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Let G be a locally compact abelian group and $B^+(G)$ the family of continuous, complex-valued non-negative definite functions on G. Set

A complex-valued function defined on the open unit disk is said to *operate* on $\{B_1^+(G), B^+(G)\}$ if $f \in B_1^+(G)$ implies $F(f) \in B^+(G)$, similarly for $\{\Phi(G), \Phi(G)\}$. Recently C. S. Herz has given a proof of a conjecture of W. Rudin that F operates on $\{B_1^+(G), B^+(G)\}$ if and only if

(*)
$$F(z) = \sum_{m,n=0}^{\infty} c_{mn} z^m \bar{z}^n, c_{mn} \ge 0, |z| < 1.$$

for a certain class of G. We shall show by independent methods that F operates on $\mathcal{P}(R^1)$ if F is given by (*) for $|z| \leq 1$ and F(1) = 1. This answers a question posed by E. Lukacs and provides in addition an alternate proof of Herz's theorem.

Let $\mathfrak{A}, \mathfrak{B}$ denote two familes of functions $a, b: X \to Y$. A function $F: Z \subseteq Y \to Y$ is said to operate on $(\mathfrak{A}, \mathfrak{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteq Z$ we have $F(a) \in \mathfrak{B}$. If $\mathfrak{A} = \mathfrak{B}$ we say simply that F operates on \mathfrak{A} . Recently there has been considerable interest in determining, for particular families $(\mathfrak{A}, \mathfrak{B})$ the class of functions which operate.

If \mathfrak{A} is the family of complex-valued 2π -periodic functions on R^1 which have absolutely convergent Fourier series

$$\mathfrak{A} = \left\{a: a(heta) \thicksim \sum_{k=-\infty}^{\infty} a_k e^{ik heta} ext{ with } \sum_{k=-\infty}^{\infty} |a_k| < \infty
ight\}$$

then a classic result of N. Wiener [10] states that $1/a \in \mathfrak{A}$ provided that $a(\theta) \neq 0 \ (0 \leq \theta < 2\pi)$. P. Lévy [3] generalized Wiener's theorem by proving that analytic functions operate on \mathfrak{A} .

If \mathfrak{A} is the family of all non-negative-definite matrices $(a_{i,j})$ with $-1 < a_{i,j} < 1$ then I. J. Schoenberg [8] proved that any continuous function F which operates on \mathfrak{A} , $F: (a_{i,j}) \to (F(a_{i,j}))$ must be of the form

$$egin{aligned} F(x) &= \sum\limits_{n=0}^{\infty} c_n x^n \ (c_n &\geq 0 \ -1 < x < 1) \end{aligned}$$

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The theorem of Wiener-Lévy can be obtained in a more general setting. Let G be a locally compact abelian group and \hat{G} its dual group, i.e. the set of continuous homomorphisms of G into the multiplicative group of complex numbers of modulus one, endowed with the weak topology. For μ a complex-valued, regular measure on G with finite total variation we define its Fourier-Stieltjes transform by

$$\widehat{\mu}(\widehat{x}) = \int_{g} \widehat{x}(x) \mu(dx) \quad (\widehat{x} \in \widehat{G})$$

and denote by $B(\hat{G})$ the family of such transforms. Then

THEOREM. Real entire functions operate on $B(\hat{G})$ (see [7] for definition).

In particular by taking G = Z (the group of integers) we obtain the Wiener-Lévy theorem.

A few years ago a converse to this theorem was obtained by H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin [1]. They proved that if F operates on $B(\hat{G})$ then F is a real-entire function.

In probability theory the elements of $B(\hat{G})$ which are of most direct interest are those $\hat{\mu}$ which arise from nonnegative measures μ , i.e. according to Bochner's theorem the $\hat{\mu}$ which are nonnegative-definite on \hat{G} . Let $B^+(\hat{G})$ denote this family. Rudin has conjectured [6] that the functions which operate on $(B_1^+(Z), B^+(Z))^1$ must have the form

$$F(z)=\sum\limits_{n,m=0\atop (c_{m,n}\geq 0)}^{\infty}c_{n,m}z^{n}\overline{z}^{m}$$
 .

