

Pacific Journal of Mathematics

FUNCTIONS WHICH OPERATE ON CHARACTERISTIC FUNCTIONS

ALAN G. KONHEIM AND BENJAMIN WEISS

FUNCTIONS WHICH OPERATE ON CHARACTERISTIC FUNCTIONS

ALAN G. KONHEIM AND BENJAMIN WEISS

Let G be a locally compact abelian group and $B^+(G)$ the family of continuous, complex-valued non-negative definite functions on G . Set

$$B_1^+(G) = \{f \in B^+(G); f(0) < 1\}$$

$$\Phi(G) = \{f \in B^+(G); f(0) = 1\}$$

A complex-valued function defined on the open unit disk is said to operate on $\{B_1^+(G), B^+(G)\}$ if $f \in B_1^+(G)$ implies $F(f) \in B^+(G)$, similarly for $\{\Phi(G), \Phi(G)\}$. Recently C. S. Herz has given a proof of a conjecture of W. Rudin that F operates on $\{B_1^+(G), B^+(G)\}$ if and only if

$$(*) \quad F(z) = \sum_{m,n=0}^{\infty} c_{mn} z^m \bar{z}^n, c_{mn} \geq 0, |z| < 1.$$

for a certain class of G . We shall show by independent methods that F operates on $\Phi(R^1)$ if F is given by $(*)$ for $|z| \leq 1$ and $F(1) = 1$. This answers a question posed by E. Lukacs and provides in addition an alternate proof of Herz's theorem.

Let $\mathfrak{A}, \mathfrak{B}$ denote two families of functions $a, b: X \rightarrow Y$. A function $F: Z \subseteq Y \rightarrow Y$ is said to operate on $(\mathfrak{A}, \mathfrak{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteq Z$ we have $F(a) \in \mathfrak{B}$. If $\mathfrak{A} = \mathfrak{B}$ we say simply that F operates on \mathfrak{A} . Recently there has been considerable interest in determining, for particular families $(\mathfrak{A}, \mathfrak{B})$ the class of functions which operate.

If \mathfrak{A} is the family of complex-valued 2π -periodic functions on R^1 which have absolutely convergent Fourier series

$$\mathfrak{A} = \left\{ a : a(\theta) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \text{ with } \sum_{k=-\infty}^{\infty} |a_k| < \infty \right\}$$

then a classic result of N. Wiener [10] states that $1/a \in \mathfrak{A}$ provided that $a(\theta) \neq 0$ ($0 \leq \theta < 2\pi$). P. Lévy [3] generalized Wiener's theorem by proving that analytic functions operate on \mathfrak{A} .

If \mathfrak{A} is the family of all non-negative-definite matrices $(a_{i,j})$ with $-1 < a_{i,j} < 1$ then I. J. Schoenberg [8] proved that any continuous function F which operates on \mathfrak{A} , $F: (a_{i,j}) \rightarrow (F(a_{i,j}))$ must be of the form

$$F(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$(c_n \geq 0 \quad -1 < x < 1)$$

The theorem of Wiener-Lévy can be obtained in a more general setting. Let G be a locally compact abelian group and \hat{G} its dual group, i.e. the set of continuous homomorphisms of G into the multiplicative group of complex numbers of modulus one, endowed with the weak topology. For μ a complex-valued, regular measure on G with finite total variation we define its Fourier-Stieltjes transform by

$$\hat{\mu}(\hat{x}) = \int_G \hat{x}(x) \mu(dx) \quad (\hat{x} \in \hat{G})$$

and denote by $B(\hat{G})$ the family of such transforms. Then

THEOREM. *Real entire functions operate on $B(\hat{G})$ (see [7] for definition).*

In particular by taking $G = Z$ (the group of integers) we obtain the Wiener-Lévy theorem.

A few years ago a converse to this theorem was obtained by H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin [1]. They proved that if F operates on $B(\hat{G})$ then F is a real-entire function.

In probability theory the elements of $B(\hat{G})$ which are of most direct interest are those $\hat{\mu}$ which arise from nonnegative measures μ , i.e. according to Bochner's theorem the $\hat{\mu}$ which are nonnegative-definite on \hat{G} . Let $B^+(\hat{G})$ denote this family. Rudin has conjectured [6] that the functions which operate on $(B_1^+(Z), B^+(Z))$ ¹ must have the form

$$F(z) = \sum_{\substack{n, m=0 \\ (c_m, n \geq 0)}}^{\infty} c_{n, m} z^n \bar{z}^m.$$

Recently C. S. Herz [2] published a proof of Rudin's conjecture for $(B_1^+(G), B^+(G))$ under certain restrictions on G . His proof consists of (1) showing that if F , defined on the unit disk, operates on $(B_1^+(G), B^+(G))$ then F operates on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$ where Γ_0 is the discrete multiplicative group of complex numbers of modulus one, and (2) characterizing the functions which operate on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$.

Lukacs posed in [5] the question of determining the class of functions which operate on the set of characteric functions $\mathcal{O}(R^1)$, where $\mathcal{O}(G) = \{f \in B^+(G) : f(0) = 1\}$.

