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## **GROUP EXTENSION REPRESENTATIONS AND THE STRUCTURE SPACE**

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# GROUP EXTENSION REPRESENTATIONS AND THE STRUCTURE SPACE

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Let  $K$  be a locally compact group.  $K^*$  will denote the Jacobson structure space of  $C^*(K)$ , the group  $C^*$ -algebra of  $K$ . For any unitary representation  $V$  of  $K$  on a Hilbert space, let  $E_V$  denote the projection valued measure on the Borel sets of  $K^*$  defined by Glimm (Pacific J. Math. 12 (1962), 885-911; Theorem 1.9). A (not necessarily Borel) subset  $S$  of  $K^*$  is called  $E_V$ -thick if  $E_V(S_1) = 0$  for every Borel  $S_1 \subseteq K^* \sim S$ . For any two representations  $V_1$  and  $V_2$ ,  $\mathcal{R}(V_1, V_2)$  will denote the space of operators intertwining  $V_1$  and  $V_2$ .

Suppose  $K$  is a closed normal subgroup of the locally compact group  $G$ . If  $V$  is a representation of  $K$  and  $x \in G$ ,  $V^x$  is defined by  $V_k^x = V_{xkx^{-1}}$ ,  $k \in K$ . If  $z \in K^*$ ,  $zx = \text{Ker}(V^x)$ , where  $V$  is any irreducible representation such that  $z = \text{Ker}(V)$ . (By  $\text{Ker}$  we mean the kernel in the group  $C^*$ -algebra.) This composition turns  $(K^*, G)$  into a topological transformation group (Glimm, op. cit., Lemma 1.3). The present paper first shows that the stability subgroups of  $G$  at points  $z \in K^*$  are closed. Then the following two theorems are proved:

**THEOREM 1.** Let  $z \in K^*$  and let  $H$  be the stability subgroup of  $G$  at  $z$ . Let  $L$  be a representation of  $H$  such that  $\{z\}$  is  $E_{L|K}$ -thick. Then  $\mathcal{R}(U^L, U^L)$  is isomorphic to  $\mathcal{R}(L, L)$  and  $\{z\}G$  is  $E_{U^L|K}$ -thick.

**THEOREM 2.** Let  $M$  be a representation of  $G$  such that  $\{z\}G$  is  $E_{M|K}$ -thick for some  $z \in K^*$ . Let  $H$  be the stability subgroup of  $G$  at  $z$ . Suppose  $G/H$  is  $\sigma$ -compact. Then there is a representation  $L$  of  $H$  such that  $\{z\}$  is  $E_{L|K}$ -thick and such that  $M \simeq U^L$ .

In the above,  $U^L$  denotes the representation of  $G$  induced by  $L$ .

It is shown further that if  $C^*(K)|z$  contains an ideal isomorphic to the algebra of all compact operators on some Hilbert space, then the representation  $L|K$  of these theorems is a multiple of the (essentially unique) irreducible representation  $L^0$  of  $K$  such that  $\text{Ker}(L^0) = z$ . Finally, it is shown that if  $M$  is primary and if  $K^*/G$  is almost Hausdorff (i.e., every nonvoid closed subset contains a nonvoid relatively open Hausdorff subset), then  $M$  satisfies the hypothesis of Theorem 2.

These results generalize Mackey's Theorem 8.1 [13], in the case of the trivial multiplier. In [13], Mackey attacks the problem of

reducing the representation theory of a locally compact group  $G$  with closed normal subgroup  $K$  to the representation theories of  $K$  and  $G/K$ . His main theorem, Theorem 8.1, supposes the following restrictions on  $G$  and  $K$ :  $G$  satisfies the second axiom of countability and  $K$  is type  $I$  (in the case of the trivial multiplier). The present paper explores what happens when these restrictions are lifted.

It turns out that a great deal of Mackey's theorem remains true in modified form. The chief modifications are these:

(a) We replace the dual space  $\hat{K}$  of  $K$  by the structure space  $K^*$  of its group  $C^*$ -algebra. This is done because  $K^*$  is fairly well behaved, being a  $T_0$  topological space, while  $\hat{K}$  can be very messy when  $K$  is not type  $I$ . The first example of § 6 shows how a theory based on  $\hat{K}$  cannot get off the ground.

(b) We replace the projection valued measure based on  $\hat{K}$  which is canonically associated with the direct integral decomposition of a given representation of  $K$  (when  $K$  is type  $I$ ) with the measure based on  $K^*$  introduced by Glimm in [10].

These modifications and the lack of separability force us to replace Mackey's highly measure theoretic arguments with arguments more in the spirit of the present author's previous work [1-3] and that of Glimm's paper [10].

After the preliminaries of § 2, we prove our analogue of Mackey's theorem in §§ 3 and 4. Section 5 is concerned with what additional hypotheses are needed to make the analogue exact. The paper closes with some examples in § 6.

The problem dealt with in the last example was the starting point for this investigation. We wish to thank James Glimm for several stimulating conversations on this problem.

**2. Preliminaries.** Let  $G$  be a locally compact group and let  $C_0(G)$  be the space of continuous complex valued functions on  $G$  with compact support. If  $f, g \in C_0(G)$  set  $(f \circ g)(x) = \int_G f(y)g(xy^{-1})dy$  and  $f^*(x) = \overline{f^*(x^{-1})}\delta_\sigma(x)^{-1}$ , where  $dy$  denotes right invariant Haar measure and where  $\delta_\sigma$  is its modular function.  $+$ ,  $*$ , the usual addition of functions, and the usual inductive limit topology on  $C_0(G)$  turn it into a topological  $*$ -algebra. Let  $L$  be a unitary representation of  $G$ . Then setting  $L_f = \int_G f(x)L_x^{-1}dx$  (strong operator topology integral) gives us a continuous  $*$ -representation of  $C_0(G)$ . Moreover  $\{L_f: f \in C_0(G)\}$  has no simultaneous null vectors (we say that  $L$  is a *nondegenerate* representation of  $C_0(G)$ ). Conversely, if  $\Phi$  is a nondegenerate continuous  $*$ -representation of  $C_0(G)$ , there is a unique unitary representation  $L$  of  $G$  such that  $L_f = \Phi_f$  for  $f \in C_0(G)$ .

