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Let T be a spectral operator on a Banach space, such that its resolvent satisfies a mth order rate of growth condition. If N be the nilpotent part of T, it is known that $N^m = 0$ on Hilbert space. We show that $N^m = 0$ on an L_p space $(1 < p < \infty)$. Known examples show that N^m need not be zero even on an uniformly convex space.

We will consider a bounded spectral operator $T = \int \lambda E(d\lambda) + N$ which operates on an L_p space $(1 < p < \infty)$. $E(\circ)$ is the resolution of the identity and N is the nilpotent part of T [1; pp. 333–334]. We will denote by M a finite constant for which $M^{-1} \mathrm{ess}_\xi \cdot \inf \cdot |a(\xi)| \leqq \left| \int a(\xi) E(d\xi) \right| \leqq M \mathrm{ess}_\xi \cdot \sup \cdot |a(\xi)|$ is true for all bounded Borel functions $a(\xi)$, [1; Theorem 7, p. 330].

Suppose that T satisfies an mth order rate of growth condition on its resolvent: given any Borel subset σ of the spectrum of T, its restriction T_σ to the range of $E(\sigma)$ has $\bar{\sigma}$ as spectrum and we assume that for $|\zeta| \leqq |T| + 1$,

$$|(\zeta - T_\sigma)^{-1}| \leqq K[\text{distance } (\zeta, \sigma)]^{-m}$$

where K and m are constants independent of σ and ζ.

It is known that in Hilbert space, this implies $N^m = 0$ [1; Theorem 11, p. 337], and that in a reflexive Banach space $N^{m+1} = 0$, but in general no more [2; Theorem 3.1, p. 1226; Examples 4.4, p. 1230]. However, in the case of a reflexive L_p space, we will show that in fact $N^m = 0$. It is immaterial whether we show $N^m = 0$ or $N^{*m} = 0$, so that we may assume that $p \geqq 2$. We will dispense with the continual remarks that our L_p functions $x(s)$ are defined for only almost every s.

It is known that for any complex numbers $\lambda_1, \cdots, \lambda_n$ and $p \geqq 2$ we have

$$(1) \quad \begin{aligned} \left(\sum_{\nu=1}^n |\lambda_\nu|^2\right)^{p/2} &\leqq (2\pi)^{-n} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \, |e^{i\theta_1}\lambda_1 + \cdots + e^{i\theta_n}\lambda_n|^p \\ &\leqq C(p) \left(\sum_{\nu=0}^n |\lambda_\nu|^2\right)^{p/2} \end{aligned}$$

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where $C(p)$ is independent of n and the choice of the λ's [3; Proposition 1].

Given $\varepsilon > 0$, let the spectrum of T be decomposed into $n = O(\varepsilon^{-2})$ Borel subsets $\sigma_1, \cdots, \sigma_n$ with each σ_ν contained in the disc $|\zeta - \zeta_\nu| \leqq \varepsilon$, and let $E_\nu = E(\sigma_\nu)$. For a given function $x(s)$ in L_p, let $\lambda_\nu(s) = (E_\nu x)(s)$. For each s, apply (1) to these $\lambda_\nu(s)$ and then integrate over all s:

$$\begin{aligned}(2)\qquad &\int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x(s)]|^2\Big)^{p/2}\\ &\leqq (2\pi)^{-n}\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \int ds\,|[(e^{i\theta_1}E_1 + \cdots + e^{i\theta_n}E_n)x](s)|^p\\ &\leqq C(p)\int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2} .\end{aligned}$$

For each choice of θ_ν we have (since $\Sigma E_\nu = I$)

$$M^{-1}|x| \leqq |(e^{i\theta_1}E_1 + \cdots + e^{i\theta_n}E_n)x| \leqq M|x| ,$$

so that upon performing the integrations in the middle of (2) we have

$$\text{(3a)}\qquad \int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2} \leqq M^p|x|^p$$

and

$$\text{(3b)}\qquad M^{-p}|x|^p \leqq C(p)\int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2} .$$

Now in (3b), replace x by $N^m x$ and apply the Holder inequality to the sum on the right hand side to obtain

$$\begin{aligned}(4)\qquad |N^m x|^p &\leqq C(p)M^p\int ds \sum_{\nu=1}^{n} |[E_\nu N^m x](s)|^p \cdot n^{(p/2)-1}\\ &= C(p)M^p n^{(p/2)-1}\sum_{\nu=1}^{n} |N^m E_\nu x|^p .\end{aligned}$$

It is a standard computation that

$$|N^m E_\nu x| \leqq 2\cdot 3^m KM\varepsilon\,|E_\nu x| .$$

For completeness, we digress for a moment to include a proof: Let Γ $(=\Gamma_\nu)$ be the contour $|\zeta - \zeta_\nu| = 2\varepsilon$, so that any point of Γ is at least ε away from σ_ν, but no point of σ_ν is further than 3ε from any point in Γ. Then we have

$$N^m E_\nu = \frac{1}{2\pi i}\int_\Gamma d\zeta(\zeta - T_{\sigma_\nu})^{-1}\int_{\sigma_\nu}(\zeta - \xi)^m E(d\xi)$$

and thus

$$
\begin{aligned}
| N^m E_\nu | &\leqq \frac{1}{2\pi} \int_\Gamma | d\zeta | K\varepsilon^{-m} M(3\varepsilon)^m \\
&= 2 \cdot 3^m KM\varepsilon \ .
\end{aligned}
$$

We now insert this estimate in (4) to obtain (with lumping all inessential constants together)

$$
\begin{aligned}
| N^m x |^p &\leqq C(p) M^p n^{p/2-1} \sum_{\nu=1}^{n} (3^m KM\varepsilon)^p \, | E_\nu x |^p \\
&= C n^{p/2-1} \varepsilon^p \int ds \sum_{\nu=1}^{n} | [E_\nu x](s) |^p && \text{(since } p \geqq 2) \\
&\leqq C n^{p/2-1} \varepsilon^p \int ds \Big(\sum_{\nu=1}^{n} | [E_\nu x](s) |^2 \Big)^{p/2} && \text{(by 3a)} \\
&\leqq C n^{p/2-1} \varepsilon^p \cdot M^p \, | x |^p \ .
\end{aligned}
$$

Now we need only remember that $n = O(\varepsilon^{-2})$ to see that

$$
| N^m x |^p = O(\varepsilon^2) \, | x |^p \ .
$$

Since ε may be arbitrarily small, $N^m x = 0$ for all x, so $N^m = 0$ as was to be proved.

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