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**A CHARACTERIZATION OF CONDITIONAL PROBABILITY**

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**The conditional probability functions relative to a sub- $\sigma$ -field  $\mathcal{B}$  are shown to constitute a vector-valued measure on the  $\sigma$ -field of the probability space, and it is proved that the conditional expectation relative to  $\mathcal{B}$  of an  $\mathcal{L}_1$  random variable  $X$  is the integral of  $X$  with respect to this vector-valued measure. A complete characterization is given of those vector-valued measures which are conditional probabilities. This machinery is illustratively applied to give alternative derivations of results of Moy and Rota on the characterization of conditional expectation operators.**

Probabilists often regret that, in general, conditional probabilities do not define probability measures almost everywhere. This defect arises naturally from the fact that conditional probabilities are Radon-Nikodym derivatives of certain set functions; and, hence, they are defined only up to equivalence. It seems advantageous to relinquish the concept of conditional probabilities as point functions, almost everywhere determined, and consider them as they are—elements of a function space. Part of the awkwardness of conditional probabilities then disappears: The conditional probabilities form a vector-valued measure such that conditional expectation of an integrable function is its integral with respect to this conditional probability measure.

Throughout this paper  $(\Omega, \mathcal{A}, \mu)$ , will be a fixed probability space and  $\mathcal{B}$  a fixed sub- $\sigma$ -field of  $\mathcal{A}$ .  $\mathcal{L}_1(\Omega, \mathcal{A}, \mu)$ , or simply  $\mathcal{L}_1$ , denotes the Banach space of all complex-valued,  $\mu$ -integrable,  $\mathcal{A}$ -measurable functions on  $\Omega$ .  $\mathcal{E}^{\mathcal{B}}$  will be the conditional expectation operator in  $\mathcal{L}_1$  relative to  $\mathcal{B}$ .  $I_A$  is the indicator of the set  $A$ .

In §1, we show that if  $\phi^{\mathcal{B}}(A) = \mathcal{E}^{\mathcal{B}} I_A$  for  $A \in \mathcal{A}$ , then  $\phi^{\mathcal{B}}$  is a vector-valued measure on  $\mathcal{A}$  with values in  $\mathcal{L}_1$ . Furthermore, we show that if  $X \in \mathcal{L}_1$ , then

$$(0.1) \quad \mathcal{E}^{\mathcal{B}} X = \int X d\phi^{\mathcal{B}},$$

the notion of integral here being defined as in [1, p. 323].

Over the last ten years, several characterizations of  $\mathcal{E}^{\mathcal{B}}$  have appeared. The papers of Moy [3] and Rota [4] characterized  $\mathcal{E}^{\mathcal{B}}$  in

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terms of its properties as a linear operator; Sidak [5] characterized it by means of its action on  $\mathcal{L}_2$ . In §2, our characterization of conditional probability will be completed by identifying those countably additive measures on  $\mathcal{A}$  with values in  $\mathcal{L}_1$  which are conditional probabilities. For such a measure, a formula like (0.1) will hold, and the corresponding sub- $\sigma$ -field will be determined.

In §3, this characterization of conditional probability will be used to obtain the characterization of conditional expectation of Moy [3] and of Rota [4] using a theorem on the representation of operators on a Lebesgue space. In a subsequent publication, further results in [3] and [4] as well as some in [5] will be obtained as applications of our characterization of conditional probabilities.

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**1. Representation of conditional expectation as an integral with respect to a vector-valued measure.** Let the notational conventions introduced above continue to hold. By definition [1, p. 318], a function  $\psi$  on the  $\sigma$ -field  $\mathcal{A}$  to  $\mathcal{L}_1$  is called a vector-valued measure if  $\psi(\phi) = 0$  and  $\psi$  is countably additive on disjoint sets belonging to  $\mathcal{A}$ . If  $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$  such that  $A_i \cap A_j = \phi$  when  $i \neq j$ , then we have from properties of conditional expectations (see [2, p. 347]),

$$(1.1) \quad \varphi^{\mathcal{E}}\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \varphi^{\mathcal{E}}(A_i) \quad \text{a.s.}$$

Since the partial sums  $\sum_{i=1}^n \varphi^{\mathcal{E}}(A_i)$  are uniformly bounded a.s. by 1, it follows that their integrals are uniformly continuous with respect to  $\mu$ . Therefore, by the  $\mathcal{L}_r$ -convergence theorem [2, p. 163],

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \varphi^{\mathcal{E}}(A_i) = \sum_{i=1}^{\infty} \varphi^{\mathcal{E}}(A_i)$$

in  $\mathcal{L}_1$ -norm. Noting (1.1), countable additivity follows.  $\varphi^{\mathcal{E}}(\phi) = 0$  a.s. is immediate. Thus,  $\varphi^{\mathcal{E}}$  is a vector-valued measure.

The total variation of a vector-valued measure need not be finite. Its role in the theory of integration with respect to a vector-valued measure  $\psi$  is assumed by a finite positive set function  $||| \psi |||$  called the semi-variation of  $\psi$ . In general,  $||| \psi |||$  is not additive.  $||| \psi |||$  is additive if and only if  $||| \psi ||| = \nu[\psi]$ , the total variation of  $\psi$  (see [1, p. 320]).