If  $L$  is unitary representation of  $G$ , then  $\|L_f\| \leq \|f\|_1$  for  $f \in C_0(G)$ .

Therefore  $\|f\| = LUB\{\|L_f\| : L \text{ unitary representation of } G\}$  exists.  $\|\cdot\|$  is a norm on  $C_0(G)$  and the completion of  $C_0(G)$  with respect to  $\|\cdot\|$  is a  $C^*$ -algebra, called the  $C^*$ -group algebra of  $G$  and denoted by  $C^*(G)$ . Clearly there is a one-to-one correspondence between non-degenerate  $*$ -representations of  $C^*(G)$  and nondegenerate continuous  $*$ -representations of  $C_0(G)$ . Thus the representation theory of  $C^*(G)$  is "the same" as that of  $G$ .

In what follows,  $G^*$  will denote the Jacobson structure space of  $C^*(G)$ ; i.e. the space of kernels of irreducible nondegenerate  $*$ -representations of  $C^*(G)$  equipped with the hull-kernel topology.  $G^*$  is a  $T_0$ -space.

Let  $\mathfrak{A}$  be a  $C^*$ -algebra,  $Z$  its structure space, and  $\Phi$  a nondegenerate  $*$ -representation of  $\mathfrak{A}$  on a Hilbert space  $\mathfrak{H}$ . Glimm [10] has shown that there is unique projection valued measure  $E$  on the Borel field generated by the topology of  $Z$  with the following property: if  $S$  is a closed subset of  $Z$ , then  $E(S)$  is the projection on the manifold of  $v \in \mathfrak{H}$  such that  $\Phi(a)v = 0$  for all  $a \in \mathfrak{A} \cap S$ .  $E$  takes its values in the center of the von Neumann algebra generated by  $\Phi(\mathfrak{A})$ . In our case, if  $L$  is a unitary representation of  $G$ ,  $E_L$  will denote the Glimm measure on  $G^*$  associated with the representation of  $C^*(G)$  determined by  $L$ .

For the formulation of induced representations used in this paper the reader is referred to [1]. If  $L$  is a representation of the closed subgroup  $H$  of  $G$ , we define a regular Borel projection valued measure  $E^L$  on  $G/H$  as follows: if  $S$  is a Borel subset of  $G/H$ , then  $E^L(S)f = (\chi_S \circ \pi) \cdot f$ , where  $\chi_S$  is the characteristic function of  $S$ ,  $\pi$  is the canonical projection of  $G$  onto  $G/H$ , and  $f \in \mathfrak{H}(U^L)$ . This  $E^L$  determines, and is determined by, the  $*$ -representation of  $C_0(G/H)$  defined by

$$E^L(h) = \int_{G/H} h(p) dE^L(p),$$

$h \in C_0(G/H)$  (cf. [3]).

Finally, if  $E$  is any projection valued measure on the measurable space  $(Z, \mathcal{B})$ , any subset  $S \subseteq Z$  (not necessarily in  $\mathcal{B}$ ) will be called *E-thick* if  $E(T) = 0$  whenever  $T \cap S = \emptyset$  (cf. [11], p. 74).

Let  $G$  be a locally compact group and let  $K$  be a closed normal subgroup. For  $f \in C_0(K)$  and  $x \in G$ , we define  $xf \in C_0(K)$  by the formula  $(xf)(\xi) = f(x^{-1}\xi x)\Delta(x)$  for  $\xi \in K$ , where  $\Delta(x)$  is the (constant) Radon-Nikodym derivative  $[d(x^{-1}\xi x)]/d\xi$ . If  $L$  is a unitary representation of  $K$  and if  $L^x$  is defined by  $L^x_\xi = L_{x\xi x^{-1}}$ ,  $\xi \in K$ , then  $L^x_f = L_{xf}$ . From this it follows readily that  $f \rightarrow xf$  is an automorphism of  $C_0(K)$  which is isometric in the  $C^*$ -norm  $\|\cdot\|$ . Therefore this map extends to an automorphism of  $C^*(K)$ . We define an action of  $G$  on  $K^*$  by setting  $zx = \{f \in C^*(K) : xf \in z\}$  for  $z \in K^*$ ,  $x \in G$ . Glimm [10] and Fell [7] have

shown that the map of  $K^* \times G \rightarrow K^*$  given by  $(z, x) \rightarrow zx$  is continuous, giving us a topological transformation group.

**LEMMA 1.** *Let  $(S, G)$  be a topological transformation group. Suppose  $S$  is a  $T_0$ -space. Then the stability subgroups of  $G$  are closed.*

*Proof.* Let  $H$  be the stability subgroup of  $G$  at  $p \in S$ . Then  $\{p\}H = \{p\}$ , so that  $\{p\}^-H^- \subseteq \{p\}^-$ . If  $x \in H^-$ , we have  $\{px\}^- = \{p\}^-x \subseteq \{p\}^-$ . Since  $x^{-1} \in H^-$ , we have  $\{p\}^-x^{-1} \subseteq \{p\}^-$  and hence  $\{p\}^- \subseteq \{p\}^-x = \{px\}^-$ ; i.e.,  $\{px\}^- = \{p\}^-$ . But  $S$  is  $T_0$ . Therefore  $px = p$  and  $x \in H$ .

We may now state our main theorems.

