

# Pacific Journal of Mathematics

**AN INEQUALITY FOR THE NUMBER OF ELEMENTS IN A  
SUM OF TWO SETS OF LATTICE POINTS**

BETTY KVARDA

# AN INEQUALITY FOR THE NUMBER OF ELEMENTS IN A SUM OF TWO SETS OF LATTICE POINTS

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For a fixed positive integer  $n$ , let  $Q$  be the set of all  $n$ -dimensional lattice points  $(x_1, \dots, x_n)$  with each  $x_i$  a nonnegative integer and at least one  $x_i$  positive. A finite nonempty subset  $R$  of  $Q$  is called a *fundamental set* if for every  $(r_1, \dots, r_n)$  in  $R$ , all vectors  $(x_1, \dots, x_n)$  of  $Q$  with  $x_i \leq r_i$ ,  $i=1, \dots, n$ , are also in  $R$ . If  $A$  is any subset of  $Q$  and  $R$  is any fundamental set, let  $A(R)$  denote the number of vectors in  $A \cap R$ . Finally, if  $A$  is any proper subset of  $Q$ , let the *density* of  $A$  be the quantity

$$\alpha = \text{glb} \frac{A(R)}{Q(R) + 1},$$

taken over all fundamental sets  $R$  for which  $A(R) < Q(R)$ . Then the theorem proved in this paper can be stated as follows.

**THEOREM.** Let  $A$  and  $B$  be subsets of  $Q$ , let  $C$  be the set of all vectors of the form  $a$ ,  $b$ , or  $a + b$  where  $a \in A$  and  $b \in B$ , let  $\alpha$  be the density of  $A$ , and let  $R$  be any fundamental set such that (1) there exists at least one vector in  $R$  which is not in  $C$ , and (2) for each  $b$  in  $B \cap R$  (if any) there exists  $g$  in  $R$  but not in  $C$  such that  $g - b$  is in  $Q$ . Then

$$C(R) \geq \alpha[Q(R) + 1] + B(R).$$

It will be seen that for  $n = 1$  this theorem implies a result of H. B. Mann [2].

Let  $A$  and  $B$  be sets of positive integers, and for any positive integer  $x$  denote by  $A(x)$  the number of integers in  $A$  which are not greater than  $x$ . Let the *modified density* (or *Erdős density*) of  $A$  be the quantity

$$\alpha = \text{glb}_{x \geq k} \frac{A(x)}{x + 1}$$

where  $k$  is the smallest positive integer not in  $A$ . If  $C = A + B$  is the set of all integers of the form  $a$ ,  $b$ , or  $a + b$ , where  $a$  is in  $A$  and  $b$  is in  $B$ , and if  $x$  is a positive integer not in  $C$ , then Mann has shown [2] that

$$C(x) \geq \alpha x + B(x).$$

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(Actually, Mann's work is sufficient to establish  $C(x) \geq \alpha(x + 1) + B(x)$ .) We will show that this theorem, with somewhat weaker hypotheses, can be extended to certain sets of  $n$ -dimensional lattice points.

Let  $Q$  be the set of all lattice points  $\mathbf{x} = (x_1, \dots, x_n)$  for which each component is a nonnegative integer and at least one component is positive. Define the sum of subsets of  $Q$  in the same manner as was done for sets of positive integers, addition of lattice points being done componentwise, and for any subsets  $A$  and  $B$  of  $Q$  let  $A - B$  denote the set of all elements of  $A$  which are not in  $B$ . If  $A$  and  $S$  are subsets of  $Q$  and  $S$  is finite let  $A(S)$  be the number of elements in  $A \cap S$ . Let  $\omega_i$  be that element of  $Q$  for which the  $i$ th component is 1 and the others are 0.