By definition,

$$(1.2) \quad ||| \varphi^{\mathcal{E}}(A) ||| = \sup_{\mathcal{P}, \{\alpha_i\}} \left\| \sum_{i=1}^n \alpha_i \varphi^{\mathcal{E}}(A_i) \right\|, \quad A \in \mathcal{A}$$

where  $\mathcal{P}$  is the collection of all finite partitions of the set  $A$  by elements of  $\mathcal{A}$  and the  $\alpha_i$  are complex numbers such that  $|\alpha_i| \leq 1$ . Recall that

$$(1.3) \quad \begin{aligned} \|\varphi^{\mathcal{P}}(A)\| &= \int |\varphi^{\mathcal{P}}(A)| d\mu \\ &= \int \mathcal{E}^{\mathcal{P}} I_A d\mu = \int_A d\mu = \mu(A), \quad A \in \mathcal{A}. \end{aligned}$$

Therefore, for any partition  $\{A_i\}_{i=1}^n$  of  $A$

$$(1.4) \quad \begin{aligned} \left\| \sum_{i=1}^n \alpha_i \varphi^{\mathcal{P}}(A_i) \right\| &\leq \sum_{i=1}^n |\alpha_i| \|\varphi^{\mathcal{P}}(A_i)\| \\ &\leq \sum_{i=1}^n \mu(A_i) = \mu(A) \end{aligned}$$

since the numbers  $\{\alpha_i\}_{i=1}^n$  all have moduli less than one. But equality of the first and last members of (1.4) is achieved when  $\alpha_1 = \dots = \alpha_n = 1$ , so that  $\|\varphi^{\mathcal{P}}\| = \nu(\mu) = \mu$ .

Now let  $X \in \mathcal{L}_1$ ; we shall show that  $X$  is integrable with respect to  $\varphi^{\mathcal{P}}$  (see [1, p. 323]) and that its integral over  $\Omega$  is  $\mathcal{E}^{\mathcal{P}} X$ . Let  $\{X_n\}_1^{\infty}$  be a sequence of simple functions converging  $\mu$  a.s. to  $X$  and such that  $|X_n| \uparrow X$ . It follows that  $X_n \rightarrow X \varphi^{\mathcal{P}}$  a.e. since  $\|\varphi^{\mathcal{P}}\| = \mu$ . Then [2, p. 348],  $\mathcal{E}^{\mathcal{P}} X_n \rightarrow \mathcal{E}^{\mathcal{P}} X$   $\mu$  a.s. Since

$$(1.5) \quad |\mathcal{E}^{\mathcal{P}} X_n| \leq \mathcal{E}^{\mathcal{P}} |X_n| \leq \mathcal{E}^{\mathcal{P}} |X| \in \mathcal{L}_1,$$

the integrals of the  $\mathcal{E}^{\mathcal{P}} X_n$  are uniformly continuous and therefore

$$(1.6) \quad \mathcal{E}^{\mathcal{P}} X_n \longrightarrow \mathcal{E}^{\mathcal{P}} X \quad \text{in } \mathcal{L}_1\text{-norm.}$$

But if  $X_n = \sum_{i=1}^r \alpha_i I_{A_i}$ , we have

$$(1.7) \quad \begin{aligned} \int X_n d\varphi^{\mathcal{P}} &= \sum_{i=1}^r \alpha_i \varphi^{\mathcal{P}}(A_i) \\ &= \sum_{i=1}^r \alpha_i \mathcal{E}^{\mathcal{P}} I_{A_i} = \mathcal{E}^{\mathcal{P}} \left( \sum_{i=1}^r \alpha_i I_{A_i} \right) = \mathcal{E}^{\mathcal{P}} X_n. \end{aligned}$$

By (1.6) and (1.7), it follows that

$$(1.8) \quad \lim_{n \rightarrow \infty} \int X_n d\varphi^{\mathcal{P}} = \mathcal{E}^{\mathcal{P}} X \quad \text{in } \mathcal{L}_1\text{-norm.}$$

Since (1.8) holds for any function in  $\mathcal{L}_1$ , it follows that

$$(1.9) \quad \lim_{n \rightarrow \infty} \int I_A X_n d\varphi^{\mathcal{P}} = \mathcal{E}^{\mathcal{P}} I_A X \quad \text{in } \mathcal{L}_1\text{-norm}$$

for all  $A \in \mathcal{A}$ .

These last two limit assertions establish, according to definition [1, p. 323], the integrability of  $X$  and the formula

$$(1.10) \quad \int_A X d\varphi^{\mathcal{B}} = \mathcal{E}^{\mathcal{B}} I_A X, \quad A \in \mathcal{A}.$$

In particular, (0.1) holds as asserted.

We gather together the results of this section in the form of a theorem.

**THEOREM 1.** *Let  $(\Omega, \mathcal{A}, \mu)$  be a probability space and  $\mathcal{B}$  a sub- $\sigma$ -field of  $\mathcal{A}$ . Then the conditional probability functions relative to  $\mathcal{B}$ , denoted  $\varphi^{\mathcal{B}}(A)$  for  $A \in \mathcal{A}$ , form a vector-valued measure on  $\mathcal{A}$  with values in  $\mathcal{L}_1(\Omega, \mathcal{A}, \mu)$  having semi-variation  $\mu$ . Furthermore, if  $X \in \mathcal{L}_1$*

$$(1.11) \quad \mathcal{E}^{\mathcal{B}} X = \int X d\varphi^{\mathcal{B}}.$$

2.  $\mathcal{L}_1$ -valued measures which are conditional probabilities. In this section, we determine the  $\mathcal{L}_1(\Omega, \mathcal{A}, \mu)$ -valued measures on  $\mathcal{A}$  which are conditional probabilities.