We shall answer here the question posed by Lukacs, directly and by quite independent methods. This will actually yield an alternate proof of Herz's more general result by making use of some of his preliminary propositions. In § 1 we state the main theorem and outline the proof. The details occupy us in § 2-§ 4. In § 5 we show how to obtain the more general result.

¹ $Z =$ the additive group of integers with discrete topology, $B_1^+(G) = \{f \in B^+(G) : f(0) < 1\}$

1. Statement of the main theorem and outline of the proof.

THEOREM 1. *If F operates on $\Phi(R^1)$ then F is given by*

$$(*) \quad F(z) = \sum_{\substack{n,m=0 \\ (c_{n,m} \geq 0)}}^{\infty} c_{n,m} z^n \bar{z}^m \quad (|z| \leq 1).$$

with $\sum_{n,m=0}^{\infty} c_{n,m} = 1$.

Assuming that F is continuous it is first shown that F operates on $B_1^+(R^1)$. It then follows that

$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k(r) \exp(ik\theta)$$

($0 \leq r \leq 1$) where $a_k(r) \geq 0$ ($k = 0, \pm 1, \pm 2, \dots$). Having obtained this representation we prove that not only is $a_k(r)$ nonnegative, but also absolutely monotonic. Thus

$$(1) \quad F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} a_{k,n} r^n \exp(ik\theta)$$

with $a_{k,n} \geq 0$. On the other hand, if the theorem is to be true, then

$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \left\{ \sum_{\substack{n,m \geq 0 \\ n-m=k}} c_{n,m} r^{n+m} \right\} \exp(ik\theta).$$

In order to pass from (1) to (*) $a_k(r)$ must actually be of the form

$$a_k(r) = r^{|k|} \sum_{n=0}^{\infty} b_{k,n} r^{2n}$$

with $b_{k,n} \geq 0$. To prove that the exponents of r in $a_k(r)$ increase by two can be done directly (Lemma 5). To prove that $a_k(r) = O(r^{|k|})$ (near $r = 0$) we introduce the more general representation of F

$$\begin{aligned} & F(r_1 \exp(i\lambda_1 t) + r_2 \exp(i\lambda_2 t) + \dots + r_n \exp(i\lambda_n t)) \\ &= \sum_{\substack{k_j = -\infty \\ 1 \leq j \leq n}}^{\infty} \alpha_{k_1, k_2, \dots, k_n}(r_1, r_2, \dots, r_n) \exp\left\{i \sum_{j=1}^n k_j \lambda_j t\right\} \end{aligned}$$

where (r_1, r_2, \dots, r_n) varies in a suitable cube of R^n . The vanishing of $a_k(r)$ to the correct order is then deduced from the simple observation that $\alpha_{k_1, k_2, \dots, k_n}(r_1, r_2, \dots, r_n) = O(r_1 r_2 \dots r_n)$ if all $k_j \neq 0$ (Lemma 4).

Finally we turn to the question of continuity. Since $F(\phi)$ is a continuous function for every $\phi \in \Phi(R^1)$, the natural approach would be to prove directly that $z_n \rightarrow z_0$ implies $F(z_n) \rightarrow F(z_0)$ by constructing a

ch.f. ϕ together with a bounded sequence $\{t_n\}$ such that $\phi(t_n) = z_n$.² However, as the referee has observed it suffices to prove a slightly weaker interpolation property; namely that some $\phi \in \mathcal{O}(R^1)$ exists which interpolates, on a bounded sequence, some subsequence of the $\{z_n\}$. His lemma and proof are given in § 4.

2. Several lemmata. In this section we assume that F is continuous on $\mathcal{A} = \{z: |z| \leq 1\}$ and operates on $\mathcal{O}(R^1)$.

LEMMA 1. *If $p \in B_1^+(R^1)$ then $F(p) \in B_1^+(R^1)$.*

Proof. It suffices by Cramey's criterion [5, p. 65] to show that

$$\int_0^A \int_0^A F(p(t-u)) \exp(ix(t-u)) dt du \geq 0$$

for all real x and $A > 0$. If the lemma were false there would exist therefore and $A_0 > 0$ and x_0 such that

$$(2) \quad \int_0^{A_0} \int_0^{A_0} F(p(t-u)) \exp(ix_0(t-u)) dt du = -d < 0^3.$$

The function

$$p_\varepsilon(t) = \begin{cases} (1-p(0))\left(1 - \frac{|t|}{\varepsilon}\right) & \text{if } |t| \leq \varepsilon \\ 0 & \text{if } |t| > \varepsilon \end{cases}$$

is in $B_1^+(R^1)$ for every $\varepsilon > 0$, [5, p. 70] and thus $\phi_\varepsilon = p_\varepsilon + p \in B^+(R^1)$. It is, in fact, in $\mathcal{O}(R^1)$ since $\phi_\varepsilon(0) = 1$. Because F operates on $\mathcal{O}(R^1)$.

$$(3) \quad \int_0^{A_0} \int_0^{A_0} F(\phi_\varepsilon(t-u)) \exp(ix_0(t-u)) dt du \geq 0.$$

On the other hand

$$\begin{aligned} & \left| \int_0^{A_0} \int_0^{A_0} \{F(p(t-u)) - F(\phi_\varepsilon(t-u))\} \exp(ix_0(t-u)) dt du \right| \\ &= \left| \int_{G_\varepsilon} \{F(p(t-u)) - F(\phi_\varepsilon(t-u))\} \exp(ix_0(t-u)) dt du \right| \leq 4A_0\varepsilon \\ G_\varepsilon &= \{(t, u): 0 \leq t \leq A_0, 0 \leq u \leq A_0, |t-u| \leq \varepsilon\} \end{aligned}$$

since $|F(z)| \leq 1$ on \mathcal{A} . If we take $\varepsilon < d/4A_0$ then (3) contradicts (2).