**THEOREM 1.** *Let  $G$  be a locally compact group and let  $K$  be a closed normal subgroup of  $G$ . Let  $z \in K^*$  and let  $H$  be the (closed) stability subgroup of  $G$  at  $z$ . Let  $L$  be a representation of  $H$  such that  $\{z\}$  is  $E_{L|K}$ -thick. Then  $\mathcal{R}(U^L, U^L)$  is isomorphic to  $\mathcal{R}(L, L)$  and  $\{z\}G$   $E_{U^L|K}$ -thick.*

**THEOREM 2.** *Let  $G$  be a locally compact group and let  $K$  be a closed normal subgroup of  $G$ . Let  $M$  be a representation of  $G$ . Assume that  $\{z\}G$  is  $E_{M|K}$ -thick for some  $z \in K^*$ . Let  $H$  be the (closed) stability subgroup of  $G$  at  $z$ . Suppose  $G/H$  is  $\sigma$ -compact. Then there is a representation  $L$  of  $H$  such that  $\{z\}$  is  $E_{L|K}$ -thick and such that  $M \simeq U^L$ .*

**3. Proof of Theorem 1.** We begin our proofs with the following lemma.

**LEMMA 2.** *Let  $H$  and  $K$  be closed subgroups of the locally compact group  $G$ ,  $K$  normal, and  $K \subseteq H \subseteq G$ . Let  $L$  be a representation of  $H$ . Then for every  $f \in C^*(K)$  and  $g \in \mathfrak{S}(U^L)$  we have  $[(U^L | K)_f g](x) = (L | K)_f^* g(x)$  for locally almost all  $z \in G$ .*

*Proof.* Suppose first that  $f \in C_0(K)$  and  $g$  is continuous with compact support modulo  $H$ . Set

$$u(x) = (L | K)_f^* g(x) = \int_K f(\xi) L_{x\xi^{-1}x^{-1}} g(x) d\xi = \int_K f(\xi) g(x\xi^{-1}) d\xi .$$

Clearly  $u$  is continuous with compact support modulo  $H$  and belongs to  $\mathfrak{S}(U^L)$ . Let  $v \in \mathfrak{S}(U^L)$  be continuous with compact support modulo  $H$ , and choose  $h \in C_0(G)$  such that  $\int_H h(\eta x) d\eta = 1$  for  $x$  in the support

of  $v$ . Then

$$\begin{aligned} (u, v) &= \int_G h(x)(u(x), v(x)) dx = \int_G \int_K h(x)f(\xi)(g(x\xi^{-1}), v(x))d\xi dx \\ &= \int_K f(\xi)(U_{\xi^{-1}}^L g, v)d\xi = ((U^L | K)_f g, v) . \end{aligned}$$

Since the set of all such  $v$  is dense in  $\mathfrak{S}(U^L)$ , our result holds in this case.

Next suppose  $f \in C_0(K)$  and  $g \in \mathfrak{S}(U^L)$ . Choose a sequence  $g_n \in \mathfrak{S}(U^L)$ , continuous with compact support modulo  $H$ , such that  $\|g_n - g\| < 2^{-n}$ . Then  $\|(U^L | K)_{f_n} g_n - (U^L | K)_f g\| < \|f\|2^{-n}$ . As in the proof of Proposition 1 of [1],  $g_n \rightarrow g$  and  $(U^L | K)_{f_n} g_n \rightarrow (U^L | K)_f g$  locally almost everywhere.

Finally, if  $f \in C^*(K)$  and  $g \in \mathfrak{S}(U^L)$ , we may choose a sequence  $f_n \in C_0(K)$  such that  $\|f_n - f\| < 2^{-n}$ . Then  $\|(L | K)_{f_n}^g - (L | K)_f^g\| < 2^{-n}$  uniformly for all  $x \in G$ , so that  $(L | K)_{f_n}^g g(x) \rightarrow (L | K)_f^g g(x)$  uniformly on  $G$ . Moreover  $\|(U^L | K)_{f_n} g - (U^L | K)_f g\| < 2^{-n}\|g\|$ , from which it follows that  $(U^L | K)_{f_n} g \rightarrow (U^L | K)_f g$  locally almost everywhere. Our lemma is thereby proved.

We now assume all the hypotheses of Theorem 1. If  $\pi$  is the natural projection of  $G$  into  $G/H$ , we define  $\alpha : G/H \rightarrow K^*$  by  $\alpha(\pi(x)) = zx$  for  $x \in G$ .  $\alpha$  is continuous and one-to-one.

LEMMA 3.  $E_{G^L|K}(S) = E^L(\alpha^{-1}(S))$  for every Borel set  $S \subseteq K^*$ .

*Proof.* In the first place, we note that  $(L | K)_f = 0$  for  $f \in z$ . In fact,  $\{z\} \cap C\{z\}^- = \emptyset$  implies that  $E_{L|K}(C\{z\}^-) = 0$  from which we get  $E_{L|K}(\{z\}^-) = I$ . But this says that  $(L | K)_f v = 0$  for all  $v \in \mathfrak{S}(L)$  and all  $f \in \cap \{z\}^- = z$ , as desired.

Let  $S$  be a closed subset of  $K^*$ . Then  $\pi(x) \in \alpha^{-1}(S)$  if and only if  $zx \in S$ , that is, if and only if  $zx \in \cap S$ . Therefore  $f \in \cap S$  implies  $xf \in z$  and hence  $(L | K)_f^g = 0$ . Let  $g \in \mathfrak{S}(U^L)$ . Let  $f \in \cap S$ . By Lemma 2, we have  $[(U^L | K)_f g](x) = 0$  for locally almost all  $x \in \pi^{-1}(\alpha^{-1}(S))$ . If, moreover,  $g \in \text{Range}(E^L(\alpha^{-1}(S)))$  then  $g(x) = 0$  for locally almost all  $x \in \pi^{-1}(\alpha^{-1}(S))$ , so by Lemma 2,  $[(U^L | K)_f g](x) = 0$  for locally almost all  $x \in \pi^{-1}(\alpha^{-1}(S))$ . We conclude that  $\text{Range}(E^L(\alpha^{-1}(S))) \subseteq \text{Range}(E_{G^L|K}(S))$ .

Suppose now that  $g \notin \text{Range}(E^L(\alpha^{-1}(S)))$ . Then  $g$  does not vanish for locally almost all  $x \in \pi^{-1}(\alpha^{-1}(S))$ . Since  $g$  is Bourbaki measurable ([4], p. 180), there exists a compact set  $C \subseteq C\pi^{-1}(\alpha^{-1}(S))$  of positive Haar measure upon which  $g$  is continuous and does not vanish. Let  $x \in C$ . Then  $z \notin Sx^{-1}$  so that  $E_{L|K}(Sx^{-1}) = 0$ . Hence there exists  $f \in \cap \{Sx^{-1}\}$  such that  $(L | K)_f g(x) \neq 0$ . Setting  $f_{(x)} = x^{-1}f$ , we have  $f_{(x)} \in S$  and  $(L | K)_{f_{(x)}}^g g(x) \neq 0$ . By continuity we have  $(L | K)_{f_{(x)}}^g g(y) \neq 0$

for  $y$  in some neighborhood  $N_x$  of  $x$  in  $C$ . Since  $C$  is compact and of positive measure,  $N_x$  has positive measure for some  $x \in C$ . It follows from Lemma 2 that for that  $x$ ,  $(U^\perp | K)_{f(x)}g \neq 0$ . Therefore  $g \notin \text{Range}(E_{U^\perp | K}(S))$ .

