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# SOME GENERAL PROPERTIES OF MULTI-VALUED FUNCTIONS

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**The object is to determine what theorems for single-valued functions can be extended to which class of multi-valued functions. It is shown that an arc cannot be mapped onto a circle by a continuous, monotone multi-valued function when the image of each point is an arc. On the other hand, the arc can be mapped onto a nonlocally connected space by a monotone, continuous function such that the image of each point is an arc. Characterizations of nonalternating functions analogous to the results in the single-valued theory are obtained, and it is shown that an nonalternating semi-single-valued continuous function on a dendrite is monotone. An analog of the monotone light factorization theorem is obtained for semi-single-valued continuous functions.**

**Some other results are: an open continuous function with finite images maps a regular curve onto a regular curve, and a continuous function with finite images maps a locally connected, compact space onto a locally connected compact space.**

A number of definitions for continuity have been proposed for multi-valued or set-valued functions, and Wayman Strother studied the problem of continuity extensively [10, 11, 12]. Also Choquet [2] has studied upper and lower semi-continuous functions. Further, Berge, unlike most authors, allows functions to be multi-valued in [1]. However, much of the work that has been done on set-valued functions has been devoted to the discovery of fixed point theorems ([3], [7] through [9], [11], [13], and [15] through [17]). The purpose of this paper is to investigate properties of multi-valued functions which are similar to the properties of single-valued functions studied in G. T. Whyburn's book, Analytic Topology, [18].

We shall use the following topology on the set of closed subsets of a space $Y$. Let

$$S(Y) = \{E \subset Y : E \text{ is closed and nonempty}\}.$$

Let $S(Y)$ have the topology used by Michael [6]; i.e., if $V_1, \cdots, V_n$ are open subsets of $Y$, then the collection $\langle V_1, \cdots, V_n \rangle = \{E \in S(Y): E \cap V_i \neq \phi$ for all $i$, and $E \subset \bigcup_{i=1}^n V_i\}$ is a basis for the open sets of $S(Y)$. We shall call this topology the finite topology. This is equivalent to

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the topology used by Strother [3] and Frink [4]. Since we shall be dealing extensively with subspaces of $S(Y)$ we shall use $\langle V_1, \cdots, V_n\rangle$ to be either a basic open set in $S(Y)$ or a basic, relatively open set in the appropriate subspace of $S(Y)$. If $\mathscr{V} = \{V_1, \cdots, V_n\}$, then set $\langle\mathscr{V}\rangle = \langle V_1, \cdots, V_n\rangle$.

In this paper we shall assume that all spaces are Hausdorff.

Given a set-valued function $F\colon X \to Y$ with $F(x)$ closed and nonempty we define the induced function $f$ on $X$ into $S(Y)$ by setting $f(x) = F(x)$ for each $x \in X$. Note that $f$ is single-valued, and $f$ will always denote the function induced by $F$ unless otherwise stated. Also we shall always use upper case letters to denote multi-valued functions.

If $A$ is a subset of $X$, then the symbols $\bar{A}$ and $Cl(A)$ are used to denote the closure of $A$, and the symbol $A^\circ$ is used to denote the interior of $A$.

Henceforth, we assume that $S(Y)$ has the finite topology, and that $F\colon X \to Y$ is, unless otherwise stated, a function such that $F(x)$ is in $S(Y)$ for each $x$ in $X$.

**1. Preliminaries.** This section will be devoted mainly to gathering known results that are needed in the development of succeeding sections.

DEFINITION. A multi-valued function $F\colon X \to Y$ is called continuous in case the induced function $f\colon X \to S(Y)$ is continuous.

NOTATION. If $A \subset X$, then $F(A) = \cup\{F(x)\colon x \in A\}$.

Now we have the following lemmas due to Strother [10].

LEMMA 1.1. *A function $F\colon X \to Y$ is continuous if and only if statements* (1) *and* (2) *hold.*

1. *If $x_0 \in X$, $V$ is open in $Y$, and if $F(x_0) \cap V \neq \phi$, then there exists an open set $U$ of $X$ with $x_0 \in U$ such that $F(x) \cap V \neq \phi$ for all $x \in U$.*

2. *If $x_0 \in X$ and $F(x_0) \subset V$ where $V$ is open in $Y$, then there exists an open set $U$ containing $x_0$ such that $F(U) \subset V$.*

LEMMA 1.2. *Let $Y$ be regular. If $F\colon X \to Y$ is continuous, if $\{x_d\}$ is a net in $X$ converging to $x_0$, and if $y_d \in F(x_d)$ such that $\{y_d\}$ converges to $y_0$, then $y_0 \in F(x_0)$.*

LEMMA 1.3. *Let $F\colon X \to Y$ be continuous, and let $X$ and $Y$ be compact. Then $F$ is closed; i.e., $F(A)$ is closed in $Y$ whenever $A$ is closed in $X$.*

We also need the following lemma from Michael [6].

LEMMA 1.4. *If $\mathscr{B}$ is a collection of subsets of $Y$ which is disjoint (a subcollection of $S(Y)$) and connected in the factor (finite) topology and all (one) of whose elements are (is) connected, then* $\cup\{E: E \in \mathscr{B}\}$ *is connected.*

Set $\mathscr{F}(Y) = \{E \in S(Y); E \text{ finite}\}$, $\mathscr{C}(Y) = \{E \in S(Y): E \text{ is compact}\}$ and $\mathscr{F}_n(Y) = \{E: E \text{ has at most } n \text{ elements}\}$.

Now we can apply the above results to obtain some further lemmas. Lemma 1.5 is a variation of a theorem in Berge [1].

LEMMA 1.5. *Let $F$ be continuous and onto, and let $X$ be compact. Then $Y$ is compact if and only if $F(x)$ is compact for each $x \in X$.*

*Proof.* Suppose $Y$ is compact; then $F(x)$ closed implies $F(x)$ compact. Suppose that $F(x)$ is compact for each $x$, and let $\mathscr{V}$ be an open cover of $Y$. Then for each $x$ we obtain a subcover $\mathscr{V}_x$ of $F(x)$, such that $F(x) \cap V \neq \phi$ for all $V \in \mathscr{V}_x$. Since $F(x)$ is compact, there is a finite subcover $\mathscr{V}'_x$ of $F(x)$ in $\mathscr{V}_x$, and $F(x) \in \langle \mathscr{V}'_x \rangle$. The collection $\{\langle \mathscr{V}'_x \rangle: x \in X\}$ is an open cover for $f(X)$ in $S(Y)$. Since $f(X)$ is compact, there is a finite subcover, say $\langle \mathscr{V}'_1 \rangle, \cdots, \langle \mathscr{V}'_n \rangle$ of $f(X)$; hence the collection $\mathscr{V}_0 = \bigcup_{i=1}^n \mathscr{V}'_i$ is a finite subcover of $Y$ and $\mathscr{V}_0 \subset \mathscr{V}$.

LEMMA 1.6. *Let $F$ be continuous and $A$ a connected subset of $X$. Then, if $F(x)$ is connected for some $x \in A$, $F(A)$ is a connected subset of $Y$.*

*Proof.* Since $F$ is continuous, $f(A)$ is connected in $S(Y)$, and for some $x$, $F(x) \in f(A)$ is connected. So by Lemma 1.4, $F(A) = \cup \{F(x): x \in A\}$ is connected.

COROLLARY 1.7. *If $F$ is continuous, if $X$ is connected, and if there is an $x \in X$ such that $F(x)$ is connected, then $F(X)$ is connected. Hence $Y$ is connected if $F$ is onto.*

COROLLARY 1.8. *Let $F$ be continuous. Then $F(A)$ is connected for every connected subset $A$ of $X$ if and only if $F(x)$ is connected for each $x \in X$.*

*Proof.* Since $\{x\}$ is connected, $F(x)$ must be connected by hypothesis. On the other hand, if $A \neq \phi$, then for any $x \in A$, $F(x)$ is connected. So Lemma 1.6 applies.

Another result from Michael's paper [6] we need is the following.