Let n be a positive integer and $2\pi, \lambda_1, \lambda_2, \dots, \lambda_n$ be rationally independent real numbers. For each vector $\mathbf{m} = (m_1, m_2, \dots, m_n)$ with

² We were not able to deduce this strong interpolation property for $\mathcal{O}(R^1)$ and this necessitated a somewhat round about argument in the original version of this paper.

³ That the integral in (2) is real follows from the easily verified identity $F(\bar{z}) = \overline{F(z)}$.

integral components and each vector $r = (r_1, r_2, \dots, r_n)$ with $0 \leq r_i < 1/n$ ($1 \leq i \leq n$) we formally define $\alpha_m(r)$ by

$$(4) \quad \alpha_m(r) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F\left(\sum_{k=1}^n r_k \exp(i\lambda_k t)\right) \exp\left\{-it \sum_{k=1}^n m_k \lambda_k\right\} dt.$$

LEMMA 2. *The limit in (4) exists and is independent of $\lambda_1, \lambda_2, \dots, \lambda_n$ (provided that $2\pi, \lambda_1, \lambda_2, \dots, \lambda_n$ are rationally independent real numbers).*

Proof. Combining Lemma 1 with the observation that

$$\sum_{k=1}^n r_k \exp(i\lambda_k \cdot) \in B_1^+(R^1)$$

we see

$$F\left(\sum_{k=1}^n r_k \exp(i\lambda_k \cdot)\right) \in B_1^+(R^1)$$

and hence the limit in (4) exists [5, p. 43].

The Kronecker-Weyl theorem [9] next shows that

$$(5) \quad \alpha_m(r) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} F\left(\sum_{k=1}^n r_k \exp(i\phi_k)\right) \times \exp\left[-i \sum_{k=1}^n m_k \phi_k\right] d\phi_1 d\phi_2 \dots d\phi_n$$

and hence $\alpha_m(r)$ is independent of the particular $\{\lambda_j\}$ chosen.

A function f defined on the cube $0 \leq x_i < a$ ($1 \leq i \leq n$) is called *absolutely monotonic function* if

$$\frac{\partial^{j_1+j_2+\dots+j_n}}{\partial x_1^{j_1} \partial x_2^{j_2} \dots \partial x_n^{j_n}} f(x_1, x_2, \dots, x_n) \geq 0$$

throughout the cube for $j_1, j_2, \dots, j_n = 0, 1, 2, \dots$ Just as in the case of one variable, an absolutely monotonic function admits a power series expansion with nonnegative coefficients.

LEMMA 3. *The pointwise limit of absolutely monotonic functions is absolutely monotonic.*

Proof. For $n = 1$ the lemma is well known. We then proceed by induction to $n + 1$. Suppose

$$\lim_{k \rightarrow \infty} f_k(r_1, r_2, \dots, r_{n+1}) = f(r_1, r_2, \dots, r_{n+1}).$$

For fixed r_1, r_2, \dots, r_n we have

$$f_k(r_1, r_2, \dots, r_{n+1}) = \sum_{j=0}^{\infty} a_{k,j}(r_1, r_2, \dots, r_n) r_{n+1}^j \rightarrow f(r_1, r_2, \dots, r_{n+1})$$

and hence

$$f(r_1, r_2, \dots, r_{n+1}) = \sum_{j=0}^{\infty} a_j(r_1, r_2, \dots, r_n) r_{n+1}^j$$

with

$$a_j(r_1, r_2, \dots, r_n) = \lim_{k \rightarrow \infty} a_{k,j}(r_1, r_2, \dots, r_n) .$$

Since $a_{k,j}(r_1, r_2, \dots, r_n)$ is an absolutely monotonic function the induction hypothesis implies $a_j(r_1, r_2, \dots, r_n)$ is likewise so and lemma is proved.