DEFINITION 1. A finite nonempty subset  $R$  of  $Q$  will be called a *fundamental set* if whenever  $\mathbf{r} = (r_1, \dots, r_n)$  is in  $R$  then all vectors  $\mathbf{x} = (x_1, \dots, x_n)$  of  $Q$  such that  $x_i \leq r_i, i = 1, \dots, n$ , are also in  $R$ .

DEFINITION 2. Let  $A$  be any proper subset of  $Q$ . Then the *density* of  $A$  is the quantity

$$\alpha = \text{glb} \frac{A(R)}{Q(R) + 1},$$

taken over all fundamental sets  $R$  for which  $A(R) < Q(R)$ .

2. Extension of Mann's result. The theorem to be proved can now be stated as follows.

THEOREM. Let  $A$  and  $B$  be subsets of  $Q$ , let  $C = A + B$ , and let  $\alpha$  be the density of  $A$ . Let  $R$  be any fundamental set such that for each  $\mathbf{b}$  in  $B \cap R$  there exists  $\mathbf{g}$  in  $R - C$  such that  $\mathbf{g} - \mathbf{b}$  is in  $Q$ , and  $Q(R - C) \geq 1$ . Then

$$C(R) \geq \alpha[Q(R) + 1] + B(R).$$

*Proof.* Let the elements of  $Q$  be ordered so that  $(x_1, \dots, x_n) > (y_1, \dots, y_n)$  if  $x_1 > y_1$ , or if  $x_1 = y_1, \dots, x_k = y_k, x_{k+1} > y_{k+1}$ . Consider a nonempty set  $S = R' - R''$ , where  $R'$  and  $R''$  are fundamental sets, and let  $\delta_1 = (\delta_{11}, \dots, \delta_{1n}), \dots, \delta_u = (\delta_{u1}, \dots, \delta_{un})$  be all the vectors of  $S$  such that for each  $i = 1, \dots, n$  and for each  $j = 1, \dots, u$  we have either (1)  $\delta_j - \omega_i$  is in  $R''$ , or (2)  $\delta_j - \omega_i = \mathbf{0} = (0, \dots, 0)$ , or (3)  $\delta_{ji} = 0$ . There must be at least one such vector in  $S$ , for  $S$  is a nonempty finite set, and hence has a least element (in our ordering). This least element will satisfy the given conditions. Also, it is easily seen that if  $(s_1, \dots, s_n)$  is any vector in  $S$  then for at least one of the  $\delta_j$  we have  $\delta_{ji} \leq s_i, i = 1, \dots, n$ .

From this it follows that if for each  $j = 1, \dots, u$  we let

$$S_j = \{ \mathbf{s} = (s_1, \dots, s_n) \mid \mathbf{s} \in S, s_i \geq \delta_{ji}, i = 1, \dots, n \} ,$$

then  $S = S_1 \cup \dots \cup S_u$ . Also, let  $S'_j = \{ \mathbf{s} - \boldsymbol{\delta}_j \mid \mathbf{s} \in S_j, \mathbf{s} \neq \boldsymbol{\delta}_j \}$  and let  $S' = S'_1 \cup \dots \cup S'_u$ . Each  $S'_j$ , and therefore also  $S'$ , is either a fundamental set or is empty.

LEMMA 1.  $Q(S') + 1 \leq Q(S)$ .

*Proof of Lemma 1.* The lemma is obvious if  $n = 1$ , since then  $u = 1$  also. Hence assume  $n \geq 2$ . Let  $\lambda_1$  be a mapping defined so that

$$\begin{aligned} S_j \lambda_1 &= \{ \mathbf{s} - \delta_{j1} \boldsymbol{\omega}_1 \mid \mathbf{s} \in S_j \}, j = 1, \dots, u , \\ S \lambda_1 &= S_1 \lambda_1 \cup \dots \cup S_u \lambda_1 . \end{aligned}$$