**THEOREM 2.** *Let  $\psi$  be an  $\mathcal{L}_1(\Omega, \mathcal{A}, \mu)$ -valued measure on  $\mathcal{A}$ . Then  $\psi$  is the vector-valued conditional probability measure relative to some sub- $\sigma$ -field  $\mathcal{B}$  if and only if the following three conditions are satisfied:*

$$(2.1) \quad \begin{aligned} & \text{(i) } \psi(A) \geq 0 \text{ a.s. } \quad A \in \mathcal{A}, \\ & \text{(ii) } \psi(\Omega) = 1 \text{ a.s.,} \\ & \text{(iii) } \| I_A \psi(B) \psi(C) \| = \| \psi(A) I_B \psi(C) \| = \| \psi(A) \psi(B) I_C \| \\ & \hspace{15em} A, B, C \in \mathcal{A}. \end{aligned}$$

*If  $\psi$  has these three properties, then it is conditional probability relative to the  $\sigma$ -field*

$$(2.2) \quad \mathcal{B} = \{A \in \mathcal{A} \mid \psi(A) = I_A \text{ a.s.}\}.$$

*Proof.* Since  $0 \leq \psi(A) \leq 1$  a.s. for all  $A \in \mathcal{A}$ , the values of  $\psi$  are equivalence classes of  $\mu$ -essentially bounded functions. Consequently the products shown in (iii) above are all  $\mu$ -integrable; and, hence, their norms are finite.

The conditions of the theorem are necessary. Let  $\varphi^{\mathcal{B}}$  be the conditional probability measure relative to a sub- $\sigma$ -field  $\mathcal{B}$ , then  $\varphi^{\mathcal{B}}$  is known to satisfy conditions (i) and (ii) above. For any  $A, B, C \in \mathcal{A}$ ,

$$\| I_A \cdot \varphi^{\mathcal{B}}(B) \cdot \varphi^{\mathcal{B}}(C) \| = \int_A \varphi^{\mathcal{B}}(B) \cdot \varphi^{\mathcal{B}}(C) d\mu$$

$$\begin{aligned}
 (2.3) \quad &= \int_{\sigma} \mathcal{E}^{\sigma} | I_A \cdot \varphi^{\sigma}(B) \cdot \varphi^{\sigma}(C) | d\mu \\
 &= \int_{\sigma} (\mathcal{E}^{\sigma} I_A) \cdot \varphi^{\sigma}(B) \cdot \varphi^{\sigma}(C) d\mu \\
 &= \int_{\sigma} \varphi^{\sigma}(A) \cdot \varphi^{\sigma}(B) \cdot \varphi^{\sigma}(C) d\mu .
 \end{aligned}$$

The right-hand side is symmetric in  $A, B$ , and  $C$ . Interchanging first  $A$  with  $B$  and then  $A$  with  $C$ , (iii) is proved for  $\varphi^{\sigma}$ . If  $B \in \mathcal{B}$ , then  $I_B$  is  $\mathcal{B}$ -measurable and  $\varphi^{\sigma}(B) = I_B$ . If  $A \in \mathcal{A}$ , but  $A \notin \mathcal{B}$ , then  $I_A$  is not  $\mathcal{B}$ -measurable and  $\varphi^{\sigma}(A) = \mathcal{E}^{\sigma} I_A \neq I_A$ . Thus,  $\mathcal{B}$  is completely characterized by (2.2). This completes the proof of necessity.

It remains to prove sufficiency. Let  $\psi$  be an  $\mathcal{L}_1$ -valued measure on  $\mathcal{A}$  satisfying conditions (i), (ii), and (iii); and let

$$\mathcal{B} = \{A \in \mathcal{A} \mid \psi(A) = I_A\} .$$

By letting  $C = \Omega$ , so that  $\psi(C) = 1$  in (iii), we have

$$(2.4) \quad \int_A \psi(B) d\mu = \int_B \psi(A) d\mu \quad A, B \in \mathcal{A} .$$

This equation can be used to prove that  $\mathcal{B}$  is closed under intersections. Let  $A, B \in \mathcal{B}$  and  $C \in \mathcal{A}$ . Then (2.4) implies

$$(2.5) \quad \int_{\sigma} \psi(A \cap B) d\mu = \int_{\sigma} \psi(C) I_{A \cap B} d\mu .$$

By repeated application of (iii) and various definitions, we have

$$\begin{aligned}
 (2.6) \quad \int_{\sigma} \psi(A \cap B) d\mu &= \int_{\sigma} \psi(C) I_A I_B d\mu \\
 &= \int_{\sigma} \psi(C) \psi(A) I_B d\mu \\
 &= \int_{\sigma} I_{\sigma} \psi(A) \psi(B) d\mu \\
 &= \int_{\sigma} I_A \cdot I_B d\mu \\
 &= \int_{\sigma} I_{A \cap B} d\mu .
 \end{aligned}$$

Since this holds for all  $C \in \mathcal{A}$ , we conclude that  $\psi(A \cap B) = I_{A \cap B}$  and that  $A \cap B \in \mathcal{B}$ .  $\mathcal{B}$  is closed under complements: Let  $A \in \mathcal{B}$ . Then  $\psi(A) = I_A$ ; also  $\psi(\Omega) = 1 = I_{\Omega}$ . By the additivity of  $\psi$ ,

$$1 = \psi(\Omega) = \psi(A) + \psi(A^c) = I_A + \psi(A^c) .$$

Therefore  $\psi(A^c) = 1 - I_A = I_{A^c}$ , and  $A^c \in \mathcal{B}$ . Thus,  $\mathcal{B}$  is a field.

