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MONOTONE APPROXIMATION

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How close can one approximate a monotone function by a monotone polynomial of degree $\leqq n$, or a convex function by a convex polynomial of degree $\leqq n$? This leads to the following general question. Let k and n be given, and suppose a real fuction f satisfies $f^{(k)}(x) \geqq 0$ throughout a closed, finite interval $[a, b]$. How close can one approximate f on $[a, b]$ by a polynomial of degree $\leqq n$ whose kth derivative, too, is $\geqq 0$ there? We give an answer to the question.

2. THEOREM 1. *Let k and p be integers, $1 \leqq k \leqq p$, and let a real function f satisfy throughout $[a, b]$*

$$f^{(k)}(x) \geqq 0 ,$$
$$|f^{(p)}(x_2) - f^{(p)}(x_1)| \leqq \lambda |x_2 - x_1| ,$$

λ being a constant. Then for every integer $n(\geqq p)$ there exists a real polynomial $Q_n(x)$ of degree[1] *$\leqq n$ such that*

(a) *$Q_n^{(k)}(x) \geqq 0$ throughout $[a, b]$,*

(b) $\displaystyle \operatorname*{Max}_{a \leqq x \leqq b} |f(x) - Q_n(x)| \leqq 2\lambda\left(\frac{\pi}{4}\right)^{p-k+1}(b-a)^{p+1}\left[k! \prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1}.$

3. To prove Theorem 1, we begin by quoting the following result of J. Favard [2] and N. Ahiezer and M. Krein [1] which strengthens a previous result of D. Jackson.

THEOREM 2. (*Favard, Ahiezer-Krein*) *Let f (with period 2π) map the reals into the reals, and satisfy for every real x_1, x_2*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1| , \tag{1}$$

λ being a constant. Then for $n = 0, 1, 2, \cdots$, there exists a trigonometric polynomial $T_n(x) \equiv \sum_{\nu=0}^{n} a_\nu^{(n)} \cos \nu x + b_\nu^{(n)} \sin \nu x$ such that $\max_{0 \leqq x \leqq 2\pi} |f(x) - T_n(x)| \leq \lambda(\pi/2)[1/(n+1)]$.

From Theorem 2 one obtains by the method of [3], pp. 13–14 the following

THEOREM 3. *Let f be a real function satisfying* (1) *throughout $[a, b]$, λ being a constant. Then for $n = 0, 1, 2, \cdots$, there exists a*

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[1] By degree of a polynomial we mean its exact degree. (The degree of the polynomial 0 is -1).

polynomial $P_n(x)$ *of degree* $\leqq n$ *such that*

$$\max_{a\leqq x\leqq b}|f(x)-P_n(x)|\leqq \lambda\frac{\pi}{4}\frac{b-a}{n+1}\,.$$

For future use, we make the following simple observation. (Compare [3], p. 16).

LEMMA. *Let* f *be a real function, continuous in* $[a, b]$ *and differentiable in* (a, b). *Let* n *be an integer* ($\geqq 0$), $q_{n-1}(x)$ *a real polynomial of degree* $\leqq n-1$, *and let* ε *be such that* $|f'(x)-q_{n-1}(x)| \leqq \varepsilon$ *throughout* (a, b). *Then there exists a polynomial* $P_n(x)$ *of degree* $\leqq n$ *such that*

$$\max_{a\leqq x\leqq b}|f(x)-P_n(x)|\leqq \varepsilon\frac{\pi}{4}\frac{b-a}{n+1}\,. \tag{2}$$

To prove the lemma, set $r(x)\equiv f(x)-\int_a^x q_{n-1}(t)dt$. Throughout (a, b), $|r'(x)|\leqq\varepsilon$, and therefore, throughout $[a, b]$, $|r(x_2)-r(x_1)|\leqq \varepsilon|x_2-x_1|$. By Theorem 3, there exists a polynomial $\pi_n(x)$ of degree $\leqq n$ such that $\max_{a\leqq x\leqq b}|r(x)-\pi_n(x)|\leqq\varepsilon(\pi/4)(b-a)/(n+1)$. Setting $P_n(x)\equiv \pi_n(x)+\int_a^x q_{n-1}(t)dt$, we obtain (2).