Recently C. S. Herz [2] published a proof of Rudin's conjecture for $(B_1^+(G), B^+(G))$ under certain restrictions on G. His proof consists of (1) showing that if F, defined on the unit disk, operates on $(B_1^+(G), B^+(G))$ then F operates on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$ where Γ_0 is the discrete multiplicative group of complex numbers of modulus one, and (2) characterizing the functions which operate on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$.

Lukacs posed in [5] the question of determining the class of functions which operate on the set of characteric functions $\mathscr{O}(R^1)$, where $\mathscr{O}(G) = \{f \in B^+(G): f(0) = 1\}$.

We shall answer here the question posed by Lukacs, directly and by quite independent methods. This will actually yield an alternate proof of Herz's more general result by making use of some of his preliminary propositions. In § 1 we state the main theorem and outline the proof. The details occupy us in § 2-§ 4. In § 5 we show how to obtain the more general result.

 $^{^1}$ Z= the additive group of integers with discrete topology, $B_1^+(G)=\{f\in B^+(G); f(0)<1\}$

1. Statement of the main theorem and outline of the proof.

THEOREM 1. If F operates on $\Phi(\mathbb{R}^1)$ then F is given by

$$(st) \qquad \qquad F(z) = \sum_{{n,m=0}\atop {(z_{n,m} \geqq 0)}}^{n,m=0} c_{n,m} z^n \overline{z}^m \qquad (\mid z \mid \leqq 1) \; .$$

with $\sum_{n,m=0}^{\infty} c_{m,n} = 1$.

Assuming that F is continuous it is first shown that F operates on $B_1^+(R^1)$. It then follows that

$$F(re^{i heta}) = \sum_{k=-\infty}^{\infty} a_k(r) \exp{(ik heta)}$$

 $(0 \leq r \leq 1)$ where $a_k(r) \geq 0$ $(k = 0, \pm 1, \pm 2, \cdots)$. Having obtained this representation we prove that not only is $a_k(r)$ nonnegative, but also absolutely monotonic. Thus

(1)
$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} a_{k,n}r^n \exp(ik\theta)$$

with $a_{k,n} \ge 0$. On the other hand, if the theorem is to be true, then

$$F(re^{i heta}) = \sum\limits_{k=-\infty}^{\infty} \left\{ \sum\limits_{\substack{n,m \geqq 0 \ n-m=k}} c_{n,m} r^{n+m}
ight\} \exp\left(ik heta
ight)$$
 .

In order to pass from (1) to (*) $a_k(r)$ must actually be of the form

$$a_k(r)=r^{{\scriptscriptstyle |k|}}\sum\limits_{n=0}^\infty b_{k,n}r^{2n}$$

with $b_{k,n} \ge 0$. To prove that the exponents of r in $a_k(r)$ increase by two can be done directly (Lemma 5). To prove that $a_k(r) = O(r^{|k|})$ (near r = 0) we introduce the more general representation of F

$$egin{aligned} F(r_1 \exp{(i\lambda_1 t)} + r_2 \exp{(i\lambda_2 t)} + \cdots + r_n \exp{(i\lambda_n t)}) \ &= \sum\limits_{\substack{k_i = -\infty \ 1 \leq i \leq n}}^\infty lpha_{k_1,k_2,\cdots,k_n}(r_1,\,r_2,\,\cdots,\,r_n) \exp{\left\{i\,\sum\limits_{j=1}^n k_j\lambda_j t
ight\}} \end{aligned}$$

where (r_1, r_2, \dots, r_n) varies in a suitable cube of \mathbb{R}^n . The vanishing of $a_k(r)$ to the correct order is then deduced from the simple observation that $\alpha_{k_1,k_2,\dots,k_n}(r_1, r_2, \dots, r_n) = O(r_1r_2 \cdots r_n)$ if all $k_j \neq 0$ (Lemma 4).

Finally we turn to the question of continuity. Since $F(\phi)$ is a continuous function for every $\phi \in \mathcal{O}(\mathbb{R}^1)$, the natural approach would be to prove directly that $z_n \to z_0$ implies $F(z_n) \to F(z_0)$ by constructing a

ch.f. ϕ together with a bounded sequence $\{t_n\}$ such that $\phi(t_n) = z_n$.² However, as the referee has observed it suffices to prove a slightly weaker interpolation property; namely that some $\phi \in \mathcal{O}(\mathbb{R}^1)$ exists which interpolates, on a bounded sequence, some subsequence of the $\{z_n\}$. His lemma and proof are given in § 4.