To prove that  $\mathcal{B}$  is a  $\sigma$ -field, it suffices to prove that  $\mathcal{B}$  is closed under countable disjoint unions. Let  $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}$  be a disjoint family. Then by the countable additivity of  $\psi$ ,

$$\begin{aligned} \psi\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \psi(A_n) \\ (2.7) \qquad &= \sum_{n=1}^{\infty} I_{A_n} \\ &= I_{\bigcup_{n=1}^{\infty} A_n}. \end{aligned}$$

So,  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$ . Therefore  $\mathcal{B}$  is a  $\sigma$ -field.

If  $B \in \mathcal{B}$  and  $A \in \mathcal{A}$ , then (2.4) implies

$$\begin{aligned} \int_B \psi(A) d\mu &= \int_A \psi(B) d\mu \\ (2.8) \qquad &= \int_A I_B d\mu \\ &= \mu(A \cap B). \end{aligned}$$

This is the defining relation for conditional probability relative to  $\mathcal{B}$  provided that  $\psi(A)$  is a  $\mathcal{B}$ -measurable function.

In reality, the statement above is imprecise:  $\psi(A)$  is an equivalence class of functions in  $\mathcal{L}_1$ . It suffices to prove that  $\psi(A)$  contains one  $\mathcal{B}$ -measurable function. (Implicitly, we are using the fact that if  $\mathcal{B}$  does not contain the entire collection of null sets in  $\mathcal{A}$ , these null sets can be adjoined to  $\mathcal{B}$  in a harmless way.) From each equivalence class  $\psi(A)$ , we select a representative. Continuing the imprecision mentioned above, we will also denote this representative by  $\psi(A)$ , and show that  $C_{A,\alpha} = \{\omega \mid \psi(A)(\omega) \leq \alpha\} \in \mathcal{B}$  for any  $A \in \mathcal{A}$  and any real number  $\alpha$ . ( $C_{A,\alpha}^c = \{\omega \mid \psi(A)(\omega) > \alpha\}$ ). This will prove that  $\psi(A)$  is  $\mathcal{B}$ -measurable.

Let  $B = C_{A,\alpha}^c$  and  $C = C_{A,\alpha}$  in (iii) above. Then,

$$\begin{aligned} \int_A \psi(C_{A,\alpha}^c) \psi(C_{A,\alpha}) d\mu &= \int_{\sigma_{A,\alpha}} \psi(A) \psi(C_{A,\alpha}^c) d\mu \\ (2.9) \qquad &\leq \alpha \int_{\sigma_{A,\alpha}} \psi(C_{A,\alpha}^c) d\mu, \end{aligned}$$

and

$$\begin{aligned} \int_A \psi(C_{A,\alpha}^c) \psi(C_{A,\alpha}) d\mu &= \int_{\sigma_{A,\alpha}^c} \psi(A) \psi(C_{A,\alpha}) d\mu \\ (2.10) \qquad &\geq \alpha \int_{\sigma_{A,\alpha}^c} \psi(C_{A,\alpha}) d\mu. \end{aligned}$$

Therefore,

$$\begin{aligned}
 (2.11) \quad \alpha \int_{\sigma_{A,\alpha}^c} \psi(C_{A,\alpha}) d\mu &\leq \int_A \psi(C_{A,\alpha}^c) \psi(C_{A,\alpha}) d\mu \\
 &\leq \alpha \int_{\sigma_{A,\alpha}^c} \psi(C_{A,\alpha}^c) d\mu .
 \end{aligned}$$

By (2.4), the extremal members of inequality (2.11) are equal, and we have

$$\begin{aligned}
 (2.12) \quad \alpha \int_{\sigma_{A,\alpha}^c} \psi(C_{A,\alpha}) d\mu &= \int_A \psi(C_{A,\alpha}^c) \psi(C_{A,\alpha}) d\mu \\
 &= \int_{\sigma_{A,\alpha}^c} \psi(A) \psi(C_{A,\alpha}) d\mu .
 \end{aligned}$$

Or,

$$(2.13) \quad \int_{\sigma_{A,\alpha}^c} (\psi(A) - \alpha) \psi(C_{A,\alpha}) d\mu = 0 .$$

But on  $C_{A,\alpha}^c$ ,  $\psi(A) - \alpha > 0$   $\mu$  a.s. Thus, since  $\psi(C_{A,\alpha}) \geq 0$   $\mu$  a.s.,  $\psi(C_{A,\alpha}) = 0$   $\mu$  a.s. on  $C_{A,\alpha}^c$ . Using (iii) again with  $A = B = \Omega$  and  $C = C_{A,\alpha}$ , we have

$$(2.14) \quad \int_{\Omega} \psi(C_{A,\alpha}) d\mu = \int_{\sigma_{A,\alpha}} 1 d\mu = \int_{\sigma_{A,\alpha}} I_{\sigma_{A,\alpha}} d\mu = \mu(C_{A,\alpha}) .$$