LEMMA 1.9. *If $A \subset Y$ is closed, the following hold.*
1. $\{E \in S(Y): E \subset A\}$ *is closed.*
2. $\{E \in S(Y): E \cap A \neq \phi\}$ *is closed.*

COROLLARY 1.10. *If $F$ is continuous, the set $\{x: y \in F(x)\}$ is closed for each $y$.*

*Proof.* The set $\{x: y \in F(x)\} = f^{-1}\{F(x): F(x) \cap \{y\} \neq \phi\}$ and the latter is closed by part 2 of Lemma 1.9.

We call $\{x: y \in F(x)\}$ the inverse of $y$ and write $F^{-1}(y)$. Similarly, for $A \subset Y$ we define

$$F^{-1}(A) = \{x: F(x) \cap A \neq \phi\} .$$

Note that if $A$ is closed, so is $F^{-1}(A)$.

NOTATION. We write $E = A \cup B$, $A \mid B$ to denote a separation of $E$, and we say that $A$ and $B$ separate $E$.

*Note.* In general, for $A \subset Y$ we need not have $F(F^{-1}(A)) = A$.

We can generalize a lemma of Whyburn's.

LEMMA 1.11. *Let $X$ be compact, $Y$ regular, $F: X \rightarrow Y$ continuous, and let $Y_0 \subset Y$. If $F^{-1}(Y_0) = A \cup B$, $A \mid B$ with $F(A)$ and $F(B)$ intersecting the same quasi-component $Q$ of $Y_0$, then there exists $y_0 \in Y_0$ such that $F^{-1}(y_0)$ intersects $A$ and $B$.*

*Proof.* Let $A_1 = F(A) \cap Y_0$ and $B_1 = F(B) \cap Y_0$. Now, by hypothesis $A_1 \cap Q \neq \phi$ and $B_1 \cap Q \neq \phi$. Therefore, $A_1$ is not separated from $B_1$, so there is a net $\{y_\alpha\}$ in $A_1$, say, such that $y_\alpha \rightarrow y_0 \in B_1$.

Now let $x_\alpha \in F^{-1}(y_\alpha) \cap A$ for each $\alpha$. This defines a net in $F^{-1}(A_1) \cap A$, and since $X$ is compact $\{x_\alpha\}$ has a limit point $x_0$ and thus a convergent subnet $x_\gamma \rightarrow x_0$. By Lemma 1.2, $y_0 \in F(x_0)$ so $x_0 \in F^{-1}(y_0)$. Further, $A \mid B$ implies that $x_0 \mid B$ and so $x_0 \in A$ or $F^{-1}(y_0) \cap A \neq \phi$. Finally, $y_0 \in B_1$ implies that $F^{-1}(y_0) \cap B \neq \phi$.

Let $X$, $Y$, and $Z$ be spaces and $F_1: X \rightarrow Y$, $F_2: Y \rightarrow Z$ be set-valued functions. The composition function $F = F_2 \circ F_1$ is defined by $F(x) = F_2(F_1(x))$ for each $x \in X$. Note that in this case $F(x)$ may not be a closed set. Also, if $z \in Z$, then $F^{-1}(z) = F_1^{-1}(F_2^{-1}(z))$. Consequently, we write $F^{-1} = F_1^{-1}F_2^{-1}$. When $X$, $Y$, and $Z$ are compact we have the following result from [10].

LEMMA 1.12. *If $F_1: X \rightarrow Y$ and $F_2: Y \rightarrow Z$ are continuous and* *if $X$, $Y$, and $Z$ are compact, then $F = F_2 \circ F_1$ is continuous.*

Let $F: X \to Y$ and let $A$ be a subspace of $X$. Then the restriction of $F$ to $A$, $F|A$, is defined by $F|A(x) = F(x)$ for all $x \in A$. An immediate consequence of Lemma 1.1 is:

LEMMA 1.13. *Let $F: X \to Y$ be continuous and let $A \subset X$. Then the restriction of $F$ to $A$ is continuous.*

**2. Monotone functions.** In this section we generalize the definition of monotone functions and investigate their elementary properties.

DEFINITION. A continuous function $F: X \to Y$ is called monotone if and only if $F^{-1}(y)$ is connected for each $y \in Y$.

Another generalization of a lemma in Whyburn [18] is:

LEMMA 2.1. *If $X$ is compact, $Y$ regular, and $F: X \to Y$ is continuous, then $F$ is monotone if and only if $F^{-1}(A)$ is connected whenever $A$ is a connected subset of $Y$.*

*Proof.* If $F^{-1}(A)$ is connected for each connected set in $Y$, then $F^{-1}(y)$ is connected for each $y \in Y$ and hence $F$ is monotone.

On the other hand, suppose that $F$ is monotone and that $A$ is a connected subset of $Y$. Further, suppose that $F^{-1}(A) = C \cup D$ with $C | D$. Both $F(C)$ and $F(D)$ meet $A$, and $A$ is a quasi-component of itself. Thus, by Lemma 1.11, there exists a $y \in A$ such that $F^{-1}(y) \cap C \neq \phi$ and $F^{-1}(y) \cap D \neq \phi$, a contradiction, as $F$ is monotone. Hence $F^{-1}(A)$ is connected.

Whyburn shows the following properties are preserved by monotone, continuous, single-valued functions, the property of being, (1) a unicoherent continuum, (2) a hereditarily locally connected continuum, (3) a regular curve, and (4) a rational curve. However, the following examples show that these properties fail to be preserved by continuous, monotone, multi-valued functions, even when fairly stringent conditions are placed on the set $F(x)$, i.e., we may require $F(x)$ to be a locally connected continuum for each $x \in X$ and have $X$ a locally connected continuum, but still not have $Y$ locally connected. See Example 6.

EXAMPLE 1. Let $X$ and $Y$ be any two spaces. Define $F(x) = Y$ for each $x \in X$. Clearly $F$ is continuous and $Y$ need not possess any property that is not shared by all spaces. Thus we see the necessity of placing restrictions on the sets $F(x)$.

EXAMPLE 2. Let $X$ be the closed interval $[(0, 0), (0, 1)]$ in the plane, let $Y$ be the circle that is tangent to the $x$-axis at $x = 2$ and the line $y = 1$ at $(2, 1)$. Denote the points of $X$ by their $y$-coordinate, and in $Y$ let the closed arc $(x, y) - (2, 0) - (x', y)$ be denoted by $yy'$, where we denote the point $(x, y)$ by $y$ and $(x', y)$ by $y'$. Then define $F$ by $F(0) = (2, 0)$, $F(y) = yy'$, $0 < y < 1$ and $F(1) = Y$. It is easily seen that $F$ is continuous. In fact $f$ is a homeomorphism and $F$ is monotone. However, $X$ is unicoherent and $Y$ is not.

EXAMPLE 3. Let $X = [(0, 0), (0, 1)]$ as above and let $Y$ be the unit square and its interior with corners $(1, 0)$, $(2, 0)$, $(1, 1)$, and $(2, 1)$. Let the closed horizontal lines $[(1, y), (2, y)]$ be denoted by $\hat{y}$ where $y$ is the common $y$-coordinate. Again identify the points of $X$ with their $y$-coordinate.

Then let $F(y) = \hat{y}$. Here again $f$ is a homeomorphism. In fact $F$ is monotone and the inverse of a single-valued continuous function of $Y$ onto $X$. Further, $X$ and $F(x)$ are locally connected continua for each $x$. Also, $X$ and $F(x)$ are hereditarily locally connected and hereditarily unicoherent, but $Y$ is neither, and $Y$ is neither rational nor regular, but $X$ is both.

EXAMPLE 4. Let $X = [0, 1]$, and $Y$ the area between and including two concentric circles $C_0$ and $C_1$. Let $C_a$, $0 \leqq a \leqq 1$, be the circle that has the same center as $C_0$ and $C_1$, and with radius $r_a = ar_1 + (1 - a)r_0$ where $r_0$, $r_1$ are the radii of $C_0$ and $C_1$, respectively. Define $F$ by $F(x) = C_x$. Then $F$ is monotone, continuous and $F(x)$ is a locally connected continuum for each $x$, and if $x_1 \neq x_2$, $F(x_1) \cap F(x_2) = \phi$. Yet $X$ is unicoherent and $Y$ is not.