LEMMA 4. *In the cube $0 \leqq r_i < 1/n (1 \leqq i \leqq n)$*

(4i) *$a_m(r)$ is an absolutely monotonic function*

$$(6) \quad a_m(r) = \sum_{\substack{0 \leqq i_j < \infty \\ 1 \leqq j \leqq n}} \alpha_{i_1, i_2, \dots, i_n}(m) r_1^{i_1} r_2^{i_2} \dots r_n^{i_n}$$

and

(4ii) *If $m_i \neq 0$ for every $i (1 \leqq i \leqq n)$ then $\alpha_{i_1, i_2, \dots, i_n}(m) = 0$ if $i_j = 0$ for some $j (1 \leqq j \leqq n)$.*

Proof. 1. Generalizing a result of Rudin [6, p. 618] we will show that if f is continuous in the cube $0 \leqq x_i < a (1 \leqq i \leqq n)$ and satisfies

$$(7) \quad \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} f(a_1 + b_1 \cos \theta_1, a_2 + b_2 \cos \theta_2, \dots, a_n + b_n \cos \theta_n) \\ \times \prod_{k=1}^n \cos j_k \theta_k d\theta_k \geqq 0$$

for all integers $j_1, j_2, \dots, j_n = 0, 1, 2, \dots$ whenever $0 \leqq b_j \leqq a_j, a_j + b_j < a$, then f is absolutely monotonic in the cube $0 \leqq x_i < a (1 \leqq i \leqq n)$.

2. To see that $a_m(r)$ satisfies (7) (with $a = 1/n$) we observe that

$$I = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} a_m(a_1 + b_1 \cos \theta_1, \dots, a_n + b_n \cos \theta_n) \\ \times \prod_{k=1}^n \cos j_k \theta_k d\theta_k \\ = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} a_m(a_1 + b_1 \cos \theta_1, \dots, a_n + b_n \cos \theta_n) \\ \times \exp -i \sum_{k=1}^n j_k \theta_k d\theta_1 d\theta_2 \dots d\theta_n$$

since the integrand in I is an even function of each of the $\{\theta_k\}$. Next, the integral representation of $a_m(r)$ and the Kronecker-Weyl theorem yields

$$\begin{aligned}
 I &= \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \\
 &\times F((a_1 + b_1 \cos \theta_1) \exp(i\phi_1) + \cdots + (a_n + b_n \cos \theta_n) \exp(i\phi_n)) \\
 &\times \exp -i \sum_{k=1}^n (j_k \theta_k + m_k \phi_k) d\theta_1 \cdots d\theta_n d\phi_1 \cdots d\phi_n .
 \end{aligned}$$

A final application of the Kronecker-Weyl theorem shows

$$\begin{aligned}
 I &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F\left(\sum_{k=1}^n (a_k + b_k \cos \zeta_k t) \exp(i\lambda_k t)\right) \\
 &\times \exp -i \sum_{k=1}^n (j_k \zeta_k + m_k \lambda_k) t dt^4
 \end{aligned}$$

and this limit is nonnegative because

$$\sum_{k=1}^n (a_k + b_k \cos \zeta_k \cdot) \exp(i\lambda_k \cdot) \in B_1^+(R^1) ,$$

Lemma 1 and [5, p. 43].

3. Suppose first that f satisfies (7) and is of class C^∞ . To show that

$$(8) \quad \frac{\partial^{j_1+j_2+\cdots+j_n}}{\partial x_1^{j_1} \partial x_2^{j_2} \cdots \partial x_n^{j_n}} f(x_1, x_2, \cdots, x_n) \geq 0$$

in the cube $0 \leq x_i < a$ ($1 \leq i \leq n$) we let $N = j_1 + j_2 + \cdots + j_n$ and write, by Taylor's theorem,

$$\begin{aligned}
 &f(a_1 + b_1 \cos \theta_1, \cdots, a_n + b_n \cos \theta_n) \\
 (9) \quad &= \sum_{k=0}^N \frac{1}{k!} \left(b_1 \cos \theta_1 \frac{\partial}{\partial x_1} + \cdots + b_n \cos \theta_n \frac{\partial}{\partial x_n} \right)^k f \Big|_{\substack{x_i = a_i \\ 1 \leq i \leq n}} \\
 &+ \frac{1}{(N+1)!} \left(b_1 \cos \theta_1 \frac{\partial}{\partial x_1} + \cdots + b_n \cos \theta_n \frac{\partial}{\partial x_n} \right)^{N+1} f \Big|_{\substack{x_i = a_i + \eta_i b_i \cos \theta_i \\ 1 \leq i \leq n}} .
 \end{aligned}$$

Multiply (9) by $\prod_{k=1}^n \cos j_k \theta_k d\theta_k$ and integrate from 0 to 2π . Set $b_i = b < \min_k a_k$ and let $b \downarrow 0$ to obtain (8).

4. If f is *a priori* only continuous, we proceed as follows: let $g: R^1 \rightarrow R^1$ satisfy

- (i) $g \in C^\infty$
- (ii) $g(t) > 0$ if $0 < t < 1$; $g(t) = 0$ otherwise
- (iii) $\int_0^1 g(t) dt = 1$.

If f satisfies (7), then so does

$$\begin{aligned}
 f_\epsilon(x_1, x_2, \cdots, x_n) &= \int_0^1 \int_0^1 \cdots \int_0^1 \\
 &\times f(x_1 + \delta y_1, \cdots, x_n + \delta y_n) \prod_{k=1}^n g(y_k) dy_k
 \end{aligned}$$

⁴ The numbers $2\pi, \lambda_1, \cdots, \lambda_n, \zeta_1, \cdots, \zeta_n$ are taken to be rationally independent real numbers.

on the cube $0 \leq x_i < a - \delta$ ($1 \leq i \leq n$). Now $f_\delta \in C^\infty$ and the argument in 3. applies to show that f_δ is absolutely monotonic. But $f_\delta \rightarrow f$ (pointwise) in the cube $0 \leq x_i < a$ ($1 \leq i \leq n$) and Lemma 3 permits us to complete the proof of 4(i).