But

$$\begin{aligned}
 (2.15) \quad \int_{\Omega} \psi(C_{A,\alpha}) d\mu &= \int_{\sigma_{A,\alpha}} \psi(C_{A,\alpha}) d\mu + \int_{\sigma_{A,\alpha}^c} \psi(C_{A,\alpha}) d\mu \\
 &= \int_{\sigma_{A,\alpha}} \psi(C_{A,\alpha}) d\mu .
 \end{aligned}$$

Comparing (2.14) and (2.15),

$$(2.16) \quad \int_{\sigma_{A,\alpha}} \psi(C_{A,\alpha}) d\mu = \int_{\sigma_{A,\alpha}} I_{\sigma_{A,\alpha}} d\mu .$$

On  $C_{A,\alpha}^c$ ,  $\psi(C_{A,\alpha}) = I_{\sigma_{A,\alpha}^c} = 0$   $\mu$  a.s. On  $C_{A,\alpha}$ ,  $0 \leq \psi(C_{A,\alpha}) \leq 1$  by (i) and  $I_{\sigma_{A,\alpha}} = 1$ , so  $I_{\sigma_{A,\alpha}} - \psi(C_{A,\alpha}) \geq 0$   $\mu$  a.s. But (2.16) implies

$$(2.17) \quad \int_{\Omega} (I_{\sigma_{A,\alpha}} - \psi(C_{A,\alpha})) d\mu = 0 .$$

Thus,  $\psi(C_{A,\alpha}) = I_{\sigma_{A,\alpha}}$   $\mu$  a.s. and  $C_{A,\alpha} \in \mathcal{B}$  for all  $A \in \mathcal{A}$  and  $\alpha$  real. This concludes the proof of Theorem 2.

**3. Characterization of conditional expectation in  $\mathcal{L}_p$ ,  $1 \leq p < \infty$ .** Throughout this section,  $T$  will be a continuous linear map of  $\mathcal{L}_p$  into  $\mathcal{L}_p$ . As usual, it will be convenient to use the same symbol for a function and its equivalence class. Thus, the statement “ $X$  is bounded



by  $M > 0$ " means that the equivalence class of  $X$  consists of functions whose  $\mu$ -essential supremum is not greater than  $M$ .

In [3], among more general results, Moy established the following which we now rephrase and state.

*Moy's result:* Let  $T$  be a continuous linear transformation of  $\mathcal{L}_1$  into  $\mathcal{L}_1$  having the following four properties:

- (i) if  $X$  is bounded, then  $TX$  is bounded;
- (ii) if  $X$  and  $Y$  are bounded, then

$$T(X \cdot TY) = TX \cdot TY;$$

- (iii)  $T1 = 1$ ;

- (iv)  $T$  is a contraction, that is, for every  $X \in \mathcal{L}_1$

$$\|TX\| \leq \|X\|.$$

Then, we have for every  $X \in \mathcal{L}_1$ ,

$$TX = \mathcal{E}^{\mathcal{B}} X,$$

where  $\mathcal{B}$  is the  $\sigma$ -field of all sets  $A \in \mathcal{A}$  such that  $T(I_A \cdot X) = I_A \cdot TX$  for all bounded  $X$ .

In [4], Rota made the following definition which we will use below.

**DEFINITION 1.** An averaging operator  $T$  in  $\mathcal{L}_p(\Omega, \mathcal{A}, \mu)$  (where  $p$  is a fixed real number  $1 \leq p \leq \infty$ ) is a linear operator in  $\mathcal{L}_p(\Omega, \mathcal{A}, \mu)$  with the following three properties:

- (i)  $T$  is a contraction operator:

$$(3.1) \quad \|TX\|_{\mathcal{L}_p} \leq \|X\|_{\mathcal{L}_p} \quad \text{for } X \in \mathcal{L}_p;$$

- (ii) if  $Y$  is of class  $\mathcal{L}_p$  and  $X$  is an essentially bounded function on  $(\Omega, \mathcal{A}, \mu)$ , then the function  $(TX) \cdot (TY)$  is of class  $\mathcal{L}_p$  and

$$(3.2) \quad T(X \cdot TY) = (TX) \cdot (TY);$$

$$(3.3) \quad \text{(iii)} \quad T1 = 1.$$

We note that the operator  $T$  in Moy's result is an averaging operator, following a standard argument of approximation of a function in  $\mathcal{L}_1$  by a sequence of bounded measurable functions. In the case  $1 < p < \infty$ , Rota [4] proved the following result which we rephrase and state.

*Rota's Result:* Let  $T$  be an averaging operator in  $\mathcal{L}_p(\Omega, \mathcal{A}, \mu)$  (for fixed  $p$ ). Then there exists a unique sub- $\sigma$ -field  $\mathcal{B} \subset \mathcal{A}$  such

that  $TX = \mathcal{E}^{\mathcal{A}}X$  for  $X \in \mathcal{L}_p$ .