In Whyburn [18] it is shown that the image of a simple arc under a continuous, monotone transformation is again a simple arc, and similarly for a simple closed curve. However, in the case of multi-valued functions neither of these results holds. Example 5 is a counterexample for the former, and the function that maps each point of the circle onto the entire unit interval serves nicely as a counterexample for the latter. We shall, however, subsequently show that the unit interval cannot be mapped onto the circle by a continuous, monotone, multi-valued function $F$, for which $F(x)$ is a simple arc for each $x$. (Here and in the following $F(x)$ may be degenerate, i.e., a point.)

EXAMPLE 5. Let $I$ be the unit interval. Let $I_1$, $I_2$, and $I_3$ be copies of $I$. Form $Y$ by erecting $I_2$ perpendicular to $I_1$ at 1/4 and by erecting $I_3$ prependicular to $I_1$ at 3/4 (the 0 of $I_2$ is identified with 1/4 in $I_1$ and the 0 of $I_3$ is identified with 3/4 in $I_1$). Define $F\colon I \rightarrow Y$ by

$F(0) = [0, 1/4] \cup I_2$, $F(1/4) = I_2$, $F(1/2) = I_2 \cup [1/4, 3/4] \cup I_3$, $F(3/4) = I_3$, and $F(1) = I_3 \cup [3/4, 1]$ (where intervals are subsets of $I_1$ unless otherwise stated). For other points in $I$, $F$ is defined by ratios. The function $F$ constructed in this manner is monotone and continuous. Also $F(x)$ is an arc for each $x \in I$. Note that the range of $F$ is a space with two branch points and that $F$ is also nonalternating (see § 3) but not open.

EXAMPLE 6. A construction similar to that of Example 5 can be used to define a continuous, monotone function with $F(x)$ an arc for each $x$ on the unit interval onto the following nonlocally connected planar space. The space consists of the union of the following subsets of the plane: $\{(x, 0): 0 \leqq x \leqq 1\}$, $\{(0, y): 0 \leqq y \leqq 1\}$, and $\{(1/n, y): n \geqq 2, 0 \leqq y \leqq 1\}$.

DEFINITION. A continuum $X$ is called a *multi-arc* in case there exists a continuous, monotone, set-valued function $F$ on the unit interval onto $X$, such that $F(x)$ is a simple arc for each $x$ in the interval. (Here $F(x)$ may be degenerate, i.e., a point.)

DEFINITION. A continuum $X$ is called *circularly reducible* if and only if there exists a continuous, monotone function $F$ from $X$ onto the circle, such that $F(x)$ is a simple arc for each $x \in X$ ($F(x)$ may be a point).

REMARK. By extending the construction in Example 5, it can be shown that any dendrite with a finite number of branch points is a multi-arc. Note, however, that Example 6 shows that not all multi-arcs are locally connected, and that Example 3 shows that the disc is a multi-arc.

From Wallace [14] we have:

DEFINITION. A continuous function $F: X \to Y$ is *anarthric* if and only if for each $y \in Y$ no $x \in X - F^{-1}(y)$ separates $F^{-1}(y)$.

Then from the definition of monotone and anarthric we obtain

LEMMA 2.2. *Let $X$ be a totally ordered, compact, connected space, and let $F: X \to Y$ be a continuous function on $X$ into $Y$. Then $F$ is anarthric if and only if $F$ is monotone.*

Also from [14] we have

THEOREM (*Wallace*): *Let $X$ be compact. A necessary and sufficient condition that a function $F$ on $X$ be anarthric is: If*

*$X = M \cup N$, where $M$ and $N$ are continua meeting in a cutpoint $x$, and $K$ is any continuum meeting $M$, then $F(M \cap K) = F(M) \cap F(K)$.*

COROLLARY 2.3. *The circle is not a multi-arc.*

*Proof.* Suppose $F: [0, 1] \rightarrow C$ is a monotone continuous function on the unit interval onto a circle such that $F(x)$ is an arc for each $x \in [0, 1]$. By Lemma 2.2 $F$ is anarthric. Thus if $x \in (0, 1)$ we have by the theorem $F(x) = F([0, x]) \cap F([x, 1])$. Also $F([0, x]) \cup F([x, 1]) = C$ and $F(x)$ is a subarc of $C$. Thus either $F([0, x])$ or $F([x, 1])$ is equal to $C$ for otherwise their intersection would not be connected. Hence we may assume that there exists an $x'$ such that $F([0, x']) = F(x')$ and $F([x, 1]) = C$. Let $x_0 = \sup \{x: F([0, x]) = F(x')$ and $F([x, 1]) = C\}$. If $y \in F(x_0) - F(x')$, and if $U$ is an open set containing $y$ which does not meet $F(x')$, then $F^{-1}(U)$ is an open set containing $x_0$ which does not meet $\{x: F([0, x]) = F(x')\}$. This contradicts the choice of $x_0$. Hence $F(x_0) = F(x')$ and $F([x_0, 1]) = C$. Note $F([x_0, 1]) = C$ implies that $x_0 \neq 1$. Now if $x > x_0$, $F([0, x]) = C$ since $x_0$ is the sup $\{x: F([x, 1]) = C$ and $F([0, x]) = F(x') = F(x_0)\}$. Thus for $y \in C - F(x_0)$ there is a decreasing sequence $\{x_n\}$ such that $x_n \rightarrow x_0$ and $y \in F(x_n)$ for all $n$. But this implies that $x_0 \in F^{-1}(y)$ since $F^{-1}(y)$ is closed, a contradiction.

We can derive more corollaries to Theorem 2.3.

COROLLARY 2.4. *A hereditarily unicoherent multi-arc is not circularly reducible.*

*Proof.* Suppose that $x$ is circularly reducible, and that $F_2: X \rightarrow C$ is a continuous, monotone function on $X$ onto the circle $C$ such that $F_2(x)$ is a simple arc for each $x \in X$. Since $X$ is a multi-arc there exists a continuous, monotone function $F_1$ on the unit interval $I$ onto $X$ such that $F_1(r)$ is a simple arc for each $r \in I$. Then by Lemma 2.1 the function $F = F_2 \circ F_1$ is continuous and monotone, and $F$ maps $I$ onto $C$. Now let $M$ be an arc contained in $X$. Then $F_2 | M$ is continuous. Further, if $y \in C$, either $F_2^{-1}(y) \cap M = \phi$ or $F_2^{-1}(y) \cap M$ is connected since $X$ is hereditarily unicoherent. Therefore $F_2 | M$ is monotone. Hence $F_2(M) \neq C$. Further, if $M \in I$, $F_1(r)$ is at most an arc, and hence, $F_2 \circ F_1(r) \neq C$. Note that $F_2 \circ F_1(r)$ is connected. Consequently, $F$ is a continuous, monotone function on $I$ onto $C$ such that $F(r)$ is a simple arc for each $r \in I$; this is a contradiction. Hence the result holds.

COROLLARY 2.5. *A hereditarily unicoherent, arcwise connected continuum is not circularly reducible.*

*Proof.* We sketch the proof of this result. Let $X$ be an hereditarily unicoherent, arcwise connected continuum. First observe that the set $\{F(x): x \in X\}$ has maximal elements, where $F: X \to C$ is a monotone function on $X$ onto $C$ such that $F(x)$ is an arc. If $x'$, $x''$ are such that $F(x')$ and $F(x'')$ are maximal, then $F(x') \cap F(x'') \neq \phi$ and $F(x') \cup F(x'') \neq C$. From Corollary 2.4, if we have $x_1, x_2, \cdots, x_n$ such that $F(x_1), F(x_2), \cdots, F(x_n)$ are maximal, then $\bigcup_{i=1}^{n} F(x_i) \neq C$. Then the fact that $X$ is compact is used to complete the proof.

