

Pacific Journal of Mathematics

**ON THE STRICT AND UNIFORM CONVEXITY OF CERTAIN
BANACH SPACES**

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Let (X, S, μ) be a σ -finite non-atomic measure space let N be a real valued continuous convex even function defined on the real line such that

- (1) $N(u)$ is nondecreasing for $u \geq 0$,
- (2) $\lim_{u \rightarrow \infty} N(u)/u = \infty$,
- (3) $\lim_{u \rightarrow 0} N(u)/u = 0$.

Let L_N be the set of all real valued μ -measurable functions f such that $\int_X N(f) d\mu < \infty$. It is known that if there exists a constant k such that $N(2u) \leq kN(u)$ for all $u \geq 0$ then L_N is a linear space; in fact, L_N is a B -Space if a norm $\|\cdot\|$ is defined by setting

$$(*) \quad \|f\| = \inf \left\{ 1/\zeta \mid \zeta > 0, \int_X N(\eta, f) d\mu \leq 1 \right\}.$$

Denoting the B -space $(L_N, \|\cdot\|)$ by L_N^* it is proposed to obtain the necessary and sufficient conditions in order that L_N^* may be (1) Strictly Convex (2) Uniformly Convex.

The linear space L_N admits another norm $\| \cdot \|_{(N)}$ known as the Orlicz norm defined by setting

$$\|f\|_{(N)} = \sup \int_X |fg| d\mu$$

for such that $\int_X M(|g|) d\mu \leq 1$, M being the function complementary to N in the sense of Young. For a discussion of this class of Banach spaces we refer to Mazur and Orlicz [2]. Convexity properties of the Orlicz norm have been studied in Milnes [3].

The space L_N^* may be considered as a modular linear space defined in Nakano [4]. A nonnegative extended real valued function m defined on a linear space is called a *modular* if

- (i) $m(0) = 0$;
- (ii) for any $x \in L$ there exists $\xi > 0$ such that $m(\xi x) < \infty$;
- (iii) $m(\xi x) = 0$ for all $\xi > 0$ implies $x = 0$;
- (iv) $m(x) = \sup_{0 \leq \xi < 1} m(\xi x)$;
- (v) m is convex (i.e., $\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1, x, y \in L$ imply $m(\alpha x + \beta y) \leq \alpha m(x) + \beta m(y)$).

The *modulated linear space* may be considered as a normed linear space if a norm $\| \cdot \|$ is defined by setting

$$(**) \quad \|x\| = \inf \{1/\xi \mid \xi > 0 \text{ and } m(\xi x) \leq 1\}.$$

We note that the linear space L_N is a modulated space if

$$m(f) = \int_x N(f) d\mu,$$

and the norm $\| \cdot \|$ defined by $(**)$ is the same as the norm defined in $*$. In fact, the modulated space L_N is a *finite modulated space*, meaning that $m(f) < \infty$, for all $f \in L_N$.

A Banach space B is said to be strictly convex if $x, y \in B, \|x\| = \|y\| = \|(x + y)/2\| = 1$ imply $x = y$. It is uniformly convex if to each $\epsilon, 0 < \epsilon \leq 2$, there corresponds a $\delta(\epsilon) > 0$ such that conditions $\|x\| = \|y\| = 1, \|x - y\| \geq \epsilon$ imply that $\|x + y\| < 2 - \delta(\epsilon)$.

We shall start by characterizing the strict convexity of L_N^* .

LEMMA 1. *The modulated norm defined in $(**)$ associated with a finite modulated space is strictly convex if and only if $m(x) = m(y) = m\{(x + y)/2\} = 1$ imply $x = y$.*

The proof is an easy consequence of the fact that in a finite modulated space, $m(x) = 1$ if and only if $\|x\| = 1$ where $\| \cdot \|$ is the related modulated norm.