In the proof, Rota defines  $\mathcal{B}$  as the smallest  $\sigma$ -field containing all the  $\sigma$ -fields of inverse images of the Borel sets by bounded functions  $Y$  fixed under  $T$ . It is clear that this procedure will yield a  $\sigma$ -field  $\mathcal{B}$  which is completed with respect to  $\mu$ -null sets, just as Moy's construction of  $\mathcal{B}$  does. The above statement must be interpreted in this light. Comparing the recipe for construction of  $\mathcal{B}$  in the two results, we see that Rota's criterion for membership in  $\mathcal{B}$  is, on the surface, weaker than Moy's. On the other hand, if  $TI_A = I_A$ , then  $TI_A \cdot X = I_A \cdot X$  for  $X$  bounded and  $T(TI_A \cdot X) = T(I_A \cdot X)$ . By the smoothing property (3.2) of averaging operators,  $T((TI_A) \cdot X) = TI_A \cdot TX = T(I_A \cdot X)$ . Hence  $I_A \cdot TX = T(I_A \cdot X)$  for all  $X$  bounded, and the two recipes define the same family of sets.

A theorem can now be stated which contains both Moy's result and Rota's. This general theorem will be proved using a theorem on the representation of operators on a Lebesgue space and the characterization of conditional probabilities given in § 2.

**THEOREM 3.** *Let  $T$  be an averaging operator in  $\mathcal{L}_p(\Omega, \mathcal{A}, \mu)$  (where  $p$  is a fixed real number  $1 \leq p < \infty$ ). Then there exists a unique sub- $\sigma$ -field  $\mathcal{B} \subset \mathcal{A}$  completed with respect to the  $\mu$ -null sets of  $\mathcal{A}$  such that  $TX = \mathcal{E}^{\mathcal{B}}X$  for  $X \in \mathcal{L}_p$ . Furthermore,*

$$\mathcal{B} = \{A \in \mathcal{A} \mid TI_A = I_A\} .$$

*Proof.* The first part of the proof consists in showing that  $T^*X = TX$  for all  $X$  bounded. A brief explanation of the meaning of this "self-adjointness" relation is given for the sake of completeness: As is well known for  $1 \leq p < \infty$ ,  $\mathcal{L}_p^*$  is isometrically isomorphic to  $\mathcal{L}_q$ , where  $q$  is determined by the relation  $1 = 1/p + 1/q$ . Every linear operator  $T: \mathcal{L}_p \rightarrow \mathcal{L}_p$  determines a mapping  $T^*: \mathcal{L}_q \rightarrow \mathcal{L}_q$ . Since  $\mathcal{L}_\infty \subset \mathcal{L}_p \cap \mathcal{L}_q$  as sets for  $1 \leq p < \infty$ , it makes sense to compare the action of  $T$  and  $T^*$  on bounded functions. To avoid confusion, the norm will be shown explicitly: thus,  $\|\cdot\|_{\mathcal{L}_p}$ .

The fact that  $\|TX\|_{\mathcal{L}_p} \leq \|X\|_{\mathcal{L}_p}$  and  $\|T1\|_{\mathcal{L}_p} = \|1\|_{\mathcal{L}_p} = 1$  implies

$$(3.4) \qquad \|T\|_{\mathcal{B}(\mathcal{L}_p)} = 1 ,$$

where  $\|\cdot\|_{\mathcal{B}(\mathcal{L}_p)}$  is operator norm in  $\mathcal{B}(\mathcal{L}_p)$ , the bounded linear operators on  $\mathcal{L}_p$ .

Rota [3, p. 58] has shown that for  $1 < p < \infty$  the averaging operator hypotheses imply that  $T^*1 = 1$ . In the case  $p = 1$ , let  $Y_0 = T^*1$ . Then,

$$(3.5) \quad \int Y_0 d\mu = \int T^*1 \cdot 1 d\mu = \int 1 \cdot T1 d\mu = 1 .$$

Further, since  $T^*: \mathcal{L}_\infty \rightarrow \mathcal{L}_\infty$ ,  $T^*1 = Y_0 \in \mathcal{L}_\infty$ ; and

$$(3.6) \quad \begin{aligned} \|Y_0\|_{\mathcal{L}_\infty} &= \sup_{\substack{X \in \mathcal{L}_1 \\ \|X\|_{\mathcal{L}_1} \leq 1}} \left| \int Y_0 \cdot X d\mu \right| \\ &= \sup_{\substack{X \in \mathcal{L}_1 \\ \|X\|_{\mathcal{L}_1} \leq 1}} \left| \int T^*1 \cdot X d\mu \right| \\ &= \sup_{\substack{X \in \mathcal{L}_1 \\ \|X\|_{\mathcal{L}_1} \leq 1}} \left| \int TX d\mu \right| \leq \|T\|_{\mathcal{L}(\mathcal{L}_1)} = 1 . \end{aligned}$$

This means  $-1 \leq Y_0 \leq +1$   $\mu$  a.s.; and, taken together with (3.5), this means  $Y_0 = 1$   $\mu$  a.s. Hence, we have for  $1 \leq p < \infty$

$$(3.7) \quad T^*1 = 1 .$$

Since  $\int 1 \cdot TX d\mu = \int T^*1 \cdot X d\mu$ , this gives immediately

$$(3.8) \quad \int TX d\mu = \int X d\mu , \quad X \in \mathcal{L}_p .$$

And so, finally, for any bounded  $X$  and  $A \in \mathcal{A}$ , we may calculate

$$(3.9) \quad \begin{aligned} \int_A TX d\mu &= \int I_A \cdot TX d\mu = \int T(I_A \cdot TX) d\mu && \text{(by 3.8)} \\ &= \int (TI_A) \cdot (TX) d\mu && \text{(by 3.2).} \end{aligned}$$