From Theorem 3 and the Lemma one gets readily (cf. [3], pp. 16–17) the following

THEOREM 4. *Let* f *be a real function satisfying throughout* $[a, b]$, *for some constant integer* $p(\geqq 0)$ *and some constant* λ,

$$|f^{(p)}(x_2)-f^{(p)}(x_1)|\leqq\lambda|x_2-x_1|\,.$$

Then for every integer $n(\geqq p)$ *there exists a polynomial* $P_n(x)$ *of degree* $\leqq n$ *such that*

$$\max_{a\leqq x\leqq b}|f(x)-P_n(x)|\leqq\lambda\left[\frac{\pi}{4}(b-a)\right]^{p+1}\left[\prod_{\nu=0}^{p}(n+1-\nu)\right]^{-1}.$$

3. *Proof of Theorem* 1. Let n be an integer $\geqq p$. Set $f_n(x)\equiv f^{(k)}(x)+\lambda[(\pi/4)(b-a)]^{p-k+1}[\prod_{\nu=k}^{p}(n+1-\nu)]^{-1}$. Then throughout $[a, b]$, $|f_n^{(p-k)}(x_2)-f_n^{(p-k)}(x_1)|\leqq\lambda|x_2-x_1|$. By Theorem 4, there exists a real polynomial $P_{n-k}(x)$ of degree $\leqq n-k$ such that

$$\max_{a\leqq x\leqq b}|f_n(x)-P_{n-k}(x)|\leqq\lambda\left[\frac{\pi}{4}(b-a)\right]^{p-k+1}\left[\prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1}.$$

So, throughout $[a, b]$, $P_{n-k}(x)\geqq f^{(k)}(x)\geqq 0$. Let

$$Q_n(x) \equiv \left[\sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(a)}{\nu!}(x-a)^\nu\right] + \int_a^{t_{k+1}} \int_a^{t_k} \cdots \int_a^{t_2} P_{n-k}(t_1) dt_1 dt_2 \cdots dt_k$$

(t_{k+1} being here and below, x). Then $Q_n(x)$ is a real polynomial of degree $\leqq n$, and $Q_n^{(k)}(x) = P_{n-k}(x) \geqq 0$ throughout $[a, b]$. Furthermore, throughout that interval, we have

$$f(x) = \left[\sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(a)}{\nu!}(x-a)^\nu\right] + \int_a^{t_{k+1}} \int_a^{t_k} \cdots \int_a^{t_2} f^{(k)}(t_1) dt_1 \cdots dt_k ,$$

and therefore

$$\begin{aligned} |f(x) - Q_n(x)| &\leqq \int_a^{t_{k+1}} \int_a^{t_k} \cdots \int_a^{t_2} |f^{(k)}(t_1) - P_{n-k}(t_1)|\, dt_1 \cdots dt_k \\ &\leqq 2\lambda\left[\frac{\pi}{4}(b-a)\right]^{p-k+1}\left[\prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1}\frac{(x-a)^k}{k!} \\ &\leqq 2\lambda\left(\frac{\pi}{4}\right)^{p-k+1}(b-a)^{p+1}\left[k!\prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1} . \end{aligned}$$

4. The following Theorem 5 deals with a somewhat more general situation than that of Theorem 1.

THEOREM 5. *Let k and p be integers, $1 \leqq k \leqq p$, and let a real function f satisfy throughout $[a, b]$*

$$f^{(k)}(x) \geqq 0 ,$$
$$|f^{(p)}(x)| \leqq M ,$$

M being a constant. Let $\omega(x)$ be the modulus of continuity of $f^{(p)}$ in $[a, b]$. Then for every integer $n\,(\geqq p)$ there exists a real polynomial $Q_n(x)$ of degree $\leqq n$ such that

(a) $$Q_n^{(k)}(x) \geqq 0 \quad \textit{throughout } [a, b] ,$$

(b) $$\max_{a \leqq x \leqq b} |f(x) - Q_n(x)| \leqq 2\left(1+\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right)^{p-k}(b-a)^p\left[k!\prod_{\nu=k}^{p-1}(n+1-\nu)\right]^{-1}\omega\left(\frac{b-a}{n-p+1}\right)$$

(an "empty" product means always 1).

