldsymbol{b} \in R - (T_1 \cup \cdots \cup T_J)$ implies $\boldsymbol{b} \in T_{J+1}$. We observe that $J < k$, since $\boldsymbol{b} \in R - (T_1 \cup \cdots \cup T_k)$ would imply that there does not exist $\boldsymbol{g}$ in $R - C$ such that $\boldsymbol{g} - \boldsymbol{b}$ is in Q, contrary to hypothesis. Also, $\boldsymbol{g}_{J+1} \notin T_1 \cup \cdots \cup T_J$.

Let $W_0 = T_1 \cup \cdots \cup T_J$. If $R - W_0$ is not of type II, there exists $\boldsymbol{b} \in B \cap T_{J+1}$ and a subscript i_1 such that $i_1 > J + 1$, $\boldsymbol{b} \notin T_{i_1}$. Let $W_1 = W_0 \cup T_{i_1}$. If $R - W_1$ is not of type II, we may repeat the above to form $W_2 = W_1 \cup T_{i_2}$, and so on. Eventually we must obtain a set W_m such that $R - W_m$ is of type II, $m \geqq 0$.

But W_m is a fundamental set satisfying the hypotheses of the theorem, and $Q(W_m - C) < k$ since $g_{J+1} \notin W_m$. Hence

$$C(W_m) \geqq \alpha[Q(W_m) + 1] + B(W_m) .$$

Also,

$$C(R - W_m) \geqq \alpha Q(R - W_m) + B(R - W_m) .$$

Adding, we obtain

$$C(R) \geqq \alpha[Q(R) + 1] + B(R) .$$

References

1. B. Kvarda, *On densities of sets of lattice points*, Pacific J. Math. **13** (1963), 611-615.
2. H. B. Mann, *On the number of integers in the sum of two sets of positive integers*, Pacific J. Math. **1** (1951), 249-253.

San Diego State College

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 15, No. 2 October, 1965