But

$$(3.10) \quad \begin{aligned} \int_A T^*X d\mu &= \int (T^*X) \cdot I_A d\mu = \int X \cdot (TI_A) d\mu \\ &= \int T(X \cdot TI_A) d\mu && \text{(by 3.8)} \\ &= \int (TX) \cdot (TI_A) d\mu && \text{(by 3.3).} \end{aligned}$$

Therefore, since  $\int_A T^*X d\mu = \int_A TX d\mu$  for all  $A \in \mathcal{A}$ , we can conclude

$$(3.11) \quad T^*X = TX \quad \text{for all } X \text{ bounded.}$$

In the next part of the proof, the principal instrument will be a theorem on the representation of operators on a Lebesgue space whose general form Dunford and Schwartz [1, p. 540] ascribe to Kantorovich and Vulich:

Result of Kantorovich and Vulich: Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space and let  $T$  be a continuous linear map of the Banach space  $\mathfrak{X}$  into  $\mathcal{L}_1(\Omega, \mathcal{A}, \mu)$ . Then there is a uniquely determined function  $\psi(\cdot)$  on  $\mathcal{A}$  to  $\mathfrak{X}^*$  such that

- (i) for each  $X \in \mathfrak{X}$ , the set function  $\psi(\cdot)[X]$  is  $\mu$ -continuous and countably additive on  $\mathcal{A}$ ;
- (ii) for every  $X \in \mathfrak{X}$ , we have

$$(3.12) \quad TX = \frac{d[\psi(\cdot)X]}{d\mu}$$

(iii) the norm of  $T$  satisfies the relations

$$(3.13) \quad \sup_{A \in \mathcal{A}} \|\psi(A)\|_{\mathfrak{X}^*} \leq \|T\|_{op} \leq 4 \sup_{A \in \mathcal{A}} \|\psi(A)\|_{\mathfrak{X}^*}.$$

Conversely, if the function  $\psi(\cdot)$  on  $\mathcal{A}$  to  $\mathfrak{X}^*$  satisfies (i), then (ii) defines an operator  $T$  on  $\mathfrak{X}$  to  $\mathcal{L}_1$  whose norm satisfies (iii).

Furthermore,  $T$  is weakly compact if and only if  $\psi(\cdot)$  is countably additive on  $\mathcal{A}$  in the strong topology of  $\mathfrak{X}^*$ .

The proof of this important theorem can be found in [1, p. 498]. To apply the result above to the operator  $T$ , we choose to consider  $T$  as a continuous linear map of  $\mathcal{L}_p(1 \leq p < \infty, p \text{ fixed})$  into  $\mathcal{L}_1$ . Thus,  $\mathfrak{X}$  is  $\mathcal{L}_p$  and  $\mathfrak{X}^*$  can be identified with  $\mathcal{L}_q$ . If  $\psi(\cdot)$  is the function given in the result above, then for any  $X \in \mathcal{L}_1$  and  $A \in \mathcal{A}$ , we have

$$(3.14) \quad \begin{aligned} \int_A TX d\mu &= \int_A \frac{d\psi(\cdot)[X]}{d\mu} d\mu = \psi(A)[X] \\ &= \int \psi(A) \cdot X d\mu. \end{aligned}$$

On the other hand,  $T = T^*$  on bounded functions, so

$$(3.15) \quad \begin{aligned} \int_A TX d\mu &= \int I_A \cdot (TX) d\mu \\ &= \int (T^* I_A) \cdot X d\mu = \int (T I_A) \cdot X d\mu. \end{aligned}$$

Hence, for each  $A \in \mathcal{A}$ , putting  $X = I_B$ , we get

$$(3.16) \quad \int_B T I_A d\mu = \int_B \psi(A) d\mu \quad B \in \mathcal{A}.$$

And therefore,

$$(3.17) \quad T I_A = \psi(A), \quad A \in \mathcal{A}.$$

Rota shows [4, p. 58] that an averaging operator in  $\mathcal{L}_p, 1 < p < \infty$

maps bounded measurable functions into bounded measurable functions. For  $p = 1$ ,  $\psi(A) \in \mathcal{L}_\infty$ ; and, hence  $TI_A = \psi(A)$  is a bounded measurable function for  $1 \leq p < \infty$ .  $\psi(\phi) = TI_\phi = 0$ . A simple domination argument shows that if  $\{A_j\}_{j=1}^\infty$  is a disjoint family,  $\sum_{j=1}^n I_{A_j} \rightarrow I_{\cup_{j=1}^\infty A_j}$  in  $\mathcal{L}_p$ -norm. Therefore

$$(3.18) \quad T\left(\sum_{j=1}^n I_{A_j}\right) = \sum_{j=1}^n \psi(A_j) \longrightarrow \psi\left(\bigcup_{j=1}^\infty A_j\right) \quad \text{in } \mathcal{L}_p\text{-norm.}$$