## 3. Nonalternating functions.

The purpose of this section is to generalize the definition of nonalternating functions to set-valued functions and to derive some characterizations of such functions.

DEFINITION. A function $F: X \to Y$ is called nonalternating if and only if for any pair $y_1, y_2 \in F(X)$ there does not exist a separation $X - F^{-1}(y_1) = A \cup B$ such that $y_2 \in F(A) \cap F(B)$.

EXAMPLE 7. Let $X = [0, 1]$ and define $F: X \to X$ by $F(\frac{1}{2}) = \{0\}$, $F(x) = [0, 2(x - \frac{1}{2})]$ for $x > \frac{1}{2}$ and $F(x) = [0, 2(\frac{1}{2} - x)]$ for $x < \frac{1}{2}$. Then $F$ is continuous and nonalternating, but not monotone. Further, this serves as a counterexample to theorems which are true for single-valued functions [18, pp. 138–140].

DEFINITION. A multi-valued function $F: X \to Y$ is called *semi-single-valued* (s.s.v.) if and only if $F(x_1) \cap F(x_2) \neq \phi$ implies that $F(x_1) = F(x_2)$.

A very small change will allow us to get the counterpart to Theorem 2.1 [18, p. 138].

THEOREM 3.1. *Let $F: X \to Y$ be continuous. Then $F$ is nonalternating if and only if for each $y \in Y$, and each quasi-component $Q$ of $X - F^{-1}(y)$, $F^{-1}(F(Q)) \cap (X - F^{-1}(y)) = Q$.*

*Proof.* Suppose that $F$ is nonalternating and that $Q$ is a quasi-component of $X - F^{-1}(y)$ for $y \in Y$. Then, if

$$x \in F^{-1}(F(Q)) \cap (X - F^{-1}(y)) - Q ,$$

there exists a separation $X - F^{-1}(y) = A \cup B$ such that $x \in A$ and $Q \subset B$, as $Q$ is a quasi-component. However, this implies that $F(A) \cap F(B) \neq \phi$, as $x \in F^{-1}(F(Q))$ implies $F(x) \cap F(Q) \neq \phi$ which implies there exists an $x' \in Q$ such that $F(x) \cap F(x') \neq \phi$. This is contrary to the assumption that $F$ is nonalternating. If $F$ is not nonalternating, there exist points $y_1, y_2 \in Y$, and a separation $X - F^{-1}(y_1) = A \cup B$ such that $y_2 \in F(A) \cap F(B)$. Let $x \in A$ with $y_2 \in F(x)$ and let $Q$ be the quasi-component of $X - F^{-1}(y_1)$ containing $x$. Since $y_2 \in F(B)$, there exists $x' \in B$ such that $y_2 \in F(x')$. Hence,

$$x' \in F^{-1}(F(Q)) \cap (X - F^{-1}(y_1)) - Q ,$$

and the condition fails.

We also obtain

THEOREM 3.2. *Let $F: X \to Y$ be continuous, and let $y \in Y$. Let $Q$ be any quasi-component of $Y - \{y\}$. Then if $F^{-1}(Q) \cap (X - F^{-1}(y))$ is contained in a quasi-component of $X - F^{-1}(y)$, $F$ is nonalternating.*

*Proof.* Let $y_1, y_2 \in Y$, and let $Q$ be the quasi-component of $Y - y_1$ which contains $y_2$. Then, since

$$F^{-1}(y_2) \cap (X - F^{-1}(y_1)) \subset F^{-1}(Q) \cap (X - F^{-1}(y_1)) ,$$

the hypothesis implies that for any separation

$$X - F^{-1}(y_1)\ A \cup B,\ A \mid B ,$$

$F^{-1}(y_2) \cap (X - F^{-1}(y_1))$ is contained in $A$ or in $B$. Thus, $F$ is nonalternating.

DEFINITIONS. Denote the set of all points that separate $a$ and $b$ by $E(a, b)$. Then call $a, b$ conjugate in case $E(a, b) = \phi$. Then, if $x$ is neither a cutpoint nor an end point, the set containing $x$ and all points which are conjugate to $x$ is called a *simple link*. Finally, a *cyclic element* of $X$ is either a cutpoint, an end point, or a simple link.

DEFINITION. A connected space is called *semi-locally-connected* (s.l.c.) at a point $x$ in case $x$ has arbitrarily small neighborhoods whose complements have only a finite number of components. If $X$ is s.l.c. at each of its points, it is called s.l.c.

Using a result of Wallace [14] we can generalize a result on single-valued functions in [18] to multi-valued functions.

THEOREM (Wallace). *A function $F: X \to Y$ on a continuum $X$ into a continuum $Y$ is anarthric if and only if for any subcontinuum $H$ of $Y$ and any subcontinuum $K$ of $X$ such that $K \cap F^{-1}(H) = P \cup Q$, $P \mid Q$, there exist points $p \in P$, $q \in Q$, such that $p$ and $q$ are conjugate.*

THEOREM 3.3. *Let $F: X \to Y$ be continuous and semi-single valued, and let $X$ be a semi-locally-connected, metric continuum and $Y$ a* *metric continuum. Then $F$ is nonalternating if and only if the following hold.*

(i) *$F$ is anarthric,*

(ii) *$F$ is nonalternating on each cyclic element of $X$.*

*Proof.* Suppose that $F$ is nonalternating. Let $y \in Y$, and suppose there exists a point $p \in E(a, b) - F^{-1}(y)$, where $a, b \in F^{-1}(y)$. Now $y \notin F(p)$, thus $(F(a) \cup F(b)) \cap F(p) = \phi$ since $F$ is semi-single-valued. Moreover, there exists a separation $X - p = A \cup B$, $A \mid B$, with $a \in A$ and $b \in B$. Let $y' \in F(p)$. Then, there exists a separation $A'$, $B'$ of $X - F^{-1}(y')$ such that $a \in A'$ and $b \in B'$, which implies that $F(A') \cap F(B') \neq \phi$. This contradicts the hypothesis that $F$ is nonalternating. Thus, (i) holds.

In order to show (ii) holds, let $E$ be a true cyclic element of $X$ (i.e., a simple link). Let $F(E) = E' \subset Y$, and let $y_1, y_2 \in E'$. If $E - F^{-1}(y_1) \cap E = A \cup B$, $A \mid B$ such that $y_2 \in F(A) \cap F(B)$, then by [18, IV, 3.22 and 6.81], there exists a separation of $X - F^{-1}(y_1) = A' \cup B'$, $A' \mid B'$, with $y_2 \in F(A') \cap F(B')$, a contradiction.

Suppose (i) and (ii) hold. Let $y \in Y$. If $X - F^{-1}(y) = A \cup B$, $A \mid B$, and if $x_1 \in A$, $x_2 \in B$ such that $F(x_1) \cap F(x_2) \neq \phi$, then by the result of Wallace there exist $x_1'$ and $x_2'$ which are separated by $F^{-1}(y)$ and which are contained in the same cyclic element, but this contradicts (ii). Thus $F$ is nonalternating.

COROLLARY 3.4. *Any nonalternating semi-single-valued fnnction on a dendrite is monotone.*

*Proof.* If $a, b \in F^{-1}(y)$ then by (i), $E(a, b) \subset F^{-1}(y)$, and $E(a, b)$ is a simple arc from $a$ to $b$.

## 4. Composite functions and factorization.

In this section some of the properties of composite functions are investigated and a factorization theorem is obtained.

DEFINITION. A function $F\colon X \to Y$ is called *open* in case whenever $U$ is open in $X$, $F(U)$ is open in $Y$.

Let $X$, $Y$ and $Z$ be compact spaces, and let $F$, $F_1$, and $F_2$ be continuous functions such that $F_1\colon X \to Z$, $F_2\colon Z \to Y$ and $F = F_2 \circ F_1$, $F\colon X \to Y$.

Lemmas 4.1 and 4.2 are extensions of results which hold for single-valued functions. The proofs are straightforward and are omitted.