We have proved Lemma 3 for closed  $S$ . The general case then follows from the fact that a projection valued Borel measure on a topological space is uniquely determined by its values on closed sets.

**LEMMA 4.** *Let  $H$  be a closed subgroup of the locally compact group  $G$  and let  $E$  be a regular Borel projection valued measure on  $G/H$ . Let  $\mathcal{T}$  be a  $T_0$  topology on  $G/H$  weaker than the natural topology such that  $((G/H)_{\mathcal{T}}, G)$  is a topological transformation group. Then  $E$  takes its values in the von Neumann algebra generated by  $\{E(S) : S \in \mathcal{T}\}$ .*

*Proof.* Let  $B$  be a Borel set in  $G/H$  and let  $T$  be a self-adjoint bounded operator commuting with all  $E(S)$ ,  $S \in \mathcal{T}$ . We must show that  $E(CB)TE(B) = 0$ . Since  $E$  is regular, it will suffice to show that  $E(C_1)TE(C_2) = 0$  for every disjoint compact pair  $C_1, C_2 \subseteq G/H$ . A standard compactness argument reduces the problem to the following: if  $p_1, p_2 \in G/H$ ,  $p_1 \neq p_2$ , find disjoint neighborhoods  $N_1$  of  $p_1$  and  $N_2$  of  $p_2$  such that  $E(N_1)TE(N_2) = 0$ . To do this we find a  $\mathcal{T}$ -closed  $S$  which separates  $p_1$  and  $p_2$ ; say,  $p_1 \notin S$ ,  $p_2 \in S$ . Since  $S$  is closed in the natural topology of  $G/H$ , we can find a compact neighborhood  $N$  of  $e$  in  $G$  such that  $p_1NN^{-1} \cap S = \emptyset$ . Set  $N_1 = p_1N$ ,  $N_2 = p_2N$ . Clearly  $N_1 \subseteq CSN$  and  $N_2 \subseteq SN$ . Since  $S$  is  $\mathcal{T}$ -closed and  $N$  is compact,  $SN$  is  $\mathcal{T}$ -closed. By hypothesis  $E(CSN)TE(SN) = 0$ , and our result follows.

*Proof of Theorem 1.* Let  $\mathcal{T} = \alpha^{-1}$  (topology of  $K^*$ ).  $\mathcal{T}$  and  $E^\perp$  satisfy the hypotheses of Lemma 4. According to Lemma 3  $\{E^\perp(S) : S \in \mathcal{T}\} \subseteq \{\text{values of } E_{U^\perp | K}\}$ . This, in turn, is contained in the center  $\mathcal{C}$  of the von Neumann algebra generated by  $U^\perp | K$ . By Lemma 4,  $\{\text{values of } E^\perp\} \subseteq \mathcal{C}$ . Therefore  $\mathcal{R}(U^\perp, U^\perp) = \mathcal{R}((E^\perp, U^\perp), (E^\perp, U^\perp)) \cong \mathcal{R}(L, L)$  by [2]. Finally,  $\{z\}G$  is  $E_{U^\perp | K}$ -thick by Lemma 3.

4. **Proof of Theorem 2.** For the proof of Theorem 2 we need the following lemmas;

**LEMMA 5.** *Let  $H$  be a closed subgroup of the locally compact group  $G$  such that  $G/H$  is  $\sigma$ -compact. Let  $\mathcal{T}$  be a  $T_0$  topology on  $G/H$  weaker than the natural topology such that  $((G/H)_{\mathcal{T}}, G)$  is a topological transformation group. Let  $\mathcal{B}$  be the Borel field generated by  $\mathcal{T}$ . Let  $f \in C_0(G/H)$ . Then  $f$  is  $\mathcal{B}$ -measurable.*

*Proof.* As is well known, it is enough to show that  $C \in \mathcal{B}$ , where  $C = \bigcap_i^\infty O_i$ ,  $C$  is compact and the  $O_i$  are open in  $G/H$ . (See [11], p. 220. Such a set is called a compact  $G_\delta$ .) Since  $CO_i$  is closed, it is  $\sigma$ -compact, whence  $CC$  is  $\sigma$ -compact. Therefore  $CC$  has the Lindelöf property. Let  $p \in C, q \in CC$ . As in the proof of Lemma 4, there exist open neighborhoods  $N_{pq}$  of  $p$  and  $M_{pq}$  of  $q$  and a set  $S_{pq} \subseteq G/H$  such that either  $S_{pq}$  or  $CS_{pq} \in \mathcal{S}$ ,  $N_{pq} \subseteq S_{pq}, M_{pq} \subseteq CS_{pq}$ . Since  $C$  is compact, we find  $p_1, \dots, p_n$  such that  $C \subseteq \bigcup_i^n N_{p_i q}$  and set  $S_q = \bigcup_i^n S_{p_i q}$  and  $M_q = \bigcap_i^n M_{p_i q}$ . Then  $S_q \in \mathcal{B}, C \subseteq S_q, M_q \subseteq CS_q$ , and  $M_q$  is an open neighborhood of  $q$ . Since  $CC$  is Lindelöfian, we find  $q_1, q_2, \dots$  such that  $CC \subseteq \bigcup_i^\infty M_{q_i}$  and set  $S = \bigcap_i^\infty S_{q_i}$ . Then  $S \in \mathcal{B}. C \subseteq S, CC \subseteq CS$ ; that is,  $C \in \mathcal{B}$ .