5. If $m_k \neq 0$ ($1 \leq k \leq n$) then from (5) we see

$$\begin{aligned} \alpha_m(0, r_2, \dots, r_n) &= \alpha_m(r_1, 0, r_3, \dots, r_n) = \dots \\ &= \alpha_m(r_1, r_2, \dots, r_{n-1}, 0) = 0 \end{aligned}$$

and this yields (4)ii.

LEMMA 5. *If*

$$(10) \quad \begin{aligned} \alpha_k(r) &= \frac{1}{2\pi} \int_0^{2\pi} F(r \exp(i\phi)) \exp(-ik\phi) d\phi \\ k &= 0, \pm 1, \pm 2, \dots \end{aligned}$$

then

$$5(i) \quad \alpha_k(-r) = (-1)^k \alpha_k(r)$$

and

$$5(ii) \quad \alpha_k(r) = \sum_{j=0}^{\infty} a_{k,j} r^j \quad -1 \leq r \leq 1$$

with

$$a_{k,j} \geq 0 \quad \sum_{j=0}^{\infty} a_{k,j} < \infty .$$

Thus

$$\alpha_k(r) = \begin{cases} \sum_{j=0}^{\infty} a_{k,2j} r^{2j} & \text{if } k \text{ is an even integer} \\ \sum_{j=0}^{\infty} a_{k,2j+1} r^{2j+1} & \text{if } k \text{ is an odd integer.} \end{cases}$$

Proof. For 5(i) note

$$\alpha_k(-r) = \frac{1}{2\pi} \int_0^{2\pi} F(r \exp i(\phi + \pi)) \exp(-ik\phi) d\phi = (-1)^k \alpha_k(r) .$$

Proceeding as in the proof of Lemma 4, we show that

$$\begin{aligned} \int_0^{2\pi} \alpha_k(\cos \theta) \exp -i\nu\theta d\theta &\geq 0 \\ \nu &= 0, \pm 1, \pm 2, \dots \end{aligned}$$

so that $\alpha_k(\cos \cdot) \in B^+(R^1)$. It follows from [4, p. 202] that

$$\alpha_k(\cos \theta) = \sum_{j=0}^{\infty} b_{k,j} \cos j\theta$$

with

$$b_{k,j} \geq 0 \sum_{j=0}^{\infty} b_{k,j} < \infty .$$

If T_j denotes the j th Tchebychev polynomial then

$$(11) \quad a_k(x) = \sum_{j=0}^{\infty} b_{k,j} T_j(x) \quad -1 \leq x \leq 1 .$$

But for $0 \leq x \leq 1$, Lemma 4 yields the representation

$$a_k(x) = \sum_{j=0}^{\infty} a_{k,j} x^j$$

with

$$a_{k,j} \geq 0 \sum_{j=0}^{\infty} a_{k,j} < \infty .$$

Using elementary properties of the Tchebychev polynomials and the fact that the Fourier series of a C^∞ function may be differentiated term-by-term, 5(i) and (11) imply that the equality

$$\sum_{j=0}^{\infty} a_{k,j} x^j = \sum_{j=0}^{\infty} b_{k,j} T_j(x)$$

extends to $-1 \leq x \leq 1$, and this proves 5(ii).

3. Proof of Theorem 1 with hypothesis of continuity.
 $F(r \exp(i\phi))$ is a continuous, periodic, nonnegative definite function. We can therefore write

$$(12) \quad F(r \exp(i\phi)) = \sum_{k=-\infty}^{\infty} a_k(r) \exp(ik\phi) \\ 0 \leq r \leq 1 \quad 0 \leq \phi \leq 2\pi$$

with

$$a_k(r) \geq 0 \quad (k = 0, \pm 1, \pm 2, \dots) \quad \sum_{k=-\infty}^{\infty} a_k(r) = F(r) .$$

In (12) we set $z = r \exp(i\phi)$ and use Lemma 5 to conclude that

$$(13) \quad F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m + \sum_{1 \leq m \leq n < \infty} (d_{n,m} z^n / \bar{z}^m + e_{n,m} \bar{z}^n / z^m)$$

with

$$c_{n,m} \geq 0 \quad (n, m = 0, 1, 2, \dots) \\ d_{n,m} \geq 0 \quad e_{n,m} \geq 0 \quad (1 \leq m \leq n < \infty) \\ \sum_{n,m=0}^{\infty} c_{n,m} + \sum_{1 \leq m \leq n < \infty} (d_{n,m} + e_{n,m}) = 1 .$$