LEMMA 4.1. *If $F_1$ is single valued:*

(i) *$F$ open implies that $F_2$ is open;*
(ii) *$F$ monotone implies that $F_2$ is monotone;*
(iii) *$F$ nonalternating implies that $F_2$ is nonalternating.*
In addition to this we can obtain:

LEMMA 4.2. *The following statements hold.*

(i) *$F_1$, $F_2$ open implies $F$ is open;*
(ii) *$F_1$, $F_2$ monotone implies $F$ is monotone;*
(iii) *$F_1$ monotone and s.s.v., and $F_2$ nonalternating imply $F$ is nonalternating.*

We now turn to the problem of factoring functions. First we have the known Theorem A, Whyburn [18, pp. 141–142], which is stated below. (Note that Theorem A holds for any compact Hausdorff space, as well as for metric spaces.)

DEFINITION. A function $F: X \to Y$ is called *light* in case $F^{-1}(y)$ is totally disconnected for each $y \in Y$.

THEOREM A. *Let $g$ be a single-valued, continuous function from $X$ onto $Y$. Then there exist a space $Z$ and continuous functions $g_1$, $g_2$; $g_1: X \to Z$, $g_2: Z \to Y$, such that $g_1$ is monotone, $g_2$ is light, and $g = g_2 \circ g_1$.*

We can extend this theorem to semi-single-valued functions, but first we need the following lemma.

LEMMA 4.3. *Let $\mathscr{S} \subset S(X)$, and let $\mathscr{S}$ have the finite topology. Define a function $F: \mathscr{S} \to X$ by $F(S) = S$ for all $S \in \mathscr{S}$. Then $F$ is continuous.*

*Proof.* Let $U$ be an open set contained in $X$. If $S \in \mathscr{S}$ and $S \cap U \neq \phi$, the set $\langle U, X \rangle = \{S \in \mathscr{S}: S \cap U \neq \phi\}$ is an open set in $\mathscr{S}$ such that $F(S) \cap U \neq \phi$ for all $S \in \langle U, X \rangle$. If $S \subset U$, then $\langle U \rangle = \{S \in \mathscr{S}: S \subset U\}$ is an open set in $\mathscr{S}$ such that $F(\langle U \rangle) \subset U$. Thus, by Lemma 1.1, $F$ is continuous.

*Note.* If $\mathscr{S}$ is a decomposition, then $F^{-1}(x) = \{S\}$ where $x \in S$, and if $F: X \to Y$ is semi-single-valued, then $\mathscr{S} = \{F(x): x \in X\}$ is a decomposition.

THEOREM 4.5. *Let $F: X \to Y$ be continuous and semi-single-valued. Then there exist a space $Z$ and continuous functions $F_1$, $F_2$ with $F_1: X \to Z$ single-valued, $F_2: Z \to Y$, $F = F_2 \circ F_1$, and such that $F_1$ is monotone, and $F_2$ is light.*

*Proof.* Let $f$ be the induced single-valued function on $X$ into $S(Y)$. Then $f(X) = \{F(x): x \in X\}$ is a decomposition of $Y$. Then by Theorem A there exist a space $Z$ and continuous functions $f_1, f_2$ such that $f_1$ is monotone, $f_2$ is light, and $f = f_2 \circ f_1$. Let $F^*$ be the function of Lemma 4.3. Then set $F_1 = f_1$ and $F_2 = F^* \circ f_2$. Thus, $F_1$ is single valued and monotone and, from the remark following Lemma 4.3, $F_2$ is continuous and light. Finally, $F = F_2 \circ F_1$.

Finally, with Lemma 4.1 (i), we get:

COROLLARY 4.6. *If $F: X \rightarrow Y$ is semi-single-valued, continuous and open, then there exist continuous functions $F_1$, $F_2$ such that $F_1$ is single-valued and monotone, and $F_2$ is light and open, and such that $F = F_2 \circ F_1$.*

5. **Semi-single-valued functions.** Let $F: X \rightarrow Y$ be a semi-single-valued continuous function from $X$ onto $Y$, and define the collections $Q = \{F(x): x \in X\}$, and $P = \{F^{-1}(y): y \in Y\}$. That $P$ and $Q$ are decompositions into disjoint closed sets follows from the definition of a continuous, semi-single-valued function.

Let $q: Y \rightarrow Q$ and $p: X \rightarrow P$ be the projections of $Y$ onto $Q$ and $X$ onto $P$, respectively. Define $F^\#$ on $P$ onto $Y$ by $F^\#(D) = F(x)$ for $D \in P$ and $x \in D$, and define $f'$ on $X$ onto $Q$ by $f'(x) = F(x)$. Note that $f'$ and $f$ are essentially the same but $Q$ as a decomposition has the quotient topology rather than the finite topology. When $F$ is the inverse of a single-valued function, we have by Theorem 5.10 [6], that the quotient and the finite topologies are equivalent. We shall generalize this result in Corollary 5.3. Finally, define $f^*: P \rightarrow Q$ by $f^*(D) = F(x)$ for $D \in P$ and $x \in D$.

THEOREM 5.1. *If $X$ and $Y$ are compact, the decompositions $P$ and $Q$ are upper semi-continuous. Further, $P$ and $Q$ are Hausdorff in the quotient topology.*

*Proof.* Let $V_1$, $V_2$ be disjoint open subsets of $Y$ such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Then, for $i = 1, 2$, $Y - FF^{-1}(Y - V_i)$ is an open set containing $F(x_i)$ which is contained in $V_i$ and which is the union of members of $Q$. Similarly, if $F^{-1}(y_1) \neq F^{-1}(y_2)$ are in $P$ and if $U_1$ and $U_2$ are open and disjoint with $F^{-1}(y_i) \subset U_i$, then $X - F^{-1}F(X - U_i)$, $i = 1, 2$, are the required open sets. This shows that $P$ and $Q$ are upper semi-continuous, and Hausdorff in the quotient topology.

THEOREM 5.2. *The functions $F^\#$ and $f'$ are continuous when $P$ and $Q$ have the quotient topology.*

*Proof.* Since $F^{\#}(D) = F(p^{-1}(D))$ for $D \in P$, Theorem 5.1 implies that $F^{\#}$ is continuous. Also $f' = q \circ F$ and hence is continuous by Lemma 1.12.

Now we obtain a generalization of Theorem 5.10 [6].

COROLLARY 5.3. *If $X$ and $Y$ are compact and $F: X \to Y$ is a semi-single-valued continuous function, then the finite and quotient topologies agree on $Q = \{F(x): x \in X\}$, and $f$ and $f'$ are equivalent functions.*

*Proof.* The function $F^{\#}$ is the inverse of a single-valued function. Hence, Theorem 5.10 [6] applies.

THEOREM 5.4. *The function $f^*: P \to Q$ is a homeomorphism onto, when $X$ and $Y$ are compact.*

*Proof.* That $f^*$ is a single-valued function which is 1 to 1 and onto follows immediately from the fact that $F$ is semi-single-valued. That $f^*$ is continuous follows from $f^* = q \circ F^{\#}$, Theorem 5.2 and Corollary 5.3.

We associate with each multi-valued function $F: X \to Y$ the induced function $f$ on $X$ into $S(Y)$ and we can define a function $F^*$ on $f(X)$ into $Y$ by $F^*(f(x)) = F(x)$. Then $F = F^* \circ f$. We consider briefly the relationships between $F$, $f$ and $F^*$ and the properties of being monotone, open, and nonalternating. A typical question is: "Does $F$ monotone imply that $f$ is monotone, and conversely?" Simple examples show that $f$ monotone does not imply that $F$ is monotone, and Example 8 shows that the converse fails.

EXAMPLE 8. Let $X$ be the rectangle with corners $(0, -1)$, $(1, -1)$, $(1, 1)$ and $(0, 1)$ together with its interior. Let $Y$ be the unit interval. Let $(x, y) \in X$ and define $r_1 = \frac{1}{2}(1 - x)$, $r_2 = \frac{1}{2}(1 + x)$. Then define $z_1 = r_1(1 - |y|)$, $z_2 = r_1 + |y|(\frac{1}{2} - r_1)$, $z_3 = r_2 - |y|(r_2 - \frac{1}{2})$ and $z_4 = r_2 + |y|(1 - r_2)$ with $r_i, z_j \in Y$. Define $F: X \to Y$ by $F((x, y)) = [z_1, z_2] \cup [z_3, z_4]$. Then $F$ is monotone and continuous but $f$ is not monotone.