LEMMA 6. *Let  $G$  be a locally compact group,  $K$  a closed normal subgroup, and  $U$  a representation of  $G$ . Let  $G$  act on  $K^*$  as above. Then, for any Borel set  $S$  in  $K^*$  and  $x \in G, E_{U|K}(Sx) = U_x^{-1}E_{U|K}(S)U_x$ .*

*Proof.* By the uniqueness of Glimm measure, it suffices to prove this for  $S$  closed in  $K^*$ . We note that  $(U|K)^{x^{-1}} = U_x^{-1}(U|K)U_x$ . Therefore  $v \in \text{Range } E_{U|K}(Sx)$  if and only if  $(U|K)_f v = 0$  for all  $f \in \cap Sx$ , if and only if  $(U|K)_f^{x^{-1}} v = 0$  for all  $f \in \cap S$ , if and only if  $U_x^{-1}(U|K)_f U_x v = 0$  for all  $f \in \cap S$ . But this is true if and only if  $U_x v \in \text{Range } E_{U|K}(S)$ , if and only if  $v \in \text{Range } U_x^{-1} E_{U|K}(S)U_x$ .

LEMMA 7. *Let  $U$  be a representation of the locally compact group  $K$ . Let  $\mathcal{S}$  be a collection of closed subsets of  $K^*$ . Then  $\text{Range } E_U(\cap \mathcal{S}) = \cap \{\text{Range } E_U(S); S \in \mathcal{S}\}$ .*

*Proof.* Let  $S_0 = \cap \mathcal{S}$ . Then  $\cap S_0 =$  the closed ideal of  $C^*(K)$  generated by  $\cup \{\cap S : S \in \mathcal{S}\} =$  the closed linear span in  $C^*(K)$  of  $\cup \{\cap S : S \in \mathcal{S}\}$ . Now let  $v \in \cap \{\text{Range } E_U(S) : S \in \mathcal{S}\}$ . Then, for every  $S \in \mathcal{S}$  and every  $f \in \cap S$ , we have  $U_f v = 0$ ; that is,  $U_f v = 0$  for every  $f \in \cup \{\cap S : S \in \mathcal{S}\}$ . By linearity and continuity,  $U_f v = 0$  for every  $f \in \cap S$ . Therefore  $v \in \text{Range } E_U(S_0)$ . The opposite inclusion is clear, since  $E_U$  is monotonic.

*Proof of Theorem 2.* Let  $\alpha$  be as above, let  $\mathcal{S} = \alpha^{-1}$  (topology of  $K^*$ ), and let  $\mathcal{B}$  be the Borel field generated by  $\mathcal{S}$ . Then  $\mathcal{B} = \alpha^{-1}$  (Borel field of  $K^*$ ). As in [11], p. 75,  $E_0(\alpha^{-1}(S)) = E_{M|K}(S)$  for all Borel  $S$  in  $K^*$  defines a projection valued measure  $E_0$  on  $\mathcal{B}$ . According to Lemma 5, every function in  $C_0(G/H)$  is  $\mathcal{B}$ -measurable. Define  $E(f) = \int_{G/H} f dE_0$ . Clearly  $E$  is a  $*$ -representation of  $C_0(G/H)$  in the sense of [3]. We assert that  $(E, M)$  is a represen-



tation of the locally compact transformation group  $(G/H, G)$  as defined in [3].

(1)  $E(C_0(G/H))\mathfrak{S}(M)$  is dense in  $\mathfrak{S}(M)$ . In fact, since  $G/H$  is  $\sigma$ -compact, there exists a sequence of functions  $f_n \in C_0(G/H)$  such that  $0 \leq f_n \uparrow 1$ . By the monotone convergence theorem,  $E(f_n) \rightarrow I$  weakly, and (1) is established.

(2)  $M_x E(f) M_x^{-1} = E(R_x f)$  for all  $f \in C_0(G/H)$  and all  $x \in G$ . Here  $(R_x f)(p) = f(px)$ . For this, it suffices to show that  $M_x E_0(B) M_x^{-1} = E_0(Bx^{-1})$  for all  $B \in \mathcal{B}$  and  $x \in G$ . But this follows immediately from Lemma 6 and the definition of  $E_0$ .

According to the Corollary of Theorem 2 in [3],  $(E, M)$  is unitarily equivalent to an induced representation of  $(G/H, G)$ ; that is, there is a representation  $L$  of  $H$  such that  $(E, M)$  is unitarily equivalent to  $(E^L, U^L)$ . We shall henceforth assume  $E = E^L$  and  $M = U^L$ . In particular, we have  $E_0 \circ \alpha^{-1} = E_{\sigma L|K}$  and also  $E_0(C) = E^L(C)$  for every compact  $G_\delta$   $C$  in  $G/H$ .

We must now show that  $\{z\}$  is  $E_{L|K}$ -thick.

Let  $S$  be  $K^*$  closed. First suppose  $z \in S$ . We assert that  $E_{L|K}(S) = I$ . By Lemma 7, it suffices to show that  $E_{L|K}(SN) = I$  for every compact neighborhood  $N$  of  $e$  in  $G$ . Let  $g \in C_0(G)$ ,  $v \in \mathfrak{S}(L)$ . As in [1], we define  $\varepsilon(g, v) \in \mathfrak{S}(U^L)$  by

$$\varepsilon(g, v)(x) = \int_H g(\xi x) \delta_H(\xi)^{-1/2} \delta_G(\xi)^{1/2} L_\xi^{-1} v d\xi .$$

$\varepsilon(g, v)$  is continuous and has compact support modulo  $H$ . Let  $C$  be a compact  $G_\delta$  neighborhood of  $\pi(e) \subseteq \pi(N)$ . Suppose  $\text{Support}(g) \subseteq N \cap \pi^{-1}(C)$ . Then

$$\begin{aligned} \varepsilon(g, v) \in \text{Range } E^L(C) &= \text{Range } E_0(C) \subseteq \text{Range } E_0(\alpha^{-1}(SN)) \\ &= \text{Range } E_{\sigma L|K}(SN) . \end{aligned}$$

According to Lemma 2,  $f \in \cap SN$  implies  $(L|K)_f^z \varepsilon(g, v)(x) = 0$  for locally almost all  $x \in G$ , hence for all  $x \in G$  by continuity. In particular,  $(L|K)_f \varepsilon(g, v)(e) = 0$ . Letting  $g \delta_H^{-1/2} \delta_G^{1/2} | H$  approach the Dirac  $\delta$  function on  $H$ , we get  $(L|K)_f v = 0$ . Since  $v$  is arbitrary,  $E_{L|K}(SN) = I$ .

