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L. Carlitz

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L. CARLITZ

Put

$$S(c) = \sum_{x,y=1}^{p-1} e(x + y + cx'y') ,$$

Where $e(x) = e^{2\pi i/p}$ **and** $xx' \equiv yy' \equiv 1$ **(mod** p**), Mordell has conjectured that** $S(c) = O(p)$**. The writer shows first, by an elementary argument that** $S(c) = O(p^{3/2})$**. Next he proves, using a theorem of Lang and Weil that** $S(c) = O(p^{11/8})$**. Finally he proves that** $S(c) = O(p^{5/4})$**; the proof makes use of the estimate**

$$\sum_{x=0}^{p-1} \psi(f(x)) = O(p^{1/2}) ,$$

where $\psi(a)$ **is the Legendre symbol and** $f(x)$ **is a polynomial of the fourth degree.**

If we put

$$K(a, b) = \sum_{x=1}^{p-1} e(ax + bx') ,$$

where $ab \not\equiv 0 \pmod{p}$, it is known that

$$|K(a, b)| \leqq 2p^{1/2} . \tag{2}$$

For proof of (2) see [1], [4].

Since

$$S = \sum_{x=1}^{p-1} e(ax) \sum_{y=1}^{p-1} e(by + cx'y')$$
$$= \sum_{x=1}^{p-1} e(ax)K(b, cx') ,$$

it follows that

$$|S| \leqq \sum_{x=1}^{p-1} |K(b, cx')| \leqq 2(p-1)p^{1/2}$$

by (2). Thus, assuming (2), we get

$$S = O(p^{3/2}) . \tag{3}$$

However it is not difficult to prove (3) directly without making use of (2). Put

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$$(4) \qquad S(c) = \sum_{x,y=1}^{p-1} e(x + y + cx'y') .$$

There is evidently no loss in generality in taking $a = b = 1$. Then we have

$$\sum_{c=0}^{p-1} | S(c) |^2 = \sum_{c=0}^{p-1} \sum_{x,y=1}^{p-1} \sum_{u,v=1}^{p=1} e\{x + y - u\, v + c(x'y' - u'v')\}$$
$$= p \sum_{xy \equiv uv (\mathrm{mod}\ p)} e(x + y - u - v) .$$

But

$$\sum_{xy \equiv uv (\mathrm{mod}\ p)} e(x + y - u - v) = \sum_{x,y,u=1}^{p-1} e(x + y - u - xyu')$$
$$= \sum_{y,u=1}^{p-1} e(y - u) \sum_{x=1}^{p-1} e\{x(1 - yu')\}$$
$$= - \sum_{y,u=1}^{p-1} \mathrm{e}(y - u) + \sum_{y,u=1}^{p-1} e(y - u) \sum_{x=0}^{p-1} e\{x(1 - yu')\}$$
$$= - 1 + p \sum_{y=1}^{p-1} 1 = p^2 - p - 1 ,$$

so that

$$(5) \qquad \sum_{c=0}^{p-1} | S(c) |^2 = p^3 - p^2 - p .$$

It follows at once from (5) that

$$(6) \qquad | S(c) | < p^{3/2} ,$$

so that we have proved (3).

2. Generalizing (4) we define

$$(7) \qquad S_n(c) = \sum_{x_1, \cdots, x_n = 1}^{p-1} e(x_1 + \cdots + x_n + cx_1' \cdots x_n') .$$

We shall show that

$$(8) \qquad S_n(c) = O(p^{1/2(n+1)}) .$$

Exactly as above we have

$$(9) \qquad \sum_c | S_n(c) |^2 = p \sum_{x_1, \cdots, x_n} \sum_{y_1, \cdots, y_n} e(x_1 + \cdots + x_n - y_1 - \cdots - y_n) ,$$

where the summation is over all x_j, y_j such that

$$x_1 x_2 \cdots x_n \equiv y_1 y_2 \cdots y_n , \qquad x_j \not\equiv 0 , \quad y_j \not\equiv 0 \pmod{p} .$$