However, if $F$ is semi-single-valued, we have

THEOREM 5.5. *If $F: X \to Y$ is a semi-single-valued, continuous function from $X$ onto $Y$, then $F$ is monotone if and only if $f$ is monotone.*

*Proof.* If $y \in Y$, then there exists a unique $S$ in $F(X)$ such that

$y \in S$. Thus $F^{-1}(y) = \{x: F(x) = S\} = f^{-1}(S)$. So $F^{-1}(y)$ is connected if and only if $f^{-1}(S)$ is connected.

THEOREM 5.6. *The following statements hold.*
(i) *$F$ monotone implies $F^*$ is monotone.*
(ii) *$F$ open implies $F^*$ is open.*
(iii) *If $F$ is semi-single-valued, $F$ open implies $f$ is open.*
(iv) *$F$ nonalternating implies $F^*$ is nonalternating.*
Further, we may state a partial converse to (i), (ii) and (iv).

THEOREM 5.7. *If $f$ is monotone, then*
(i) *$F^*$ monotone implies that $F$ is monotone; and*
(ii) *$F^*$ nonalternating implies that $F$ is nonalternating.*

THEOREM 5.8. *If $f$ is open, $F^*$ open implies $F$ is open.*

**6. Open functions.** The purpose of this section is to show that certain results in Whyburn [18] on open mappings can be generalized to semi-single-valued functions and in some cases to arbitrary multi-valued functions. In this section all spaces will be separable, metric spaces.

REMARK 1. The definition of terms used in this section are those of Whyburn [18].

REMARK 2. If $X$ is compact, then a collection of subsets $G$ of $X$ is continuous if and only if it is continuous in the limit sense.

THEOREM 6.1. *Let $F: X \to Y$ be a continuous, semi-single-valued function of $X$ onto $Y$. If $F$ is open, then the collection $\{F^{-1}(y): y \in Y\}$ is continuous in the limit sense. Conversely, if $X$ is compact, and if the collection $\{F^{-1}(y): y \in Y\}$ is continuous, then $F$ is open.*

*Proof.* By Theorem 5.6, $F$ open implies $f$ open and since $F$ is s.s.v., $f^{-1}(F(x)) = F^{-1}(y)$, $y \in F(x)$. Thus, the first statement follows from the theorem for single-valued functions [18, Theorem 4.31, p. 130], and minor modifications of the proof in [18] will yield a proof of the converse.

COROLLARY 6.2. *Let $X$ be compact and let $F$ be as in Theorem 6.1. Then $F$ is open if and only if the collection $\{F^{-1}(y): y \in Y\}$ is continuous.*

We can also generalize a theorem due to Eilenberg, [18, p. 138].

THEOREM 6.3. *Let $F: X \to Y$ be continuous, semi-single-valued, and onto. Then $F$ is open if and only if for each sequence $\{y_n: n = 1, \cdots\}$ in $Y$ such that* $\lim_n y_n = y_0$, $\lim F^{-1}(y_n) = F^{-1}(y_0)$.

*Proof.* Suppose that $F$ is open, and that $\{y_n\}$ is a sequence in $Y$ such that $\lim_n y_n = y_0$. In view of Theorem 6.1 we need only show that $F^{-1}(y_0) \cap \liminf F^{-1}(y_n) \neq \phi$. If $x \in F^{-1}(y_0)$, if $U$ is an open set containing $x$, and if $U \cap F^{-1}(y_n) = \phi$ for infinitely many $n$, then $F(U)$ is an open set containing $y_0$ such that $y_n \notin F(U)$ for infinitely many $n$, a contradiction to $\lim_n y_n = y_0$.

Now suppose that $\lim_n y_n = y_0$ implies $\lim F^{-1}(y_n) = F^{-1}(y_0)$, and let $U$ be open in $X$. If $F(U)$ is not open in $Y$, there exists $y_0 \in U$ and a sequence $\{y_n\} \subset Y - F(U)$ such that $\lim y_n = y_0$. Now $y_0 \in F(U)$ implies that there exists an $x \in F^{-1}(y_0) \cap U$, and from the hypothesis $U \cap F^{-1}(y_n) \neq \phi$ for all but finitely many $n$. Thus $y_n \in F(U)$ for all but finitely many $n$, a contradiction. Hence $F$ is open.

The proof of the following lemma is straightforward and is omitted. Note that in many of the following results the restriction to separable metric spaces is unnecessary.

LEMMA 6.4. *Let $F: X \to Y$ be continuous and onto. Then $Q \subset X$ is an inverse set if and only if $F(A \cap Q) = F(A) \cap F(Q)$ for each $A \subset X$.*

LEMMA 6.5. *Let $F: X \to Y$ be continuous and open. If $Q \subset X$ is an inverse set, then $F$ restricted to $Q$ is open.*

*Proof.* Let $V$ be open in $Q$. Then there exists an open set $U$ in $X$ such that $V = Q \cap U$. Then, by Lemma 6.4, $F(V) = F(U \cap Q) = F(U) \cap F(Q)$, which is open in $F(Q)$ since $F(U)$ is open.

In order to establish the next result we need a theorem of Michael's [6, Theorem 2.5.1].

THEOREM B. *If $X$ is regular, and $B \subset S(X)$ is compact, then $\cup\{E: E \in B\}$ is closed.*

THEOREM 6.6. *Let $F: X \to Y$ be onto and continuous. Then, if $A \subset X$ is conditionally compact:*

(i) $\overline{F(A)} = F(\bar{A})$;

(ii) $\overline{F(A)} - F(A) \subset F(\bar{A} - A)$.

*Further, if $F$ is an open function, and $A$ is an open set,*

(iii) $b(F(A)) \subset F(b(A))$

*where $b(A)$ denotes the boundary of $A$.*

*Proof.*

(i) Let $A \subset X$ be conditionally compact. Then, by Theorem B, $F(\bar{A})$ is closed. Hence $\overline{F(A)} \subset F(\bar{A})$. Also $F(\bar{A}) \subset \overline{F(A)}$ since $F$ is continuous.

(ii) From (i), $\overline{F(A)} - F(A) = F(\bar{A}) - F(A) \subset F(\bar{A} - A)$.

(iii) With $A$ open and $F$ open this is immediate from (ii).

LEMMA 6.7. *Let $U$, $U_1$, $U_2$ be open sets such that $U = U_1 \cup U_2$. If $U_1 \cap U_2 = \phi$, then $b(U) = b(U_1) \cup b(U_2)$.*

*Proof.* If $x \in b(U)$, then $x \in \bar{U}_1$ or $x \in \bar{U}_2$ and $x \notin U_1 \cup U_2$. Therefore $x \in b(U_1)$ or $x \in b(U_2)$. On the other hand $x \in b(U_i)$ implies $x \in \bar{U}_i$ and $x \notin U_1 \cup U_2$. Thus $x \in b(U)$.

THEOREM 6.8. *Let $X$ and $Y$ be continua, and let $F: X \rightarrow Y$ be continuous, open and onto. Then:*

(i) *If $X$ is a curve of order less than or equal to $n$, and if $F(x)$ contains at most $m$ points for each $x \in X$, then $Y$ is a curve of order less than or equal to $nm$;*

(ii) *If $X$ is a regular curve and if $F(x)$ is finite for each $x$, then $Y$ is a regular curve; and*

(iii) *If $X$ is a rational curve and $F(x)$ countable for each $x$, then $Y$ is a rational curve.*

*Proof.*

(i) Let $y \in Y$, and let $V$ be an open set containing $y$. Since $F$ is onto, there exists an $x \in X$ such that $y \in F(x)$. Let $F(x) = \{y_1, \cdots, y_k\}$, $k \leqq m$. Suppose $y = y_1$. Then there exist open sets $V$, $V'$ of $Y$ such that $y_1 \in V$, $\{y_2, \cdots, y_k\} \subset V'$ and $V \cap V' = \phi$. Further there exists an open set $U$ containing $x$ such that $F(U) \subset V \cup V'$ and such that $b(U)$ contains at most $n$ points (as $X$ is a curve of order less than or equal to $n$). Let $V_1 = F(U) \cap V$ and $V_2 = F(U) \cap V'$, $V_1$ and $V_2$ are open and disjoint. Thus, by Lemma 6.7, $b(F(U)) \subset b(V_1) \cup b(V_2)$. Therefore by Theorem 6.6, $b(V_1) \subset b(F(U)) \subset F(b(U))$ and this latter set contains at most $nm$ points. Thus $V_1$ is the required open set containing $y$.