Partition  $S$  into sets  $T_{c_2 \dots c_n}$  such that

$$T_{c_2 \dots c_n} = \{ \mathbf{s} = (x_1, c_2, \dots, c_n) \mid \mathbf{s} \in S \} ,$$

and let

$$\begin{aligned} T_{c_2 \dots c_n} \lambda_1 &= \{ \mathbf{s} = (x_1, c_2, \dots, c_n) \mid \mathbf{s} \in S \lambda_1 \} , \\ k_{c_2 \dots c_n} &= \max_{1 \leq j \leq u} (\max \{ x_1 - \delta_{j1} \mid (x_1, c_2, \dots, c_n) \in S_j \}) . \end{aligned}$$

Then  $Q(T_{c_2 \dots c_n} \lambda_1) = k_{c_2 \dots c_n} + 1$  or  $Q(T_{c_2 \dots c_n} \lambda_1) + 1 = k_{c_2 \dots c_n} + 1$ , according as  $\mathbf{0} \notin T_{c_2 \dots c_n} \lambda_1$  or  $\mathbf{0} \in T_{c_2 \dots c_n} \lambda_1$ , and  $k_{c_2 \dots c_n} + 1 \leq Q(T_{c_2 \dots c_n})$ .

Hence  $Q(S \lambda_1) \leq Q(S)$ , and  $Q(S \lambda_1) + 1 \leq Q(S)$  if  $\mathbf{0} \in S \lambda_1$ .

Now define mappings  $\lambda_2, \dots, \lambda_n$  such that

$$S_j \lambda_1 \dots \lambda_{i-1} \lambda_i = \{ \mathbf{s} - \delta_{ji} \boldsymbol{\omega}_i \mid \mathbf{s} \in S_j \lambda_1 \dots \lambda_{i-1} \} ,$$

$i = 2, \dots, n$ , and obtain as above

$$Q(S \lambda_1 \dots \lambda_i) + \theta_i \leq Q(S \lambda_1 \dots \lambda_{i-1}) + \theta_{i-1} \leq Q(S) ,$$

where  $\theta_1 = 0$  or 1 according as  $\mathbf{0} \notin S \lambda_1 \dots \lambda_i$  or  $\mathbf{0} \in S \lambda_1 \dots \lambda_i$ . This establishes the lemma.

DEFINITION 3. A set  $S$  will be said to be of type *I* if

- (1)  $S$  is a fundamental set,
- (2)  $Q(S - C) \geq 1$ , and
- (3) for all  $\mathbf{b}$  in  $B \cap S$  (if any) and all  $\mathbf{g}$  in  $S - C$  we have  $\mathbf{g} - \mathbf{b}$  contained in  $Q$ .

DEFINITION 4. A set  $S$  will be said to be of type *II* if

- (1) there exist fundamental sets  $R', R''$  such that  $S = R' - R''$ ,
- (2)  $B(S) \geq 1$  and  $Q(S - C) \geq 1$ , and
- (3) for all  $\mathbf{b}$  in  $B \cap S$  and  $\mathbf{g}$  in  $S - C$  we have  $\mathbf{g} - \mathbf{b}$  contained in  $Q$ .

LEMMA 2. *If  $S$  is any set of type II then*

$$C(S) \geq \alpha Q(S) + B(S).$$

*Proof of Lemma 2.* Define the sets  $S'_j$  and  $S'$  as above. Let  $\mathbf{b} = (b_1, \dots, b_n)$  be the largest vector such that

(1)  $\mathbf{b}$  is in  $B \cap S$ , and

(2)  $b_1 + \dots + b_n = \max \{x_1 + \dots + x_n \mid (x_1, \dots, x_n) \in B \cap S\}$ . Like-

wise, let  $\mathbf{g} = (g_1, \dots, g_n)$  be the largest vector such that

(1)  $\mathbf{g}$  is in  $S - C$ , and

(2)  $g_1 + \dots + g_n = \max \{y_1 + \dots + y_n \mid (y_1, \dots, y_n) \in S - C\}$ .

