SYMMETRIC DUAL NONLINEAR PROGRAMS

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Consider a function $K(x, y)$ continuously differentiable in $x \in R^n$ and $y \in R^m$. We form two problems:

PRIMAL: Find $(x, y) \geq 0$ and Min F such that

$$F = K(x, y) - y^T D_y K(x, y), \ D_y K(x, y) \leq 0$$

DUAL: Find $(x, y) \geq 0$ and Max G such that

$$G = K(x, y) - x^T D_x K(x, y), \ D_x K(x, y) \geq 0$$

where $D_y K(x, y)$ and $D_x K(x, y)$ denote the vectors of partial derivatives $D_{y_i} K(x, y)$ and $D_{x_j} K(x, y)$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Our main result is the existence of a common extremal solution (x_0, y_0) to both the primal and dual systems when (i) an extremal solution (x_0, y_0) to the primal exists, (ii) K is convex in x for each y, concave in y for each x and (iii) K, twice differentiable, has the property at (x_0, y_0) that its matrix of second partials with respect to y is negative definite.

F and G are related to the conjugate functions considered by Fenchel [6] (see also [9]) and to the Legendre transforms of K.

Special Cases.

A. If we set $K(x, y) = c^T x + b^T y - y^T A x$, we obtain von Neumann's symmetric formulation of primal and dual linear programs: see [3] or [9].

PRIMAL: Find $x \geq 0$, Min F such that $F = c^T x$, $Ax \geq b$

DUAL: Find $y \geq 0$, Max G such that $G = b^T y$, $A^T y \leq c$.

B. Symmetric dual quadratic programs [2] can be obtained by setting

$$K(x, y) = c^T x + b^T y - y^T A x + \frac{1}{2} (x^T D x - y^T E y) .$$

The dual quadratic programs of Dorn [4] result when $E = 0$.

C. If we set $K(x, y) = F_0(x) - \sum_{i=1}^{m} y_i F_i(x)$, we obtain the nonlinear statements of duality by Wolfe [12] and Huard [7]. For y not sign restricted, the problems may be simplified to

PRIMAL: Find $x \geqq 0$ and Min $F_0(x)$ such that

$$F_i(x) = 0, \quad i = 1, \cdots, m .$$

DUAL: Find $x \geqq 0$, y and Max $K(x, y)$ such that

$$D_{x_j}K(x, y) \geqq 0, \quad j = 1, \cdots, n .$$

A " Weak " Duality Theorem.

With additional conditions on $K(x, y)$ we can state a relation between solutions of the primal and dual systems.

THEOREM 1.   *If $K(x, y)$ is convex in x, for each y, and concave in y, for each x, then any (not necessarily extremal) solutions (x, y) of the primal and (u, v) of the dual satisfy the inequality*

$$F(x, y) \geqq G(u, v)$$

where equality holds only if (x, v) is orthogonal to

$$[D_u K(u, v), \, - D_y K(x, y)] .$$

Proof.   By the assumptions of convexity, concavity, and differentiability

$$K(x, v) - K(u, v) \geqq (x - u)^T D_u K(u, v)$$
$$K(x, v) - K(x, y) \leqq (v - y)^T D_y K(x, y) .$$

Subtracting and rearranging, we get

$$[K(x, y) - y^T D_y K(x, y)] - [K(u, v) - u^T D_u K(u, v)]$$
$$\geqq x^T D_u K(u, v) - v^T D_y K(x, y) \geqq 0 .$$

REMARK.   The significance of Theorem 1 is apparent when in addition it is also true that $F(x^0, y^0) = G(u^0, v^0)$ for some (x^0, y^0) and (u^0, v^0); in that case, (x^0, y^0) and (u^0, v^0) are extremal solutions of the primal and dual, respectively.   This is illustrated in the theorem below.

A Strong Duality Theorem.

Under certain assumptions, we can show that if one of the systems can be extremized, the two systems possess a solution in common. (References on duality, in addition to those mentioned above, are [11], [5], [1].)

THEOREM 2.   *If (x^0, y^0) is an extremal solution for the primal where $K(x, y)$ is twice differentiable and $- D_{yy}K(x^0, y^0)$ is positive definite, then (x^0, y^0) satisfies the dual constraints with $F(x^0, y^0) = G(x^0, y^0)$.   If, in addition, $K(x, y)$ is convex in x, for each y, and concave in y, for each x, then*

$$\text{Min } F(x, y) = F(x^0, y^0) = G(x^0, y^0) = \text{Max } G(x, y) \ .$$

Proof. By a result of John [8, Theorem 1], see also Kuhn and Tucker [10], there exist "multipliers" $v_0 \geqq 0$ (scalar) and $v \geqq 0$ (m-vector) such that $(v_0, v) \neq 0$, $v^T D_y(x^0, y^0) = 0$, and

$$v_0 D_y F(x^0, y^0) + D_{yy} K(x^0, y^0) v \geqq 0$$
$$(y^0)^T [v^0 D_y F(x^0, y^0) + D_{yy} K(x^0, y^{0)} v] = 0$$
$$v_0 D_x F(x^0, y^0) + D_{xy} K(x^0, y^0) v \geqq 0$$
$$(x^0)^T [v_0 D_x F(x^0, y^0) + D_{xy} K(x^0, y^0) v] = 0 \ .$$

Replacing F by its definition, $K(x, y) - y^T D_y K(x, y)$, yields

(1) $\qquad D_{yy} K(x^0, y^0)(v - v_0 y^0) \geqq 0$

(2) $\qquad (y^0)^T D_{yy} K(x^0, y^0)(v - v_0 y^0) = 0$

(3) $\qquad v_0 D_x K(x^0, y^0) + D_{xy} K(x^0, y^0)(v - v_0 y^0) \geqq 0$

(4) $\qquad (v_0 x^0)^T D_x K(x^0, y^0) + (x^0)^T D_{xy} K(x^0, y^0)(v - v_0 y^0) = 0 \ .$

Multiplying (1) by $v^T \geqq 0$, (2) by v_0, and then subtracting, we obtain

$$(v - v_0 y^0)^T D_{yy} K(x^0, y^0)(v - v_0 y^0) \geqq 0 \ .$$

But since the matrix $- D_{yy} K(x^0, y^0)$ is positive definite, we have $v - v_0 y^0 = 0$. Relations (3) and (4) yield

(5) $\qquad\qquad\qquad v_0 D_x K(x^0, y^0) \geqq 0 \ ,$

(6) $\qquad\qquad\qquad (v_0 x^0)^T D_x K(x^0, y^0) = 0 \ .$

Moreover, since $v^T D_y K(x^0, y^0) = 0$, we have

(7) $\qquad\qquad\qquad (v_0 y^0)^T D_y K(x^0, y^0) = 0 \ .$

But $v_0 > 0$ for otherwise $v = v_0 y^0 = 0$ and $(v_0, v) = 0$, a contradiction. Dividing (5), (6), and (7) by the positive scalar v_0 produces relations which imply that (x^0, y^0) satisfies the dual constraints as well as the equation $F(x^0, y^0) = G(x^0, y^0)$. The final assertion of the theorem is an application of Theorem 1.

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