Now suppose  $z \notin S$ . We assert that  $E_{L|K}(S) = 0$ . Let  $v \in \text{Range } E_{L|K}(S)$ . Choose a compact neighborhood  $N$  of  $e$  in  $G$  such that  $\pi(NN^{-1}) \cap \alpha^{-1}(S) = \emptyset$ . Then  $\alpha^{-1}(SN) \cap \pi(N) = \emptyset$ . Let  $f \in \cap SN$ . Let  $x \in G$ . Then  $xf \in \cap SNx^{-1}$ . Hence if  $\xi \in Nx^{-1} \cap H$ , we have  $xf \in \cap S\xi$ , so that  $\xi xf \in \cap S$ . Let  $C$  be a compact  $G_\delta$  neighborhood of  $\pi(e)$  in  $\pi(N)$ . Suppose  $\text{Support}(g) \subseteq N \cap \pi^{-1}(C)$ . Then

$$(L|K)_f^z \varepsilon(g, v)(x) = \int_{N_x^{-1} \cap H} g(\xi x) \delta_H(\xi)^{-1/2} \delta_G(\xi)^{1/2} L_\xi^{-1} (L|K)_{\xi x f} v d\xi = 0$$

(compare the proof of Lemma 6). From Lemma 2,

$$\varepsilon(g, v) \in \text{Range } E_{U|_{L|K}}(SN) = \text{Range } E_0(\alpha^{-1}(SN)) .$$

On the other hand  $\varepsilon(g, v) \in \text{Range } E^1(C) = \text{Range } E_0(C)$ . Since  $\alpha^{-1}(SN) \cap C = \emptyset$ ,  $\varepsilon(g, v) = 0$ . Therefore  $\varepsilon(g, v)(e) = 0$  because  $\varepsilon(g, v)$  is continuous. Again letting  $g\delta_H^{-1/2}\delta_d^{1/2} | H$  approach the Dirac  $\delta$  function on  $H$ , we get  $v = 0$ . Therefore  $E_{L|K}(S) = 0$ .

Finally let  $\mathcal{C}$  be the class of all Borel sets  $S$  in  $K^*$  such that either  $z \in S$  and  $E_{L|K}(S) = I$  or  $z \notin S$  and  $E_{L|K}(S) = 0$ . Clearly  $\mathcal{C}$  is a  $\sigma$ -field, and by the foregoing  $\mathcal{C}$  contains all the closed sets. Therefore  $\mathcal{C}$  consists of all the Borel sets of  $K^*$ ; that is,  $\{z\}$  is  $E_{L|K}$ -thick.

**5. Connections with Mackey's work.** In the original forms of Theorems 1 and 2 due to Mackey ([13], Theorem 8.1), it is assumed that  $G$  satisfies the second axiom of countability and that  $K$  is a type  $I$  group.  $K^*$  is replaced there by  $\hat{K}$ , the set of all unitary equivalence classes of irreducible representations of  $K$ , equipped with the Mackey Borel structure. According to Glimm ([8], Theorem 1) the natural mapping of  $\hat{K}$  onto  $K^*$ , which sends every irreducible representation into its kernel in  $C^*(K)$ , is one-to-one if  $K$  is of type  $I$  and second countable. Moreover, Fell has shown [6] that in this case the Mackey Borel structure is just the  $\sigma$ -field generated by the topology of  $\hat{K}(= K^*)$ . Our result then specializes to give Mackey's result, except for the following: Mackey shows that  $L|K$  must be a multiple of the (unique up to unitary equivalence) representation (whose kernel is)  $z$ . To get this, in our general setting, seems to require a type restriction on  $K$  (or at least on  $z$ ). The form of our restriction is suggested by Glimm's theorem ([8], Theorem 1) that a separable  $C^*$  algebra is of type  $I$  if and only if its image under every irreducible representation contains the compact operators. We are led to make the following definition:

**DEFINITION.** Let  $\mathfrak{A}$  be a  $C^*$ -algebra and let  $L$  be an irreducible representation of  $\mathfrak{A}$ .  $L$  is called *semi-compact* if  $L_{\mathfrak{A}}$  contains the compact operators on  $\mathfrak{S}(L)$ .  $\text{Ker } L$  will also be called semi-compact.

We know (see Glimm [1], p. 583) that if  $L$  is semi-compact and if  $M$  is irreducible with  $\text{Ker } L = \text{Ker } M$ , then  $L$  and  $M$  are unitarily equivalent.

**LEMMA 8.** *Let  $U$  be a representation of the  $C^*$ -algebra  $\mathfrak{A}$  with structure space  $Z$ . Let  $z$  be semi-compact in  $Z$ . Suppose  $\{z\}$  is  $E_U$ -thick. Then  $U$  is a multiple of the (essentially unique) irreducible representation  $L^0$  of  $K$  such that  $\text{Ker } L^0 = z$ .*

*Proof.* By hypothesis  $\mathfrak{A}$  contains an ideal  $\mathcal{I} \supseteq z$  such that  $\mathcal{I}/z$