(ii) A proof similar to the proof of (i) will establish (ii).

(iii) Let $x \in Y$, and let $V$ be an open set containing $y$. Pick an $x \in X$ such that $y \in F(x)$. Since $F(x)$ is countable, we may assume that $F(x) \subset V \cup (Y - \bar{V})$. Since $X$ is rational, there exists an open set $U$ containing $X$ such that $F(U) \subset V \cup (Y - \bar{V})$, with $b(U)$ countable. By part (iii) of Theorem 6.6, $b(F(U)) \subset F(b(U))$ and $F(b(U))$ is countable. Then $F(U) \cap V$ is an open set containing $y$

with countable boundary. This last since

$$b(F(U)) = b(V \cap F(U)) \cup [b(Y - \bar{V}) \cap F(U)],$$

by Lemma 6.7.

Following Whyburn's proof [18, p. 147, 7.4], we can prove

THEOREM 6.9. *Let $X$ be compact and let $F: X \to Y$ be continuous, open and onto. If $A$ is a connected open set in $Y$, and if $Q$ is any quasi-component of $F^{-1}(A)$, then $A \subset F(Q)$.*

COROLLARY 6.10. *Let $X$ and $Y$ be locally connected, compact spaces, $F: X \to Y$ open and onto, and let $A$ be any closed set in $Y$. If $C$ is any component of $Y - A$, then $F^{-1}(C)$ has only a finite number of components and each of these maps onto all of $C$ under $F$.*

*Proof.* It follows from the hypothesis that any quasi-component of $F^{-1}(C)$ is also a component of $F^{-1}(C)$. Then if $F^{-1}(C)$ has an infinite number of quasi-components, a sequence constructed by choosing one element from each quasi-component must have a limit point. However, each quasi-component is open; hence no subsequence can converge to the limit point, a contradiction. Finally, $C$ is open so Theorem 6.5 implies that $C \subset F(Q)$ for any quasi-component $Q \subset F^{-1}(C)$.

PROPOSITION 6.11. Let $F: X \to Y$ be open and onto, and let $Y$ be connected. If $X_0$ is an inverse set in $X$ which is open and closed, then $F(X_0) = Y$.

*Proof.* Since $X_0$ is an inverse set and $F$ is open, $F(X_0)$ and $F(X - X_0)$ are disjoint open sets whose union is $Y$. Therefore, $F(X_0) = Y$.

REMARK. Let $F: X \to Y$ be continuous. If $C$ is a subset of $Y$, then $F^{-1}(C)$ need not be an inverse set. However, if $F$ is an s.s.v. function, we have:

LEMMA 6.12. *Let $F: X \to Y$ be an* s.s.v. *function. If $C \subset Y$, then $F^{-1}(C)$ is an inverse set.*

*Proof.* If $x \in F^{-1}FF^{-1}(C)$, then $F(x) \cap F(F^{-1}(C)) \neq \phi$. Thus there exists an $x' \in F^{-1}(C)$ such that $F(x) \cap F(x') \neq \phi$. Then since $F$ is s.s.v., $F(x) = F(x')$. Therefore, $F(x) \cap C \neq \phi$, and hence $x \in F^{-1}(C)$. Consequently, $F^{-1}(C)$ is an inverse set.

THEOREM 6.13. *Let $X$ be compact, and let $F: X \to Y$ be a con-*

*tinuous, open semi-single-valued function on $X$ onto $Y$. Let $C$ be any compact, connected set in $Y$. Then for any component $K$ of $Q = F^{-1}(C)$, it follows that $C \subset F(K)$.*

*Proof.* Since $F$ is s.s.v., $F^{-1}(Q)$ is an inverse set in $X$ by Lemma 6.12. Hence, by Lemma 6.5, $F$ restricted to $Q$ is open and the result follows by applying Theorem 6.9 to $F$ restricted to $Q$.

*Note.* Single-valued open, continuous functions map nodal sets onto nodal sets ($A$ is nodal in case $A \cap \overline{X - A}$ is at most a single point), but Example 3 is a counterexample to this result for s.s.v. mappings. In fact, in Example 3, $F$ is the inverse of a continuous single-valued function.

## 7. Quasi-monotone functions.

In this section $X$ and $Y$ are compact and connected, and $F: X \to Y$ will always denote a continuous function of $X$ onto $Y$.

DEFINITION. A function $F$ is called *quasi-monotone* in case for each continuum $Y_0 \subset Y$ with nonvoid interior, $F^{-1}(Y_0)$ has only a finite number of components $C_n$ and $Y_0 \subset F(C_n)$ for each component $C_n$ of $F^{-1}(Y_0)$. Note that any monotone function on a continuum is quasi-monotone.

REMARK. If $g$ is a continuous single-valued function on a compact, connected, locally connected space $X$, then $g(X)$ is also compact, connected and locally connected. However, when $F$ is multi-valued this may not be the case, so it is sometimes necessary to assume that $Y$ as well as $X$ is compact, connected, and/or locally connected.

The proof of Theorem 7.1 is very much like the proof of the corresponding theorem for single-valued functions [18, p. 152, Th. 8.1] and is omitted.

THEOREM 7.1. *Let $X$ and $Y$ be locally connected continua, and let $F: X \to Y$. Then $F$ is quasi-monotone if and only if for each component $C$ of the inverse of any connected open set $V$ of $Y$, $V \subset F(C)$.*

COROLLARY 7.2. *Every open function on a locally connected continuum onto a locally connected continuum is quasi-monotone.*

*Proof.* Corollary 6.10 implies that the hypotheses of Theorem 7.1 are satisfied.

THEOREM 7.3. *If $X$ and $Y$ are locally connected continua, and if $F$ is light, then $F$ is quasi-monotone if and only if $F$ is open.*

*Proof.* If $F$ is open, then $F$ is quasi-monotone by Corollary 7.2. Suppose that $F$ is quasi-monotone, let $U$ be open in $X$, and let $y \in F(U)$. If $x \in U \cap F^{-1}(y)$, then since $F$ is light there exists a connected open set $U' \subset U$ such that $x \in U'$ and $b(U') \cap F^{-1}(y) = \phi$. Let $Q$ be the component of $Y - F(b(U'))$ containing $y$, and let $C$ be the component of $F^{-1}(Q)$ containing $x$. Then $C \subset U'$ since $C \cap b(U') = \phi$, and by Theorem 7.1, $Q \subset F(C)$. Then $Q \subset F(C) \subset F(U') \subset F(U)$ and $Q$ is an open set containing $y$. Thus $F(U)$ is open.

THEOREM 7.4. *Let $X$, $Y$ and $Z$ be locally connected, and let $F = F_2 \circ F_1$, $F_1: X \to Z$, $F_2: Z \to Y$ with $F_1$ and $F_2$ continuous and onto. Then*:

(i) *If $F$ is quasi-monotone and $F_1$ is single-valued, $F_2$ is quasi-monotone*; *and*

(ii) *If $F_1$ and $F_2$ are quasi-monotone, $F$ is quasi-monotone.*

*Proof.*

(i) Let $V$ be an open connected set in $Y$, and let $C$ be a component of $F_2^{-1}(V)$. Let $C'$ be a component of $F^{-1}(V)$ contained in $F_1^{-1}(C)$. Then, since $F_1$ is single-valued, $F_1(C') \subset C$, and since $F$ is quasi-monotone, $V \subset F_2 \circ F_1(C') = F(C)$. Therefore, $V \subset F_2(C)$, as $F_2 \circ F_1(C') \subset F_2(C)$, and $F_2$ is quasi-monotone by Theorem 7.1.