Thus,  $\psi$  is a vector-valued measure on  $\mathcal{A}$  to the bounded measurable functions on  $\mathcal{A}$ , which is countably additive in the  $\mathcal{L}_p$ -topology ( $1 \leq p < \infty$ ). A fortiori,  $\psi$  is  $\mathcal{L}_1$ -countably additive. Since  $TI_\Omega = \psi(\Omega) = 1$ , condition (ii) in (2.1) is satisfied. Furthermore, for  $A \in \mathcal{A}$

$$(3.19) \quad \mu(A) = \int (T^*I) \cdot I_A d\mu = \int \psi(A) d\mu .$$

But,

$$(3.20) \quad \int \psi(A) d\mu \leq \int |\psi(A)| d\mu \leq \int |TI_A| d\mu \\ \leq \|TI_A\|_{\mathcal{L}_1} \leq \|T\|_{\mathcal{A}(\mathcal{L}_1)} \cdot \|I_A\|_{\mathcal{L}_1} \leq \mu(A) ,$$

since  $\|T\|_{\mathcal{A}(\mathcal{L}_p)} = 1, 1 \leq p < \infty$ . Therefore, equality holds throughout (3.20), so

$$(3.21) \quad \int [|\psi(A)| - \psi(A)] d\mu = 0 .$$

Because  $|\psi(A)| - \psi(A) \geq 0$ , we have

$$(3.22) \quad \psi(A) = |\psi(A)| \geq 0 .$$

Thus, condition (ii) in (2.1) is satisfied.

The smoothing property (3.2) implies  $T$  is idempotent:  $T^2X = T(1 \cdot TX) = T1 \cdot TX = TX$ . Now let  $A, B, C \in \mathcal{A}$ , using (3.17), (3.2), (3.8), the fact that  $T$  maps bounded measurable functions into bounded measurable functions, and idempotence, we have

$$(3.23) \quad \int_A \psi(B)\psi(C) d\mu = \int_A (TI_B) \cdot (TI_C) d\mu \\ = \int I_A \cdot T(I_B \cdot (TI_C)) d\mu \\ = \int T(I_A \cdot T(I_B \cdot (TI_C))) d\mu \\ = \int (TI_A) \cdot T(I_B \cdot (TI_C)) d\mu$$

$$\begin{aligned}
 &= \int (TI_A) \cdot (TI_B) \cdot (T^2 I_C) d\mu \\
 &= \int (TI_A) \cdot (TI_B) \cdot (TI_C) d\mu .
 \end{aligned}$$

Clearly, the result is the same if  $A$  is interchanged with  $B$  or  $A$  is interchanged with  $C$ . Thus, condition (iii) in (2.1) is satisfied; and  $\psi(\cdot)$  is a conditional probability in  $\mathcal{L}_1$ , by Theorem 2. The integral with respect to  $\psi(\cdot)$  of  $X \in \mathcal{L}_1$  is  $\mathcal{E}^{\mathcal{B}} X$ , where  $\mathcal{B}$  is the sub- $\sigma$ -field defined by

$$\mathcal{B} = \left\{ A \in \mathcal{A} \mid \psi(A) = \int I_A d\psi = I_A \right\} .$$

Thus,  $\mathcal{B}$  consists of those sets in  $\mathcal{A}$  whose indicator functions are invariant under  $\mathcal{E}^{\mathcal{B}}$ . It is well known [2, p. 348] that  $\mathcal{E}^{\mathcal{B}}$  defines an  $\mathcal{L}_p$ -continuous linear transformation ( $1 \leq p < \infty$ ). Then,

$$(3.24) \qquad \mathcal{E}^{\mathcal{B}} I_A = \psi(A) = TI_A ,$$

and so the restriction of  $\mathcal{E}^{\mathcal{B}}$  to  $\mathcal{L}_p$  agrees with the  $\mathcal{L}_p$ -continuous operator  $T$  on a generating set; hence, it agrees on all of  $\mathcal{L}_p$ . We conclude that  $T$  is  $\mathcal{E}^{\mathcal{B}}$  on  $\mathcal{L}_p$ . This, with the fact then that  $\mathcal{B} = \{A \in \mathcal{A} \mid TI_A = I_A\}$ , finishes the proof of Theorem 3.

This theorem shows that the set of averaging operators in  $\mathcal{L}_r$  and  $\mathcal{L}_s$  are in one-to-one correspondence  $1 \leq r, s < \infty$ . Moreover, every averaging operator  $T$  on  $\mathcal{L}_r$  has a unique  $\mathcal{L}_s$ -continuous extension  $1 \leq s \leq r < \infty$ . In a subsequent publication, the author intends to relate these facts to the work of Sidak [5].

REFERENCES

1. N. Dunford and J. Schwartz, *Linear Operators Part I: General Theory*, New York, 1958.
2. M. Loève, *Probability Theory*, 3rd ed., Princeton, 1963.
3. S. C. Moy, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math. 4 (1954), 47-63.
4. G. C. Rota, *On the representation of averaging operators*, Rendiconti del Seminario Matematico della Università di Padova, Vol. XXX, 1960, pp. 52-64.
5. Z. Sidak, *On relations between strict-sense and wide-sense conditional expectations*, Theory of Probability and Its Applications, Vol. II, 1957, pp. 267-272.

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