Let  $B(S) = \rho \geq 1$ ,  $Q(S - C) = \sigma \geq 1$ ,  $Q(S' - A) = \tau$ . The set  $\{\mathbf{g} - \mathbf{x} \mid \mathbf{x} \in B \cap S\}$  contains  $\rho$  elements of  $Q$  (Definition 4, part 3), none of which is in  $A$ . We show that these are in  $S'$ : If  $\mathbf{x} = (x_1, \dots, x_n)$  is in  $B \cap S$  then  $\mathbf{x}$  is in  $S_j$  for some  $j$  such that  $1 \leq j \leq u$ . Hence  $\delta_{ji} \leq x_i \leq g_i$  for all  $i = 1, \dots, n$ , and  $\mathbf{g}$  is in  $S_j$ .  $\mathbf{0} \neq \mathbf{g} - \mathbf{x} = (\mathbf{g} - \delta_j) - (\mathbf{x} - \delta_j)$ . But  $\mathbf{g} - \delta_j$  is in  $S'_j$  and  $S'_j$  is a fundamental set. Hence  $\mathbf{g} - \mathbf{x}$  is in  $S'_j$ , therefore in  $S'$ .

Likewise, the (possibly empty) set  $\{\mathbf{y} - \mathbf{b} \mid \mathbf{y} \in S - C, \mathbf{y} \neq \mathbf{g}\}$  contains  $\sigma - 1$  elements, all of which are in  $S' - A$ . We must show that the two sets are disjoint. Hence suppose that for some  $\mathbf{y} \neq \mathbf{g}$  and, therefore,  $\mathbf{x} \neq \mathbf{b}$ , we have

$$\mathbf{g} - \mathbf{x} = \mathbf{y} - \mathbf{b}.$$

Equating the  $i$ th components and transposing gives the  $n$  equations

$$(V) \quad \begin{aligned} g_1 + b_1 &= y_1 + x_1 \\ g_2 + b_2 &= y_2 + x_2 \\ &\vdots \\ g_n + b_n &= y_n + x_n \end{aligned}$$

and

$$g_1 + \dots + g_n + b_1 + \dots + b_n = y_1 + \dots + y_n + x_1 + \dots + x_n.$$

Because of the way in which  $\mathbf{g}$  and  $\mathbf{b}$  were chosen, this implies

$$g_1 + \dots + g_n = y_1 + \dots + y_n \quad \text{and} \quad b_1 + \dots + b_n = x_1 + \dots + x_n.$$

Therefore  $\mathbf{g} > \mathbf{y}$  and  $\mathbf{b} > \mathbf{x}$ , and at least one of the  $n$  equations of (N) must fail to hold. We now have

$$\begin{aligned} \tau &\geq \sigma - 1 + \rho, \\ Q(S) - \sigma &\geq Q(S) - \tau - 1 + \rho, \\ Q(S) - \sigma &\geq Q(S') - \tau + Q(S) - Q(S') - 1 + \rho. \end{aligned}$$

We recall that  $Q(S) - Q(S') - 1 \geq 0$ , and that  $S'$  is a fundamental set. Hence

$$\begin{aligned} C(S) &\geq A(S') + Q(S) - Q(S') - 1 + B(S) \\ &\geq \alpha[Q(S') + 1] + \alpha[Q(S) - Q(S') - 1] + B(S) \\ &= \alpha Q(S) + B(S). \end{aligned}$$