is isomorphic to the algebra of all compact operators on  $\mathfrak{S}(L^0)$ . As in the proof of Lemma 3,  $E_v(\{z\}^-) = I$  implies that  $U_a = 0$  for all  $a \in z$ . Dividing out by  $z$ , we may therefore assume  $z = \{0\}$ . Let  $S = \{w \in Z : w \supseteq \mathcal{S}\}$ .  $S$  is closed.  $\mathcal{S} \neq \{0\}$  implies  $\{0\} \notin S$ . Therefore  $E_v(S) = 0$ . Since  $\mathcal{S} = \cap S$ , this says that  $U|_{\mathcal{S}}$  is a nondegenerate representation of  $\mathcal{S}$ . From the known representation theory of the algebra of compact operators on a Hilbert space, we obtain an orthogonal decomposition of  $\mathfrak{S}(U)$  into  $U|_{\mathcal{S}}$  invariant subspaces  $\mathfrak{S}^\gamma$ , the restriction of  $U|_{\mathcal{S}}$  to each of which is unitarily equivalent to the irreducible representation  $L^0|_{\mathcal{S}}$ . Let  $a \in \mathfrak{A}$ ,  $v \in \mathfrak{S}^\gamma$ . Since  $U|_{\mathcal{S}}$  is nondegenerate, we can choose a sequence  $b_n \in \mathcal{S}$  such that  $U_{b_n} U_a v \rightarrow U_a v$ . But  $b_n a \in \mathcal{S}$  and hence  $U_{b_n a} v \in \mathfrak{S}^\gamma$ . Therefore  $U_a v \in \mathfrak{S}^\gamma$ ; that is, the  $\mathfrak{S}^\gamma$  are invariant under  $U$ . Let  $L^\gamma = U$  restricted to act on  $\mathfrak{S}^\gamma$ .  $L^\gamma$  is irreducible since  $L^\gamma|_{\mathcal{S}}$  is. Now  $\text{Range } E_v(\{\text{Ker } L^\gamma\}^-) \supseteq \mathfrak{S}^\gamma \neq \{0\}$ . Since  $\{0\}$  is  $E_v$ -thick,  $\{0\} \in \{\text{Ker } L^\gamma\}^-$ ; that is,  $\text{Ker } L^\gamma = \{0\}$ . Since  $\text{Ker } L^\gamma = \text{Ker } L^0$ ,  $L^\gamma \simeq L^0$ . Therefore  $U \simeq$  a multiple of  $L^0$ .

As regards Theorem 2, Mackey shows that if, in addition to the hypotheses on  $G$  and  $K$  mentioned above, one assumes that  $K$  is regularly embedded in  $G$  (see [13], p. 302 for the definition), then  $E_{M|K}$  is concentrated in an orbit if  $M$  is primary. Glimm ([9], Theorem 1) has proved that, in Mackey's case, the assumption of regular embeddedness is equivalent to the topology of  $\hat{K}/G$  being almost Hausdorff, in the following sense:

DEFINITION. Let  $X$  be a topological space.  $X$  is said to be *almost Hausdorff* if every nonvoid closed subset contains a nonvoid relatively open Hausdorff subset.

We propose to turn Glimm's theorem into a definition, even when  $K$  is not of type  $I$  and second countable.

DEFINITION. Let  $K$  be a closed normal subgroup of the locally compact group  $G$ .  $K$  is *regularly embedded* in  $G$  if  $K^*/G$  is almost Hausdorff.

REMARK. It follows from [9], p. 133, that  $K$  regularly embedded in  $G$  implies that every  $G$ -orbit in  $K^*$  is a Borel set and in fact is relatively open in its closure.

LEMMA 9. *Let  $K$  be a regularly embedded closed normal subgroup of the locally compact group  $G$ . Let  $M$  be a primary representation of  $G$ . Then there is a  $G$ -orbit of  $K^*$  which is  $E_{M|K}$ -thick.*

*Proof.* If  $S$  is a  $G$ -invariant Borel set in  $K^*$ , then

$$E_{M|K}(S) \in \mathcal{E}(M, M)$$

by Lemma 6, and hence  $E_{M|K}(S)$  belongs to the center of the von Neumann algebra generated by  $M$ . Therefore  $E_{M|K}(S) = 0$  or  $I$ . Let  $\mathcal{S}$  be the collection of all closed  $G$ -invariant  $S \subseteq K^*$  such that  $E_{M|K}(S) = I$ .  $S_0 = \bigcap \mathcal{S} \in \mathcal{S}$  by Lemma 7. Let  $\theta$  be the natural projection of  $K^*$  onto  $K^*/G$ . If  $R$  is any nonvoid relatively open subset of  $\theta(S_0)$ , then  $S_0 - \theta^{-1}(R)$  is a proper closed  $G$ -invariant subset of  $S_0$ . Hence  $E_{M|K}(S_0 - \theta^{-1}(R)) = 0$  so that  $E_{M|K}(\theta^{-1}(R)) = I$ . Now  $\theta(S_0)$  is a nonvoid closed subset of  $K^*/G$ . There exists a nonvoid relatively open Hausdorff subset  $R_0$  of  $\theta(S_0)$ . We assert that  $R_0$  reduces to a point. If not, then  $R_0$  contains two nonvoid disjoint subsets  $R_1$  and  $R_2$  which are open relative to  $R_0$  and hence to  $\theta(S_0)$ . Then  $E_{M|K}(\theta^{-1}(R_i)) = I$  for  $i = 1, 2$ , an impossibility. So  $R_0$  reduces to a point,  $\theta^{-1}(R_0)$  is a  $G$ -orbit in  $K^*$ , and our lemma is proved.

REMARK. This is not the only reasonable definition of regular embeddedness. Indeed, if  $G$  satisfies the second axiom of countability, we could simply require that  $K^*/G$  be  $T_0$  or, more generally, be countably separated. The conclusion of Lemma 9 would then follow (cf. [9], p. 126). If  $K$  is not type  $I$ , the relations between these properties and the almost Hausdorff property is obscure.

6. Three examples. Our first example shows that, despite Lemma 1, the stability subgroup of  $G$  at a point in  $\hat{K}$  may be very bad. Let  $G$  be the group whose underlying topological space is  $T \times Z \times C$ , where  $T = \{\xi \in C : |\xi| = 1\}$  and  $Z$  is the discrete integers. The group multiplication is given by

$$(\xi, m, a)(\zeta, n, b) = (\xi\zeta, m + n, a\zeta e^{in} + b).$$

Let  $K = \{1\} \times Z \times C$  and  $N = \{1\} \times \{0\} \times C$ .  $N$  and  $K$  are normal subgroups of  $G$ , and  $N$  is abelian. We identify  $\hat{N} = N^*$  with  $C$  as follows: each  $\lambda \in C$  corresponds to the character  $\chi_{(\lambda)} : (1, 0, a) \rightarrow e^{i \operatorname{Re}(a\lambda)}$ . In terms of this identification, the action of  $K$  on  $N^*$  is given by  $\lambda^{(1, m, a)} = \lambda e^{-im}$ . By Theorems 1 and 2,  ${}^\lambda L = {}_K U^{\chi_{(\lambda)}}$  is irreducible if  $\lambda \neq 0$ ; moreover  ${}^\lambda L \simeq {}^\mu L$  if and only if  $\lambda$  and  $\mu$  belong to the same  $K$ -orbit in  $N^*$ .

