

Pacific Journal of Mathematics

THE NILPOTENT PART OF A SPECTRAL OPERATOR. II

CHARLES ALAN MCCARTHY

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Let T be a spectral operator on a Banach space, such that its resolvent satisfies a m th order rate of growth condition. If N be the nilpotent part of T , it is known that $N^m = 0$ on Hilbert space. We show that $N^m = 0$ on an L_p space ($1 < p < \infty$). Known examples show that N^m need not be zero even on a uniformly convex space.

We will consider a bounded spectral operator $T = \int \lambda E(d\lambda) + N$ which operates on an L_p space ($1 < p < \infty$). $E(\circ)$ is the resolution of the identity and N is the nilpotent part of T [1; pp. 333-334]. We will denote by M a finite constant for which $M^{-1} \operatorname{ess}_\varepsilon \cdot \inf \cdot |a(\xi)| \leq \left| \int a(\xi) E(d\xi) \right| \leq M \operatorname{ess}_\varepsilon \cdot \sup \cdot |a(\xi)|$ is true for all bounded Borel functions $a(\xi)$, [1; Theorem 7, p. 330].

Suppose that T satisfies an m th order rate of growth condition on its resolvent: given any Borel subset σ of the spectrum of T , its restriction T_σ to the range of $E(\sigma)$ has $\bar{\sigma}$ as spectrum and we assume that for $|\zeta| \leq |T| + 1$,

$$|(\zeta - T_\sigma)^{-1}| \leq K[\text{distance}(\zeta, \sigma)]^{-m}$$

where K and m are constants independent of σ and ζ .

It is known that in Hilbert space, this implies $N^m = 0$ [1; Theorem 11, p. 337], and that in a reflexive Banach space $N^{m+1} = 0$, but in general no more [2; Theorem 3.1, p. 1226; Examples 4.4, p. 1230]. However, in the case of a reflexive L_p space, we will show that in fact $N^m = 0$. It is immaterial whether we show $N^m = 0$ or $N^{*m} = 0$, so that we may assume that $p \geq 2$. We will dispense with the continual remarks that our L_p functions $x(s)$ are defined for only almost every s .

It is known that for any complex numbers $\lambda_1, \dots, \lambda_n$ and $p \geq 2$ we have

$$(1) \quad \left(\sum_{\nu=1}^n |\lambda_\nu|^2 \right)^{p/2} \leq (2\pi)^{-n} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n |e^{i\theta_1 \lambda_1} + \cdots + e^{i\theta_n \lambda_n}|^p \\ \leq C(p) \left(\sum_{\nu=0}^n |\lambda_\nu|^2 \right)^{p/2}$$

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where $C(p)$ is independent of n and the choice of the λ 's [3; Proposition 1].

Given $\varepsilon > 0$, let the spectrum of T be decomposed into $n = O(\varepsilon^{-2})$ Borel subsets $\sigma_1, \dots, \sigma_n$ with each σ_ν contained in the disc $|\zeta - \zeta_\nu| \leq \varepsilon$, and let $E_\nu = E(\sigma_\nu)$. For a given function $x(s)$ in L_p , let $\lambda_\nu(s) = (E_\nu x)(s)$. For each s , apply (1) to these $\lambda_\nu(s)$ and then integrate over all s :

$$\begin{aligned}
 & \int ds \left(\sum_{\nu=1}^n |[E_\nu x(s)]|^2 \right)^{p/2} \\
 (2) \quad & \leq (2\pi)^{-n} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \int ds \left| [(e^{i\theta_1} E_1 + \cdots + e^{i\theta_n} E_n)x](s) \right|^p \\
 & \leq C(p) \int ds \left(\sum_{\nu=1}^n |[E_\nu x](s)|^2 \right)^{p/2}.
 \end{aligned}$$

For each choice of θ_ν we have (since $\Sigma E_\nu = I$)

$$M^{-1} |x| \leq |(e^{i\theta_1} E_1 + \cdots + e^{i\theta_n} E_n)x| \leq M |x|,$$

so that upon performing the integrations in the middle of (2) we have

$$(3a) \quad \int ds \left(\sum_{\nu=1}^n |[E_\nu x](s)|^2 \right)^{p/2} \leq M^p |x|^p$$

and

$$(3b) \quad M^{-p} |x|^p \leq C(p) \int ds \left(\sum_{\nu=1}^n |[E_\nu x](s)|^2 \right)^{p/2}.$$

Now in (3b), replace x by $N^m x$ and apply the Holder inequality to the sum on the right hand side to obtain

$$\begin{aligned}
 (4) \quad & |N^m x|^p \leq C(p) M^p \int ds \sum_{\nu=1}^n |[E_\nu N^m x](s)|^p \cdot n^{(p/2)-1} \\
 & = C(p) M^p n^{(p/2)-1} \sum_{\nu=1}^n |N^m E_\nu x|^p.
 \end{aligned}$$

It is a standard computation that

$$|N^m E_\nu x| \leq 2 \cdot 3^m K M \varepsilon |E_\nu x|.$$

For completeness, we digress for a moment to include a proof: Let $\Gamma (= \Gamma_\nu)$ be the contour $|\zeta - \zeta_\nu| = 2\varepsilon$, so that any point of Γ is at least ε away from σ_ν , but no point of σ_ν is further than 3ε from any point in Γ . Then we have

$$N^m E_\nu = \frac{1}{2\pi i} \int_\Gamma d\zeta (\zeta - T_{\sigma_\nu})^{-1} \int_{\sigma_\nu} (\zeta - \xi)^m E(d\xi)$$

and thus

$$\begin{aligned}
 |N^m E_\nu| &\leq \frac{1}{2\pi} \int_r |d\zeta| K\varepsilon^{-m} M(3\varepsilon)^m \\
 &= 2 \cdot 3^m KM\varepsilon .
 \end{aligned}$$

We now insert this estimate in (4) to obtain (with lumping all inessential constants together)

$$\begin{aligned}
 |N^m x|^p &\leq C(p)M^p n^{p/2-1} \sum_{\nu=1}^n (3^m KM\varepsilon)^p |E_\nu x|^p \\
 &= Cn^{p/2-1}\varepsilon^p \int ds \sum_{\nu=1}^n |[E_\nu x](s)|^p \quad (\text{since } p \geq 2) \\
 &\leq Cn^{p/2-1}\varepsilon^p \int ds \left(\sum_{\nu=1}^n |[E_\nu x](s)|^2 \right)^{p/2} \quad (\text{by 3a}) \\
 &\leq Cn^{p/2-1}\varepsilon^p \cdot M^p |x|^p .
 \end{aligned}$$

Now we need only remember that $n = O(\varepsilon^{-2})$ to see that

$$|N^m x|^p = O(\varepsilon^2) |x|^p .$$

Since ε may be arbitrarily small, $N^m x = 0$ for all x , so $N^m = 0$ as was to be proved.

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