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SYMMETRIC DUAL NONLINEAR PROGRAMS

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Consider a function $K(x, y)$ continuously differentiable in $x \in R^n$ and $y \in R^m$. We form two problems:

PRIMAL: Find $(x, y) \geq 0$ and Min F such that

$$F = K(x, y) - y^T D_y K(x, y), \quad D_y K(x, y) \leq 0$$

DUAL: Find $(x, y) \geq 0$ and Max G such that

$$G = K(x, y) - x^T D_x K(x, y), \quad D_x K(x, y) \geq 0$$

where $D_y K(x, y)$ and $D_x K(x, y)$ denote the vectors of partial derivatives $D_{y_i} K(x, y)$ and $D_{x_j} K(x, y)$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. Our main result is the existence of a common extremal solution (x_0, y_0) to both the primal and dual systems when (i) an extremal solution (x_0, y_0) to the primal exists, (ii) K is convex in x for each y , concave in y for each x and (iii) K , twice differentiable, has the property at (x_0, y_0) that its matrix of second partials with respect to y is negative definite.

F and G are related to the conjugate functions considered by Fenchel [6] (see also [9]) and to the Legendre transforms of K .

Special Cases.

A. If we set $K(x, y) = c^T x + b^T y - y^T A x$, we obtain von Neumann's symmetric formulation of primal and dual linear programs: see [3] or [9].

PRIMAL: Find $x \geq 0$, Min F such that $F = c^T x, Ax \geq b$

DUAL: Find $y \geq 0$, Max G such that $G = b^T y, A^T y \leq c$.

B. Symmetric dual quadratic programs [2] can be obtained by setting

$$K(x, y) = c^T x + b^T y - y^T A x + \frac{1}{2} (x^T D x - y^T E y).$$

The dual quadratic programs of Dorn [4] result when $E = 0$.

C. If we set $K(x, y) = F_0(x) - \sum_{i=1}^m y_i F_i(x)$, we obtain the nonlinear statements of duality by Wolfe [12] and Huard [7]. For y not sign restricted, the problems may be simplified to

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PRIMAL: Find $x \geq 0$ and Min $F_0(x)$ such that
 $F_i(x) = 0, i = 1, \dots, m.$

DUAL: Find $x \geq 0, y$ and Max $K(x, y)$ such that
 $D_{x_j}K(x, y) \geq 0, j = 1, \dots, n.$

A "Weak" Duality Theorem.

With additional conditions on $K(x, y)$ we can state a relation between solutions of the primal and dual systems.

THEOREM 1. *If $K(x, y)$ is convex in x , for each y , and concave in y , for each x , then any (not necessarily extremal) solutions (x, y) of the primal and (u, v) of the dual satisfy the inequality*

$$F(x, y) \geq G(u, v)$$

where equality holds only if (x, v) is orthogonal to

$$[D_u K(u, v), -D_y K(x, y)].$$

Proof. By the assumptions of convexity, concavity, and differentiability

$$\begin{aligned} K(x, v) - K(u, v) &\geq (x - u)^T D_u K(u, v) \\ K(x, v) - K(x, y) &\leq (v - y)^T D_y K(x, y). \end{aligned}$$

Subtracting and rearranging, we get

$$\begin{aligned} [K(x, y) - y^T D_y K(x, y)] - [K(u, v) - u^T D_u K(u, v)] \\ \geq x^T D_u K(u, v) - v^T D_y K(x, y) \geq 0. \end{aligned}$$

REMARK. The significance of Theorem 1 is apparent when in addition it is also true that $F(x^0, y^0) = G(u^0, v^0)$ for some (x^0, y^0) and (u^0, v^0) ; in that case, (x^0, y^0) and (u^0, v^0) are extremal solutions of the primal and dual, respectively. This is illustrated in the theorem below.

A Strong Duality Theorem.

Under certain assumptions, we can show that if one of the systems can be extremized, the two systems possess a solution in common. (References on duality, in addition to those mentioned above, are [11], [5], [1].)

THEOREM 2. *If (x^0, y^0) is an extremal solution for the primal where $K(x, y)$ is twice differentiable and $-D_{yy}K(x^0, y^0)$ is positive definite, then (x^0, y^0) satisfies the dual constraints with $F(x^0, y^0) = G(x^0, y^0)$. If, in addition, $K(x, y)$ is convex in x , for each y , and concave in y , for each x , then*

$$\text{Min } F(x, y) = F(x^0, y^0) = G(x^0, y^0) = \text{Max } G(x, y) .$$

Proof. By a result of John [8, Theorem 1], see also Kuhn and Tucker [10], there exist “multipliers” $v_0 \geq 0$ (scalar) and $v \geq 0$ (m -vector) such that $(v_0, v) \neq 0$, $v^T D_y(x^0, y^0) = 0$, and

$$\begin{aligned} v_0 D_y F(x^0, y^0) + D_{yy} K(x^0, y^0) v &\geq 0 \\ (y^0)^T [v_0 D_y F(x^0, y^0) + D_{yy} K(x^0, y^0) v] &= 0 \\ v_0 D_x F(x^0, y^0) + D_{xy} K(x^0, y^0) v &\geq 0 \\ (x^0)^T [v_0 D_x F(x^0, y^0) + D_{xy} K(x^0, y^0) v] &= 0 . \end{aligned}$$

Replacing F by its definition, $K(x, y) - y^T D_y K(x, y)$, yields

$$\begin{aligned} (1) \quad D_{yy} K(x^0, y^0) (v - v_0 y^0) &\geq 0 \\ (2) \quad (y^0)^T D_{yy} K(x^0, y^0) (v - v_0 y^0) &= 0 \\ (3) \quad v_0 D_x K(x^0, y^0) + D_{xy} K(x^0, y^0) (v - v_0 y^0) &\geq 0 \\ (4) \quad (v_0 x^0)^T D_x K(x^0, y^0) + (x^0)^T D_{xy} K(x^0, y^0) (v - v_0 y^0) &= 0 . \end{aligned}$$

Multiplying (1) by $v^T \geq 0$, (2) by v_0 , and then subtracting, we obtain

$$(v - v_0 y^0)^T D_{yy} K(x^0, y^0) (v - v_0 y^0) \geq 0 .$$

But since the matrix $-D_{yy} K(x^0, y^0)$ is positive definite, we have $v - v_0 y^0 = 0$. Relations (3) and (4) yield

$$\begin{aligned} (5) \quad v_0 D_x K(x^0, y^0) &\geq 0 , \\ (6) \quad (v_0 x^0)^T D_x K(x^0, y^0) &= 0 . \end{aligned}$$

Moreover, since $v^T D_y K(x^0, y^0) = 0$, we have

$$(7) \quad (v_0 y^0)^T D_y K(x^0, y^0) = 0 .$$

But $v_0 > 0$ for otherwise $v = v_0 y^0 = 0$ and $(v_0, v) = 0$, a contradiction. Dividing (5), (6), and (7) by the positive scalar v_0 produces relations which imply that (x^0, y^0) satisfies the dual constraints as well as the equation $F(x^0, y^0) = G(x^0, y^0)$. The final assertion of the theorem is an application of Theorem 1.

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