We will now show that $d_{n_0, m_0} = 0$. Let $2\pi, \lambda_1, \dots, \lambda_{n_0}, \lambda$ be rationally independent real numbers and set

$$(14) \quad z = r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t)$$

in (13) where

$$0 \leq r < 2/3 \quad r_k = r/2n_0 \quad (1 \leq k \leq n_0).$$

Let $m = (m_0, \underbrace{1, 1, \dots, 1}_{n_0})$ and note by Lemma 4

$$(15) \quad \begin{aligned} a_m(r, r_1, r_2, \dots, r_{n_0}) &= C_m r r_1 r_2 \dots r_{n_0} + o(r r_1 r_2 \dots r_{n_0}) \\ &= C_m \left(\frac{1}{2n_0}\right)^{n_0} r^{n_0+1} + o(r^{n_0+1}). \end{aligned}$$

Examining the term z^α/\bar{z}^β with z as in (14) we obtain

$$(16) \quad \begin{aligned} &\frac{\left(r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t)\right)^\alpha}{\left(r \exp(-i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(-i\lambda_k t)\right)^\beta} \\ &= r^{\alpha-\beta} \left(\exp(i\lambda t) + \frac{1}{2n_0} \sum_{k=1}^{n_0} \exp(i\lambda_k t)\right)^\alpha \exp(i\beta\lambda t) \\ &\quad \times \sum_{p=0}^{\infty} b_p \left\{ \frac{1}{2n_0} \sum_{k=1}^{n_0} \exp(-i(\lambda_k - \lambda)t) \right\}^p \quad (b = 1) \end{aligned}$$

so that only the terms z^α/\bar{z}^β with $\beta = m_0 - j, \alpha = n_0 + j (0 \leq j \leq m_0 - 1)$ yield a contribution to $a_m(r, r_1, r_2, \dots, r_{n_0})$. But with z as in (14)

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T z^{n_0+j}/\bar{z}^{m_0-j} \exp(-i(m_0\lambda + \lambda_1 + \dots + \lambda_{n_0})t) dt \\ &= D_j r^{n_0-m_0+2j} \end{aligned}$$

with $D_j \neq 0$ for $j = 0$. Thus (15) implies that $d_{n_0, m_0} = 0$. A similar argument shows $e_{n_0, m_0} = 0$ and the theorem is proved with the hypothesis of continuity.

4. The continuity of F^5 . We begin with an interpolation lemma.

LEMMA 6. *Let $z_n \rightarrow z_0 (|z_n| < 1, n = 0, 1, 2, \dots)$. There exists a ch.f. ϕ , a sequence (of real numbers) $t_k \rightarrow 1$ and a sequence (of integers) $\{n_k\}$ such that $\phi(t_k) = z_{n_k}$.*

Proof. Let $\tau_n = 1 - (2/3)9^{-n}$; then $(9^n/2)\tau_n \equiv (1/6) \pmod{1}$ while $(9^{n+m}/2)\tau_n \equiv (1/2) \pmod{1}$ for $m > 0$. Hence

⁵ We wish to acknowledge our thanks to the referee for the statement and proof of Lemma 6.

$$\cos \frac{\pi}{2} 9^n \tau_n = \frac{\sqrt{3}}{2}, \cos \frac{\pi}{2} 9^{n+m} \tau_n = 0 \quad (m > 0)$$

and $\cos (\pi/2)9^n = 0$. Let $\{\eta_n\}$ be a sequence of positive numbers such that

$$|z_0| + \sum_{n=1}^{\infty} \eta_n < 1.$$

We define inductively a sequence $\{\phi_n\}$ of positive-definite functions as follows; let

$$\phi_0(t) = |z_0| e^{i(\arg z_0)t}.$$

Assume that $\phi_0, \phi_1, \dots, \phi_p$ have been defined such that $\phi_j(1) = 0$ for $j > 0$. Choose integers m_{p+1} and n_{p+1} such that

$$r_{p+1} = \left| \sum_{j=0}^p \phi_j(\tau_{m_{p+1}}) - z_{n_{p+1}} \right| < \frac{\eta_{p+1}}{2}$$

and define

$$\phi_{p+1}(t) = 2r_{p+1}(\cos \varepsilon_{p+1}t) \left(\cos \frac{\pi}{2} 9^{m_{p+1}}t \right) e^{i\lambda_{p+1}t}$$

where ε_{p+1} and λ_{p+1} are chosen such that

$$\phi_{p+1}(\tau_{m_{p+1}}) = z_{n_{p+1}} - \sum_{j=0}^p \phi_j(\tau_{m_{p+1}}).$$

We shall assume that the sequence $\{m_k\}$ is strictly increasing. If we set $t_k = \tau_{m_k}$ and

$$\phi(t) = \sum_{j=0}^{\infty} \phi_j(t) + \varepsilon \Delta(t)$$

where $\Delta(x) = \max(0, 1 - 2|x|)$ and $\varepsilon > 0$ is such that $\phi(0) = 1$ then $\phi(t_k) = z_{n_k}$ ($k = 1, 2, \dots$) and $\phi \in \mathcal{O}(R^1)$.