Let T_n denote the sum on the right of (9). Then we have

$$T_n = \sum e(x_1 + \cdots + x_n - y_1 - \cdots - y_{n-1} - x_1 \cdots x_n y_1' \cdots y_{n-1}')$$
$$= \sum_{\substack{x_1,\cdots,x_{n-1} \\ y_1,\cdots,y_{n-1}}} e(x_1 + \cdots + x_{n-1} - y_1 - \cdots - y_{n-1}) \cdot \sum_x e[(1 - x_1 \cdots x_{n-1} y_1' \cdots y_{n-1}')x] .$$

The inner sum is equal to

$$\begin{cases} p-1 & (x_1 \cdots x_{n-1} \equiv y_1 \cdots y_{n-1}) \\ -1 & (x_1 \cdots x_{n-1} \not\equiv y_1 \cdots y_{n-1}) , \end{cases}$$

so that

$$T_n = pT_{n-1} - \sum_{\substack{x_1,\cdots,x_{n-1} \\ y_1,\cdots,y_{n-1}}} e(x_1 + \cdots + x_{n-1} - y_1 - \cdots - y_{n-1}) .$$

Hence

(10) $$T_n = pT_{n-1} - 1 .$$

Now

$$T_1 = \sum_{x \equiv y} e(x - y) = p - 1 , \qquad T_2 = p(p-1) - 1 = p^2 - p - 1$$

and generally

(11) $$T_n = p^n - p^{n-1} - \cdots - 1 .$$

Thus (9) becomes

(12) $$\sum_c | S_n(c) |^2 = p^{n+1} - p^n - \cdots - p$$

and (8) follows at once.

It follows from (12) that

$$S_n(c) = o(p^{n/2})$$

cannot hold for all c.

3. Returning to (4) we shall now show that

(13) $$S(c) = O(p^{11/8}) .$$

It is convenient to put

$$S(a, b, c) = \sum_{x,y} e(ax + by + cx'y') .$$

Then

$$\sum_{a=1}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} |S(a, b, c)|^4 = p^3 N, \tag{14}$$

where N denotes the number of solutions of the system

$$\begin{cases} x_1 + x_2 \equiv x_3 + x_4 \\ y_1 + y_2 \equiv y_3 + y_4 \\ x_1'y_1' + x_2'y_2' \equiv x_3'y_3' + x_4'y_4' \\ x_1x_2x_3x_4y_1y_2y_3y_4 \not\equiv 0. \end{cases}$$

Eliminating x_4, y_4 it follows that N is the number of solutions of

$$\begin{aligned}(x_1y_1 + x_2y_2)x_3y_3(x_1 + x_2 - x_3)(y_1 + y_2 - y_3) \\ \equiv x_1y_1x_2y_2[(x_1 + x_2 - x_3)(y_1 + y_2 - y_3) + x_3y_3]\end{aligned} \tag{15}$$

such that

$$x_1x_2x_3y_1y_2y_3(x_1 + x_2 - x_3)(y_1 + y_2 - y_3) \not\equiv 0 . \tag{16}$$

Now by a theorem of Lang and Weil [2] we have

$$N = p^5 + O(p^{5-1/2}),$$

so that (14) becomes

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} |S(a, b, c)|^4 = p^8 + O(p^{15/2}) . \tag{17}$$

On the other hand

$$\begin{aligned}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} |S(a, b, c)|^4 &= |S(0, 0, 0)|^4 + 3\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} |S(a, b, 0)|^4 \\ &+ 3\sum_{a=1}^{p-1} |S(a, 0, 0)|^4 + \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} |S(a, b, c)|^4 \\ &= (p-1)^8 + (p-1)^2 + 3(p-1)^5 + (p-1)^2 \sum_{c=1}^{p-1} |S(c)|^4 ,\end{aligned}$$

so that (17) reduces to

$$\sum_{c=1}^{p-1} |S(c)|^4 = O(p^{11/2}) . \tag{18}$$

Clearly (18) implies (13).