THEOREM. *The Banach space L_N^* is strictly convex if and only if the N -function N is strictly convex; i.e.,*

$$N\left(\frac{u + v}{2}\right) < \frac{1}{2} [N(u) + N(v)]$$

for all real u, v such that $u \neq v$.

Proof. Let N be a strictly convex N -function. Let $f, g \in L_N^*$ such that

$$m(f) = m(g) = m\left(\frac{f + g}{2}\right) = 1.$$

By definition of m it follows that

$$\int_x \left[\frac{N(f) + N(g)}{2} \right] - N\left(\frac{f + g}{2}\right) d\mu = 0.$$

whence the convexity of N together with the restrictions on f , and g imply that $f = g$ a.e. Thus by Lemma 1, L_N^* is strictly convex.

To prove the "only if" part, let L_N^* be strictly convex. If possible let N be not strictly convex so that there exist $a, b \geq 0$ $a \neq b$ such that $N\{(a + b)/2\} = 1/2 [N(a) + N(b)]$. The continuity of N together with the condition $\lim_{u \rightarrow 0} N(u)/u = 0$ imply that N is linear on the interval $[a, b]$ and $a \neq 0, b \neq 0$. For $u \in [a, b]$ let $N(u) = pu + q$, where p and q are reals.

Since μ is a nonatomic positive measure there exist pairwise disjoint measurable sets A, B, C of arbitrarily small measure such that

$$\mu(A) = \mu(B) = \mu(C) .$$

Let us define functions f, g as follows. Let $f(x) = a$ for $x \in A, f(x) = b$ for $x \in B$, and $f(x) = 0$ for all $x \notin A \cup B$. Let $g(x) = b$ for $x \in A, g(x) = a$ for $x \in B$, and $g(x) = 0$ for $x \notin A \cup B$, and $g(x) = 0$ for $x \in C$. Then

$$\begin{aligned} m(f) &= \int_x N(f)d\mu = [p(a + b) + 2q]\mu(A) , \\ m(g) &= \int_x N(g)d\mu = [p(a + b) + 2q]\mu(B) , \\ m\left(\frac{f + g}{2}\right) &= \frac{1}{2} [m(f) + m(g)] , \end{aligned}$$

and $m(f) = m(g) = m\{(f + g)/2\}$. By a suitable choice of A, B, C we can assume that

$$m(f) = m(g) = m\left(\frac{f + g}{2}\right) = K < \frac{1}{2} .$$

Now let h be a function on X defined by setting

$$h(x) = 0 \text{ if } x \in C , \quad h(x) = t \text{ if } x \in C$$

where t is such that $N(t)\mu(C) = 1 - K$. Let $f_1 = h + f$, and $g_1 = h + g$; since $h \wedge f = 0 = h \wedge g$, we obtain

$$m(f_1) = m(h) + m(f) = (1 - K) + K = 1 .$$

Similarly $m(g_1) = 1$, and further

$$m\left(\frac{f_1 + g_1}{2}\right) = m\left(\frac{f + g}{2} + h\right) = m\left(\frac{f + g}{2}\right) + m(h) = 1 .$$

Thus we have $f_1 \in L_N^*, g_1 \in L_N^*$ and $m(f_1) = m(g_1) = m\{(f_1 + g_1)/2\} = 1$; however $f_1 \neq g_1$. Thus L_N^* is not strictly convex, a contradiction.

We next proceed to characterize the uniform convexity of L_N^* .

It is known [5] that in a modular semiordered linear space, the modular norm is uniformly convex if and only if the associated norm

is uniformly convex. The modulated linear spaces L_N are modulated semiordered linear spaces under the natural pointwise ordering, and the above two norms are respectively the norms $\|\cdot\|_{(N)}$ and $\|\|\cdot\|\|_{(N)}$.

With this remark we conclude that the Theorem 8 in Milnes [3] which characterizes the uniform convexity of the norm $\|\|\cdot\|\|_{(N)}$ also characterizes the uniform convexity of the norm $\|\cdot\|_{(N)}$.

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