(ii) Let $V$ be an open connected set in $Y$. Let $C$ be a component of $F^{-1}(V)$, and let $Q$ be a component of $F_2^{-1}(V)$ such that $C$ contains a component of $F_1^{-1}(Q)$. Then, since $F_1$ is quasi-monotone, $Q \subset F_1(C)$. Further, $F_2$ quasi-monotone implies that $V \subset F_2(Q)$. Thus $V \subset F_2 \circ F_1(C) = F(C)$.

THEOREM 7.5. *Let $X$ and $Y$ be locally connected and let $F: X \to Y$ be* s.s.v. *Then $F$ is quasi-monotone if and only if there exists a locally connected continuum $Z$, a continuous monotone function $F_1$ of $X$ onto $Z$ and a continuous, light, open function $F_2$ of $Z$ onto $Y$ such that $F = F_2 \circ F_1$.*

*Proof.* If such a $Z$, $F_1$, and $F_2$ exist, then $F$ is quasi-monotone by Corollary 7.2 and by Theorem 7.4, Part (ii). If $F$ is quasi-monotone, then by Theorem 4.5 there exists a continuum $Z$, and a monotone, single-valued function $F_1$ of $X$ onto $Z$, and there exists a continuous, light function $F_2$ of $Z$ onto $Y$, such that $F = F_2 \circ F_1$. By Theorem 7.4, $F_2$ is quasi-monotone and therefore, by Theorem 7.3, $F_2$ is open.

Finally, combining the results of Theorem 7.3, the fact that monotone functions are quasi-monotone, and Theorem 7.5, we have the following result for semi-single-valued functions.

THEOREM 7.6. *A topological property of locally connected continua is invariant under quasi-monotone, semi-single-valued functions if and only if it is invariant under both monotone and light, open, semi-single-valued functions.*

**8. Local properties and functions with finite images.** In previous sections we have exhibited examples of functions that did not preserve local properties. We saw that even if $F(x)$ was an arc for each $x$, the image of the unit interval need not be locally connected. The purpose of this section is to show that if $F(x)$ is finite for each $x \in X$, then local properties may be preserved. The main theorem is this: If $F$ is defined on a locally connected metric continuum $X$ onto a metric continuum $Y$, and if $F(x)$ is finite for each $x$, then $Y$ is locally connected.

NOTATION. Designate the number of points in $F(x)$ by $N(F(x))$, and if $N(F(x)) \leqq n$ for all $x$, then write $N(F) \leqq n$. $N(F) = n$ means that $N(F) \leqq n$ and there is at least one $x$ such that $N(F(x)) = n$. If $F(x)$ is finite for each $x$, write $N(F) < \infty$. Finally, $N(F) \equiv n$ means $N(F(x)) = n$ for all $x \in X$.

LEMMA 8.1. *Let $F: X \to Y$ be continuous with $N(F) < \infty$. If $K$ is a connected subset of $X$, then $F(K)$ has at most $n$ components, where $n =$ min $N(F(x))$. If $C$ is a component of $F(K)$ and if $x \in K$, then $F(x) \cap C \neq \phi$.*

*Proof.* Let $C$ be a component of $F(K)$ and let $x \in K$. Suppose $F(x) \cap C = \phi$. Define $K_1 = \{x \in K: F(x) \cap C = \phi\}$ and $k_2 = \{x \in K: F(x) \cap C \neq \phi\}$. Clearly $K_1$, $K_2 \neq \phi$ and $K = K_1 \cup K_2$. Also $K_2 \subset F^{-1}(\bar{C})$ and so $\bar{K}_2 \cap K_1 = \phi$. If $x \in \bar{K}_1 \cap K_2$, then $F(x) \cap C \neq \phi$ and $x \in \bar{K}_1$ implies there is an $x' \in K_1$ such that $F(x') \cap C \neq \phi$, a contradiction. Hence $F(x) \cap C \neq \phi$. Finally since $n = \min N(F(x))$, $x \in K$, there can be at most $n$ components of $F(K)$.

PROPOSITION 8.2. Let $F: X \to Y$ be open, continuous, and onto with $N(F) < \infty$. Then the following statements hold:

(i) $X$ locally compact implies $Y$ locally compact;

(ii) $X$ locally connected implies $Y$ locally connected.

*Proof.* Both proofs are done at once. Let $y \in Y$ and $x \in X$ such that $y \in F(x)$. Let $F(x) = \{y, y_1, \cdots, y_k\}$, and let $V_0, V_1, \cdots, V_k$ be disjoint open sets containing $y, y_1, \cdots, y_k$, respectively. Then there exists an open set $U$ with $\bar{U}$ compact (or $U$ connected) such that $F(\bar{U}) \subset \bigcup_{i=0}^{k} V_i$, and $F(\bar{U}) \cap V_j \neq \phi$ for all $j$. Then since $F$ is open, $F(U)$ and hence $F(U) \cap V_0$ is open. Further, $F(\bar{U}) \subset \bigcup_{i=0}^{k} V_i$. Thus, when $\bar{U}$ is compact, $F(\bar{U}) \cap V_0$ is compact and (i) is proved. Moreover, when $U$ is connected, $F(U)$ has exactly $k+1$ components $C_i$, each of which is open. If $C_0$ is the component of $F(U)$ containing $y$, then $C_0 \subset V_0$ and $C_0$ is connected. Hence $Y$ is locally connected.

We now state one of the main results of this section.

THEOREM 8.3. *Let $X$ and $Y$ be compact metric spaces and let $F: X \rightarrow Y$ be continuous and onto with $N(F) < \infty$. If $X$ is locally connected, then $Y$ is locally connected.*

*Proof.* We shall show that $Y$ has property $S$. Let $\varepsilon > 0$. Let $x \in X$ and $F(x) = \{y_1, \cdots, y_k\}$. There exist open sets $V_1, \cdots, V_k$ in $Y$ with $d(V_i) < \varepsilon$ for each $i$, and $V_i \cap V_j = \phi$, $i \neq j$, such that $y_i \in V_i$ for each $i$. Since $X$ is compact and locally connected, there exists an open connected set $U_x$ containing $x$ such that $F(U_x) \subset \bigcup_{i=1}^{k} V_i$. Thus, by Lemma 8.1, $F(U_x)$ has $k$ components each of which has diameter less than $\varepsilon$. We obtain such a $U_x$ for each $x$ and extract a finite subcover $U_{x_1}, \cdots, U_{x_q}$. Then, if $F(U_{x_j})$ has components $C_{j1}, \cdots, C_{jn_j}$, the collection $\{C_{ij}: i = 1, \cdots, q, j = 1, \cdots, n_i\}$ is a finite cover of $Y$ by connected sets of diameter less than $\varepsilon$. Hence, $Y$ has property $S$ and is locally connected.

COROLLARY 8.4. *Let $X$ be a locally connected, metric continuum, $Y$ a metric space, and let $F: X \rightarrow Y$ be continuous. If $N(F) < \infty$, and* min $N(F(x)) = n$, *then $F(X)$ is the union of at most $n$ locally connected, metric continua.*

PROPOSITION 8.5. If $F: X \rightarrow Y$ is a continuous function with $N(F) \leqq n$ and if $C$ is a component of $F(X)$, then $F^*: X \rightarrow C$ defined by $F^*(x) = F(x) \cap C$ is continuous with $N(F^*) \leqq n - r$, where $r$ is the number of other components of $F(X)$.

*Proof.* Since $C$ is a component of $F(X)$ there is an open subset of $Y$ which contains $C$ and does not meet any other component of $F(X)$. By Lemma 8.1, $F(x) \cap C \neq \phi$ for all $x \in X$. Thus the result follows by Lemma 1.1.

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