4. If an exact formula for

$$\sum_{c=0}^{p-1} |S(c)|^4$$

were available we should presumably be able to prove

(19) $$S(c) = O(p^{5/4}) .$$

In this connection it may be of interest to remark that the sum

(20) $$\sum_{c=0}^{p-1} S^3(c)$$

can be evaluated. Indeed if we put

$$S(a, b, c) = \sum_{x,y} e(ax + by + cx'\,y') ,$$

then

(21) $$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} (S(a, b, c))^3 = p^3 N ,$$

where N denotes the number of solutions of the system

(22) $$\begin{cases} x_1 + x_2 + x_3 \equiv 0 \\ y_1 + y_2 + y_3 \equiv 0 \\ x_1'y_1' + x_2'y_2' + x_3'y_3' \equiv 0 \\ x_1x_2x_3y_1y_2y_3 \not\equiv 0 . \end{cases}$$

Eliminating x_3, y_3, we find that (22) reduces to

(23) $$x_1(x_1 + x_2)y_1^2 + (x_1^2 + 3x_1x_2 + x_2^2)y_1y_2 + x_2(x_1 + x_2)y_2^2 \equiv 0$$

together with

(24) $$x_1x_2y_1y_2(x_1 + x_2)(y_1 + y_2) \not\equiv 0 .$$

We may replace (23) by

(25) $$[(x_1 + x_2)y_1 + x_2y_2][x_1y_1 + (x_1 + x_2)y_2] = 0 .$$

If $x_1x_2(x_1 + x_2)y_1 \not\equiv 0$, it is clear from (25) that $y_2 \not\equiv 0$ and $y_1 - y_2 \not\equiv 0$. The two factors in (25) may vanish simultaneously. This will happen when

(26) $$x_1^2 + x_1x_2 + x_2^2 \equiv 0 ,$$

that is when -3 is a quadratic residue of p; moreover if x_1, x_2 satisfy (26) with $x_1x_2 \not\equiv 0$ then $x_1 + x_2 \not\equiv 0$. Thus the number of solutions of (26) is equal to

$$\left\{1 + \left(\frac{-3}{p}\right)\right\}\frac{p-1}{2} .$$

If -3 is a nonresidue we find that

(27) $$N = 2(p-1)^2(p-2),$$

while, if -3 is a residue,

(28) $$N = 2(p-1)^2(p-2) - (p-1)^2.$$

For $p = 3$ we have

(29) $$N = 4,$$

for it is evident from (22) that $x_1 \equiv x_2 \equiv x_3$, $y_1 \equiv y_2 \equiv y_3$.

Combining (27) and (28) we have

(30) $$N = 2(p-1)^2(p-2) - \left\{1 + \left(\frac{-3}{p}\right)\right\}\frac{(p-1)^2}{2} \qquad (p > 3).$$

On the other hand, since

$$S(0, 0, 0) = (p-1)^2 S(a, 0, 0) = -(p-1) \qquad (a \not\equiv 0),$$
$$S(a, b, 0) = 1 \qquad (ab \not\equiv 0),$$

we have

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}(S(a, b, c))^3 = (p-1)^6 - 3(p-1)^4 + 3(p-1)^2 + \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}(S(a, b, c))^3$$
$$= (p-1)^6 - 3(p-1)^4 + 3(p-1)^2 + (p-1)^2\sum_{c=1}^{p-1}(S(c))^3.$$

Therefore, using (21) and (30), we get

(31) $$\sum_{c=1}^{p-1}(S(c))^3 = 2p^3(p-2) - (p-1)^4 + 3(p-1)^2 - 3 - \frac{1}{2}\left\{1 + \left(\frac{-3}{p}\right)\right\}.$$

5. We shall now show that

(32) $$S(c) = O(p^{5/4}).$$

With the notation of § 3 we have, as above,

(33) $$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}|S(a, b, c)|^4 = p^3 N,$$

where N is the number of solutions of the system

$$(34)\qquad \begin{cases} (x_1 + x_2)x_3x_4 \equiv x_1x_2(x_3 + x_4) \\ (y_1 + y_2)y_3y_4 \equiv y_1y_2(y_3 + y_4) \\ x_1y_1 + x_2 y_2 \equiv x_3y_3 + x_4y_4 \\ x_1x_2x_3x_4y_1y_2y_3y_4 \not\equiv 0 \,. \end{cases}$$

Note that we have replaced each x_j, y_j by its reciprocal (mod p).