We next calculate  ${}^\lambda L^{(\xi, m, a)}$ . To this end, we realize  ${}^\lambda L$  in the Hilbert space of all square summable functions  $f$  on  $Z$  according to the rule:  $({}^\lambda L_{(l, n, b)} f)(k) = \exp(i \operatorname{Re}(\lambda b e^{-i(n+k)})) f(n+k)$ . Then  $({}^\lambda L_{(l, n, b)}^{(\xi, m, a)} f)(k) = ({}^\lambda L_{(l, n, b\xi^{-1}e^{-im}} f)(k) = \exp(i \operatorname{Re}(\lambda b \xi^{-1} e^{-im} e^{-i(n+k)})) f(n+k)$ . It follows that  ${}^\lambda L^{(\xi, m, a)} = \lambda \xi^{-1} e^{-im} L$ . Therefore  ${}^\lambda L^{(\xi, m, a)} \simeq {}^\lambda L$  if and only if  $\lambda$  and  $\lambda \xi^{-1}$  are in the same  $K$ -orbit. Supposing  $\lambda \neq 0$ , we see that the stability subgroup of  $G$  at  $({}^\lambda L)^\wedge$  in  $\hat{K}$  is  $\{(\xi, m, a) : \xi = e^{in} \text{ for some } n\}$ .

$n \in \mathbf{Z}$ }, a proper dense subgroup of  $G$ .

As a consequence of Lemma 1, we see that  ${}^\lambda L$  and  ${}^\mu L$  have the same kernel in  $C^*(K)$  if  $|\lambda| = |\mu|$ . (That this sufficient condition is also necessary may be seen by applying the structure theory of Glimm [10].) We also see that Theorems 1 and 2 are useless in this case in analyzing the irreducible representations  $M$  of  $G$  for which  $\{\text{Ker } {}^\lambda L\}G$  is  $E_{M|K}$ -thick. This is precisely because the stability subgroup of  $G$  at  $\text{Ker } {}^\lambda L$  in  $K^*$  is  $G$  itself.

Our second example shows that in Theorem 2 some restriction on  $G/H$ , such as  $\sigma$ -compactness, is necessary. Let  $G_1$  be the “ $ax + b$  group”; that is, the group whose underlying topological space is  $\mathbf{R} \times \mathbf{R}$  and whose group multiplication is given by  $(a, b)(c, d) = (a + c, be^c + d)$ . Let  $G$  be the same group, except that the topology is modified by making the first factor discrete. Let  $K_1 = \{0\} \times \mathbf{R} \subseteq G_1$  and  $K = \{0\} \times \mathbf{R} \subseteq G$ . Let  $\varphi$  be the (continuous) identity map of  $G$  onto  $G_1$ . Let  $\chi$  be a nontrivial character of  $K_1$ . By Theorem 1,  $M_1 = {}_{G_1}U^*$  is an irreducible representation of  $G_1$  and  $\{\chi\}G_1$  is  $E_{M_1|K_1}$ -thick in  $\hat{K}_1 = K_1^*$ . Let  $M = M_1 \circ \varphi$ , an irreducible representation of  $G$ . Because  $\varphi|K$  is an isomorphism of  $K$  onto  $K_1$  which is equivariant with respect to  $G$ , when  $G$  and  $G_1$  are identified as abstract groups under  $\varphi$ ,  $\{\chi \circ (\varphi|K)\}G$  is  $E_{M|K}$ -thick in  $\hat{K} = K^*$ . The stability subgroup of  $G$  at  $\{\chi \circ (\varphi|K)\}$  is  $K$ . But  $M$  is not induced from any representation of  $K$ , because  $\dim \mathfrak{S}(M) = \aleph_0$  while  $\dim \mathfrak{S}({}_G U^*) \geq 2^{\aleph_0}$  for any representation  $L$  of  $K$ . One may see that, in this case, the proof of Theorem 2 breaks down right at the beginning: the representation  $E$  of  $C_0(G/K)$  defined there is identically zero.

Finally, we show that the first part of Theorem 6.2 of [5] is an easy consequence of our Theorem 1. In fact, we have the following generalization: Let  $L$  be an irreducible representation of the closed normal subgroup  $K$  of the locally compact group  $G$  and let  $H$  be the stability subgroup of  $G$  at  $\hat{L}$  in  $\hat{K}$ . If  ${}_G U^L$  is irreducible, then  $H = K$ ; if  $L$  is semi-compact, the converse holds. In fact, suppose  $H \neq K$  and choose  $x \in H, x \notin K$ . Define  $T$  on  $\mathfrak{S}(U^L)$  by setting  $(Tf)(y) = Vf(xy)$ , where  $V$  implements the equivalence of  $L^x$  and  $L$ ; i.e.,  $L_\xi^x = V^{-1}L_\xi V, \xi \in K$ . Then  $(Tf)(\xi y) = Vf(x\xi y) = VL_{x\xi x^{-1}}V^{-1}Vf(xy) = L_\xi(Tf)(y)$  for  $f \in \mathfrak{S}(U^L), \xi \in K, y \in G$ , and it follows easily that  $Tf \in \mathfrak{S}(U^L)$  and that  $T$  is bounded.  $T$  clearly intertwines  $U^L$  but is not a scalar multiple of  $I$ , so the first assertion is established. The converse assertion follows from Theorem 1 together with the observation that  $H$  is the stability subgroup of  $G$  at  $\text{Ker } L$  in  $K^*$  by virtue of the comment following the definition of semi-compactness.

There remains the question of whether the converse is true without the semi-compactness condition. If  $L$  is not semi-compact, but if  $G/K$

is discrete, the converse is true ([12], Theorem 3'). However, the general case is open.

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