LEMMA 3. *If  $S$  is any set of type I then*

$$C(S) \geq \alpha[Q(S) + 1] + B(S).$$

*Proof of Lemma 3.* (i) Suppose  $B(S) = 0$ . Then

$$C(S) = A(S) \geq \alpha[Q(S) + 1] + B(S).$$

(ii) Suppose  $B(S) \geq 1$ . Define  $\mathbf{b}$  and  $\mathbf{g}$  as in the proof of Lemma 2. Let  $B(S) = \rho$ ,  $Q(S - C) = \sigma$ ,  $Q(S - A) = \tau$ . Again the two sets  $\{\mathbf{g} - \mathbf{x} \mid \mathbf{x} \in B \cap S\}$  and  $\{\mathbf{y} - \mathbf{b} \mid \mathbf{y} \in S - C, \mathbf{y} \neq \mathbf{g}\}$  give  $\sigma - 1 + \rho$  elements not in  $A$ , which now will be in  $S$ . Also  $\mathbf{g}$  is in  $S - C$ , hence is in  $S - A$ , but is in neither of the two sets above. This implies that

$$\begin{aligned} \tau &\geq \sigma + \rho, \\ Q(S) - \sigma &\geq Q(S) - \tau + \rho, \\ C(S) &\geq A(S) + B(S) \geq \alpha[Q(S) + 1] + B(S). \end{aligned}$$

We can now return to the proof of the theorem. Let  $R$  be any fundamental set satisfying the hypotheses of the theorem. We will use induction on the number of elements in  $R - C$ .

(i) Let  $Q(R - C) = 1$ . Then  $R$  is a set of type I, and we may apply Lemma 3.

(ii) Assume the the theorem holds for any fundamental set  $R'$  satisfying the hypotheses of the theorem and such that  $Q(R' - C) < k$ ,  $k \geq 2$ , and let  $Q(R - C) = k$ . If  $B(R) = 0$  then  $R$  is of type I, so assume  $B(R) \geq 1$ .

Let  $\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_k$  be the  $k$  vectors in  $R - C$ ,  $T_j = \{\mathbf{x} \mid \mathbf{x} = \mathbf{g}_j \text{ or } \mathbf{g}_j - \mathbf{x} \in Q\}$ ,  $j = 1, \dots, k$ . If  $\mathbf{b} \in T_j$  for all  $j = 1, \dots, k$  and all  $\mathbf{b}$  in  $B \cap R$  then again  $R$  is of type I, so assume (by re-numbering, if necessary) that  $B(R - T_1) > 0$ . Let  $J$  be the maximum  $j$  such that  $B(R - (T_1 \cup \dots \cup T_j)) > 0$ . Then  $\mathbf{b} \in B$  and  $\mathbf{b} \in R - (T_1 \cup \dots \cup T_j)$  implies  $\mathbf{b} \in T_{j+1}$ . We observe that  $J < k$ , since  $\mathbf{b} \in R - (T_1 \cup \dots \cup T_k)$  would imply that there does not exist  $\mathbf{g}$  in  $R - C$  such that  $\mathbf{g} - \mathbf{b}$  is in  $Q$ , contrary to hypothesis. Also,  $\mathbf{g}_{j+1} \notin T_1 \cup \dots \cup T_j$ .

Let  $W_0 = T_1 \cup \dots \cup T_j$ . If  $R - W_0$  is not of type II, there exists  $\mathbf{b} \in B \cap T_{j+1}$  and a subscript  $i_1$  such that  $i_1 > J + 1$ ,  $\mathbf{b} \notin T_{i_1}$ . Let  $W_1 = W_0 \cup T_{i_1}$ . If  $R - W_1$  is not of type II, we may repeat the above

to form  $W_2 = W_1 \cup T_{i_2}$ , and so on. Eventually we must obtain a set  $W_m$  such that  $R - W_m$  is of type II,  $m \geq 0$ .

But  $W_m$  is a fundamental set satisfying the hypotheses of the theorem, and  $Q(W_m - C) < k$  since  $g_{j+1} \notin W_m$ . Hence

$$C(W_m) \geq \alpha[Q(W_m) + 1] + B(W_m).$$

Also,

$$C(R - W_m) \geq \alpha Q(R - W_m) + B(R - W_m).$$

Adding, we obtain

$$C(R) \geq \alpha[Q(R) + 1] + B(R).$$

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