If we put

$$x_3 = x_1u_1 \,, \qquad x_4 = x_2u_2 \,, \qquad y_3 = y_1v_1 \,, \qquad y_4 = y_2v_2 \,,$$

(34) becomes

$$(35)\qquad \begin{cases} (x_1 + x_2)u_1u_2 \equiv x_1u_1 + x_2 u_2 \\ (y_1 + y_2)v_1v_2 \equiv y_1v_1 + y_2v_2 \\ x_1y_1 + x_2y_2 \equiv x_1y_1u_1v_1 + x_2y_2u_2v_2 \\ x_1x_2y_1y_2u_1u_2v_1v_2 \not\equiv 0 \,. \end{cases}$$

Now put $x_2 = x_1x$, $y_2 = y_1y$ and (35) reduces to

$$(36)\qquad \begin{cases} (1 + x)u_1u_2 \equiv u_1 + xu_2 \\ (1 + y)v_1v_2 \equiv v_1 + yv_2 \\ 1 + xy \equiv u_1v_1 + xyu_2v_2 \\ xyx_1y_1u_1v_1u_2v_2 \not\equiv 0 \,. \end{cases}$$

Finally, eliminating x, y we get the single equation

$$(37)\qquad \frac{(1 - u_1)(1 - v_1)(1 - u_1v_1)}{u_1v_1} + \frac{(1 - u_2)(1 - v_2)(1 - u_2v_2)}{u_2v_2} \equiv 0$$

subject to

$$(38)\qquad x_1y_1u_1v_1u_2v_2 \not\equiv 0 \,.$$

It should be noted that for fixed u_1, v_1, u_2, v_2 satisfying (37), x, y are uniquely determined by (36) unless $u_1 \equiv u_2 \equiv v_1 \equiv v_2 \equiv 1$; also we find that the forbidden cases $xy \equiv 0$ or xy "infinite" contribute $O(p^2)$.

Let $N'(k)$ denote the number of solutions u, $v \not\equiv 0$ of

$$(39)\qquad (1 - u)(1 - v)(1 - uv) \equiv kuv$$

and let $N(k)$ denote the total number of solutions of (39), so that

$$N(k) = N'(k) + O(1) \,.$$

Then clearly the number of nonzero solutions of (37) is equal to

$$(40)\qquad \sum_{k=0}^{p-1} N(k)N(- k) + O(p^2) \,.$$

Let $\psi(a)$ denote the Legendre symbol (a/p). Then for fixed u and k, the number of solutions of (39) is equal to

$$1 + \psi\{(1 + ku - u^2)^2 - 4u(1 - u)^2\} ,$$

so that

$$N(k) = p + \sum_{u=0}^{p-1} \psi(f(k, u)) ,$$

where

$$f(k, u) = (1 + ku - u^2)^2 - 4u(1 - u)^2 . \tag{41}$$

Thus (40) becomes

$$\begin{aligned} p^3 + 2p \sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \psi(f(k, u)) \\ + \sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \psi(f(k, u))\psi(f(-k, v)) + O(p^2) . \end{aligned} \tag{42}$$

Since $f(k, u)$ is a quadratic in k we have

$$\sum_{k=0}^{p-1} \psi(f(k, u)) = -1$$

unless $u(1 - u) \equiv 0$. It follows that

$$\sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \psi(f(k, u)) = O(p^2) . \tag{43}$$

Consider next the sum

$$\sum_{u=0}^{p-1} \psi(f(k, u)) .$$

It is easily seen from (41) that for fixed k, $f(k, u)$ is the square of a polynomial in u only when $k \equiv 0$. We therefore have the estimate

$$\sum_{u=0}^{p-1} \psi(f(k, u)) = O(p^{1/2}) , \tag{44}$$

so that

$$\sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \psi(f(k, u))\psi(f(-k, u)) = O(p^2) . \tag{45}$$

Substituting from (43) and (45) in (42) we see that the number of nonzero solutions (37) is

$$p^3 + O(p^2) .$$

Therefore N, the number of solutions of (34) is

$$p^5 + O(p^4)$$

and (33) becomes

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} | S(a, b, c) |^4 = p^8 + O(p^7) ;$$

since $S(0, 0, 0) = p^2$,

$$S(a, b, c) = S(1, 1, abc) \qquad (abc \not\equiv 0)$$

and there are $(p - 1)^2$ terms $S(a, b, c)$ in the sum that give the same $S(1, 1, c)$, (32) now follows immediately.

Note that, except for (44), the proof is elementary.

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