LEMMA 7. *F is continuous in the open unit disk $\{z: |z| < 1\}$.*

Proof. Suppose not; then there would exist a $z_0, |z_0| < 1$ and a sequence $\{z_n\}$ ($|z_n| < 1$) such that $z_n \rightarrow z_0$ and $F(z_n) \not\rightarrow F(z_0)$. By passing to a subsequence if necessary we can assume that $\{F(z_n)\}$ converges. By Lemma 6 there is a ch.f. ϕ and a sequence (of real numbers) $\{t_k\}$ with limit one such that $\phi(t_k) = z_{n_k}$. But then

$$F(z_0) = F(\phi(1)) = \lim_{k \rightarrow \infty} F(\phi(t_k)) = \lim_{k \rightarrow \infty} F(z_{n_k})$$

which is a contradiction.

REMARK. For future reference let us note that Lemma 1 now shows that F operates on $B_1^+(\mathbb{R}^1) \cup \Phi(\mathbb{R}^1)$.

LEMMA 8. F is continuous on $-1 \leq x \leq 1$.

Proof. By observing that $F(\cos \cdot) \in \Phi(\mathbb{R}^1)$, we obtain, just as in Lemma 5

$$F(x) = \sum_{n=0}^{\infty} p_n T_n(x)$$

where $p_n \geq 0$ and

$$\sum_{n=0}^{\infty} p_n = 1 .$$

Since $|T_n(x)| \leq 1$ on $-1 \leq x \leq 1$, F is continuous there.

THEOREM 2. F is continuous on \mathcal{A} .

Proof. As we have already remarked, F operates on $B_1^+(\mathbb{R}^1) \cup \Phi(\mathbb{R}^1)$. Now Lemmata 2-5 carry over mutatis mutandis to prove that

$$(20) \quad F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m$$

$$|z| < 1$$

where $c_{n,m} \geq 0$. Setting $z = x$ in (20) and using Lemma 8 we see that

$$\lim_{x \uparrow 1} \sum_{k=0}^{\infty} \sum_{\substack{n,m \geq 0 \\ n+m=k}} c_{n,m} x^k = F(1) = 1 .$$

But the $\{c_{n,m}\}$ are nonnegative and hence

$$\sum_{n,m=0}^{\infty} c_{n,m} = 1 .$$

Thus our series in (20) extends to a continuous function on \mathcal{A} . We assert that F is equal to this extension. For let $\phi \in \Phi(\mathbb{R}^1)$ $t_k \rightarrow t_0$ with $0 < |\phi(t_k)| < 1, |\phi(t_0)| = 1$. Then $F(\phi)$ is a continuous function and thus $\lim F(\phi(t_k)) = F(\phi(t_0))$. But

$$\lim F(\phi(t_k)) = \lim \sum_{n,m=0}^{\infty} c_{n,m} (\phi(t_k))^n \overline{(\phi(t_k))^m}$$

$$= \sum_{n,m=0}^{\infty} c_{n,m} (\phi(t_0))^n \overline{(\phi(t_0))^m}$$

and thus

$$F(\phi(t_0)) = \sum_{n,m=0}^{\infty} c_{n,m} (\phi(t_0))^n \overline{(\phi(t_0))^m} .$$

5. **Concluding remarks.** In order to obtain the general theorem we require two propositions due to Herz [2 p. 165, p. 167].

PROPOSITION 1. If a locally compact abelian group H has elements of arbitrarily high order then every F which operates on $(B_1^+(H), B^+(H))$ is continuous.

PROPOSITION 2. If a locally compact abelian group H has elements of arbitrarily high order, then every F which operates on $(B_1^+(H), B^+(H))$ operates on $(B_1^+(Z), B^+(Z))$.

REMARKS. 1. In Propositions 1 and 2 it is assumed that F is defined on $\{z: |z| < 1\}$.

2. Proposition 1 does not include our Lemma 7 since we assume merely that F operates on $\mathcal{P}(R^1)$, not on $(B_1^+(R^1), B^+(R^1))$.

THEOREM 2. *If a locally compact abelian group H has elements of arbitrarily high order, then F operates on $(B_1^+(H), B^+(H))$ if and only if*

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m, \quad (|z| < 1)$$

where $c_{n,m} \geq 0$.

Proof. By Propositions 1 and 2 we may assume that $H = Z$ and that F is continuous. It suffices, by the proof of Theorem 1, to show that F operates on $(B_1^+(R^1), B^+(R^1))$. Suppose $\lambda \in B_1^+(R^1)$ and set $\phi = F(\lambda)$. Since ϕ is continuous all that must be verified is that ϕ is a nonnegative-definite function. For any $\delta > 0$, the sequence $\{\lambda_n = \lambda(n\delta)\}$ is nonnegative definite and therefore by the hypothesis $\{\phi(n\delta)\}$ is a nonnegative definite sequence for any $\delta > 0$. Since ϕ is continuous

$$\begin{aligned} & \int_0^A \int_0^A \phi(u-v) \exp(ix(u-v)) du dv \\ &= \lim_{\delta \downarrow 0} \sum_{n,m=1}^{A/\delta} \phi((n-m)\delta) \exp ix\delta(n-m) \delta^2. \end{aligned}$$

But since $\{\phi(n\delta)\}$ is a nonnegative-definite sequence for each $\delta > 0$

$$\sum_{n,m=1}^{A/\delta} \phi((n-m)\delta) \exp ix\delta(n-m) \delta^2 \geq 0$$

and hence by Cramer's criterion ϕ is nonnegative definite.