Theorem 5 is proved by means of the following Theorem 6, in the same way that Theorem 1 was proved by means of Theorem 4.

THEOREM 6. *Let f be a real function having a bounded* pth *($p \geqq 0$) derivative throughout $[a, b]$. Let $\omega(x)$ be as in Theorem 5. Then for every integer n ($\geqq p$) there exists a polynomial $P_n(x)$ of degree $\leqq n$ such that throughout $[a, b]$*

$$|f(x) - P_n(x)| \leqq \left(1 + \frac{\pi}{4}\right)\left[\frac{\pi}{4}(b-a)\right]^p\left[\prod_{\nu=0}^{p-1}(n+1-\nu)\right]^{-1}\omega\left(\frac{b-a}{n-p+1}\right).$$

5. Theorem 6 follows from Theorem 3 by Jackson's method ([3], pp. 15–18). For the reader's convenience we hereby prove Theorem 6 in full. We do it by induction on p. Suppose first $p = 0$. Let n be an integer ($\geqq 0$). Let $\phi(x)$ be the function whose graph is obtained by joining successively the points $(\xi_\nu, f(\xi_\nu))$ ($\nu = 0, 1, \cdots, n+1$) of the x, y plane, where $\xi_\nu = a + [(b-a)/(n+1)]\nu$. For $\nu = 1, 2, \cdots, n+1$ we have $|\phi(\xi_\nu) - \phi(\xi_{\nu-1})| \leqq \omega[(b-a)/(n+1)]$. Hence, if $a \leqq x_1 < x_2 \leqq b$, then

$$\frac{|\phi(x_2) - \phi(x_1)|}{x_2 - x_1} \leqq \frac{n+1}{b-a}\omega\left(\frac{b-a}{n+1}\right).$$

By Theorem 3, there exists a polynomial $P_n(x)$ of degree $\leqq n$ such that throughout $[a, b]$

$$|\phi(x) - P_n(x)| \leqq \frac{n+1}{b-a}\omega\left(\frac{b-a}{n+1}\right)\frac{\pi}{4}\frac{b-a}{n+1} = \frac{\pi}{4}\omega\left(\frac{b-a}{n+1}\right).$$

Clearly, for every $x \in [a, b]$, $|f(x) - \phi(x)| \leqq \omega[(b-a)/(n+1)]$. Therefore, throughout $[a, b]$, $|f(x) - P_n(x)| \leqq [1 + (\pi/4)]\omega[(b-a)/(n+1)]$. This proves Theorem 6 when $p = 0$. Suppose the theorem was proved for some $p - 1$ ($\geqq 0$). We shall prove it for p. Let n be an integer ($\geqq p$). By our hypothesis there exists a polynomial $P_{n-1}(x)$ of degree $\leqq n - 1$ such that throughout $[a, b]$

$$|f'(x) - P_{n-1}(x)| \leqq \left(1 + \frac{\pi}{4}\right)\left[\frac{\pi}{4}(b-a)\right]^{p-1}\left[\prod_{\nu=1}^{p-1}(n+1-\nu)\right]^{-1}\omega\left(\frac{b-a}{n-p+1}\right).$$

By the lemma, there exists a polynomial $P_n(x)$ of degree $\leqq n$, such that

$$\max_{a\leqq x\leqq b}|f(x) - P_n(x)| \leqq \left(1 + \frac{\pi}{4}\right)\left[\frac{\pi}{4}(\mathrm{b}-a)\right]^p\left[\prod_{\nu=0}^{p-1}(n+1-\nu)\right]^{-1}w\left(\frac{b-a}{n-p+1}\right).$$

This completes the proof.

References

1. N. Ahiezer and M. Krein, *On the best approximation of differentiable periodic functions by trigonometric sums* (Russian), Doklady Akad. Nauk SSSR, XV (1937), 107–112.

2. J. Favard, *Sur les meilleurs procédés d'approximation de certaines classes de fonctions par des polynomes trigonométriques,* Bull. des Sciences Math. **LXI** (1937), 209-224, 243-256, in particular pp. 219, 220, and 223.
3. D. Jackson, *The theory of approximation,* Amer. Math. Soc. Colloq. Publ. Vol. XI, New York, 1930.

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