We conclude with a few remarks.

1. There is a formal relation between the result of [1] and our Theorem 1. Every real-entire function F can be written in the form

$$F = (F_1 - F_2) + i(F_3 - F_4)$$

where F_1, F_2, F_3 and F_4 satisfy (*). On the other hand every $\hat{\mu} \in B(\hat{G})$ is of the form

$$\hat{\mu} = (\hat{\mu}_1 - \hat{\mu}_2) + i(\hat{\mu}_3 - \hat{\mu}_4)$$

where $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$ and $\hat{\mu}_4$ are in $B^+(\hat{G})$. A direct proof of our theorem starting from this observation would be desirable.

2. The proof given here of Theorem 1 demonstrates in one stroke that F is real-analytic in Δ and if it is expressed as a power series in z and \bar{z} it has nonnegative coefficients. If one could prove directly that F operates on all Fourier transforms assuming values in Δ then proof of the theorem could be completed in two steps:

(A) F is real-analytic [7, Chapter VI] and thus

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m$$

(B) $c_{n,m} \geq 0$ ($n, m = 0, 1, 2, \dots$) The second step is a consequence of the explicit representation

$$c_{n,m} = \lim_{r \downarrow 0} \lim_{T \rightarrow \infty} \frac{1}{r^{n+m}} \frac{1}{2T} \int_{-T}^T F\left(\sum_{k=1}^{n+m} r_k \exp(i\lambda_k t)\right) \times \exp\left(\sum_{k=1}^n \lambda_k t - i \sum_{k=1}^m \lambda_{n+k} t\right) dt^6$$

where the inner limit exists and is positive by virtue of Lemma 1 and [5, p. 43] and the outer limit exists by (A) above.

3. For nondiscrete G with elements of arbitrarily high order one can show by using the methods used in the proof of Theorem 1, that F operates on $\mathcal{O}(G)$ if and only if F satisfies (*). If G is discrete this needn't be the case, and F needn't even be continuous as, $F(z) = 0$ ($|z| < 1$), $=1$ ($|z| = 1$), which operates on $\mathcal{O}(Z)$ already shows. For such discrete groups we don't know if it is true that F operates on $\mathcal{O}(G)$ implies that F must operate on $B_1^+(G)$. If it were true then at least the structure of F for $|z| < 1$ could be determined.

REFERENCES

1. H. Helson, J. P. Kahane, Y. Katznelson, and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. **102** (1959), 135-157.

⁶ $\lambda_1, \dots, \lambda_{n+m}$ are rationally independent real numbers

2. C. S. Herz, *Fonctions opérant sur les fonctions définies-positives*, Ann. Inst. Fourier **13** (1963), 161-180.
3. P. Lévy, *Sur la convergence absolue des séries de Fourier*, Compositio Math. **1** (1934), 1-14.
4. M. Loève, *Probability Theory*, van Nostrand, (1955).
5. E. Lukacs, *Characteristic Functions*, Griffen's Statistical Monographs And Courses, (1960).
6. W. Rudin, *Positive definite sequences and absolutely monotonic functions*, Duke Math. J. **26** (1959), 617-622.
7. ———, *Fourier Analysis On Groups*, John Wiley and Sons. (1962).
8. I. J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. **9** (1942), 96-108.
9. H. Weyl, *Über die gleichverteilung Von Zahlen mod. eins.*, Math. Ann. **77** (1916), 313-352.
10. N. Wiener, *Tauberian theorems*, Ann. of Math. **33** (1932), 1-100.

THOMAS J. WATSON RESEARCH CENTER
INTERNATIONAL BUSINESS MACHINES CORPORATION

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

*RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Basil Gordon, Acting Managing Editor until February 1, 1966.

Pacific Journal of Mathematics

Vol. 15, No. 4

December, 1965

Robert James Blattner, <i>Group extension representations and the structure space</i>	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i>	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i>	1119
Hsin Chu, <i>Fixed points in a transformation group</i>	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i>	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i>	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i>	1153
Michel Mendès France, <i>A set of nonnormal numbers</i>	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i>	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i>	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i>	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i>	1239
James Patrick Jans, <i>Some aspects of torsion</i>	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i>	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i>	1279
Ronald John Larsen, <i>Almost invariant measures</i>	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i>	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i>	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i>	1319
Mitsuru Nakai, <i>Φ-bounded harmonic functions and classification of Riemann surfaces</i>	1329
L. G. Novoa, <i>On n-ordered sets and order completeness</i>	1337
Fredos Papangelou, <i>Some considerations on convergence in abelian lattice-groups</i>	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i>	1365
John R. Ringrose, <i>On sub-algebras of a C^*-algebra</i>	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i>	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i>	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i>	1397
Hans Samelson, <i>On small maps of manifolds</i>	1401
Annette Sinclair, <i>$\varepsilon(z)$-closeness of approximation</i>	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i>	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i>	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i>	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i>	1435
Horace C. Wisner, <i>Embedding a circle of trees in the plane</i>	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i>	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i>	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i>	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in S^n be cellular</i>	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i>	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i>	1474