2. Several lemmata. In this section we assume that F is continuous on $\Delta = \{z : |z| \leq 1\}$ and operates on $\Phi(R^{i})$.

LEMMA 1. If $p \in B_1^+(R^1)$ then $F(p) \in B_1^+(R^1)$.

Proof. It suffices by Cramey's criterion [5, p. 65] to show that

$$\int_{0}^{4}\int_{0}^{4}F(p(t-u))\exp{(ix(t-u))}dtdu\geq 0$$

for all real x and A > 0. If the lemma were false there would exist therefore and $A_0 > 0$ and x_0 such that

$$(2) \qquad \int_{0}^{4_{0}} \int_{0}^{4_{0}} F(p(t-u)) \exp{(ix_{0}(t-u))} dt du = -d < 0^{3} \; .$$

The function

$$p_arepsilon(t) = egin{cases} (1-p(0))\Big(1-rac{\mid t\mid}{arepsilon}\Big) & ext{if} \mid t\mid \leq arepsilon \ 0 & ext{if} \mid t\mid > arepsilon \end{cases}$$

is in $B_1^+(R^i)$ for every $\varepsilon > 0$, [5, p. 70] and thus $\phi_{\varepsilon} = p_{\varepsilon} + p \in B^+(R^i)$. It is, in fact, in $\mathcal{O}(R^i)$ since $\phi_{\varepsilon}(0) = 1$. Because F operates on $\mathcal{O}(R^i)$.

(3)
$$\int_0^{A_0}\int_0^{A_0}F(\phi_{\varepsilon}(t-u))\exp{(ix_0(t-u))}dtdu \ge 0.$$

On the other hand

$$egin{aligned} & \left|\int_{0}^{A_{0}}\int_{0}^{A_{0}}\left\{F(p(t-u))-F(\phi_{arepsilon}(t-u))
ight\}\exp\left(ix_{0}(t-u)
ight)dtdu
ight|\ &=\left|\int_{darepsilon}\left\{F(p(t-u))-F(\phi_{arepsilon}(t-u))
ight\}\exp\left(ix_{0}(t-u)
ight)dtdu
ight|&\leq 4A_{0}arepsilon\ &G_{arepsilon}=\{(t,u)\colon 0\leq t\leq A_{0},\,0\leq u\leq A_{0},\,|\,t-u\,|\leq arepsilon\} \end{aligned}
ight.$$

since $|F(z)| \leq 1$ on Δ . If we take $\varepsilon < d/4A_0$ then (3) contradicts (2).

Let *n* be a positive integer and $2\pi, \lambda_1, \lambda_2, \dots, \lambda_n$ be rationally independent real numbers. For each vector $\boldsymbol{m} = (m_1, m_2, \dots, m_n)$ with

1282

² We were not able to deduce this strong interpolation property for $\Phi(R^1)$ and this necessitated a somewhat round about argument in the original version of this paper.

³ That the integral in (2) is real follows from the easily verified identity $F(\bar{z}) = \overline{F(\bar{z})}$.

integral components and each vector $\mathbf{r} = (r_1, r_2, \dots, r_n)$ with $0 \leq r_i < 1/n \ (1 \leq i \leq n)$ we formally define $a_m(\mathbf{r})$ by

$$(4) \qquad a_m(r) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F\left(\sum_{k=1}^n r_k \exp\left(i\lambda_k t\right)\right) \exp\left\{-it \sum_{k=1}^n m_k \lambda_k\right\} dt \ .$$

LEMMA 2. The limit in (4) exists and is independent of $\lambda_1, \lambda_2, \dots, \lambda_n$ (provided that $2\pi, \lambda_1, \lambda_2, \dots, \lambda_n$ are rationally independent real numbers).

Proof. Combining Lemma 1 with the observation that

$$\sum\limits_{k=1}^n r_k \exp{(i\lambda_k ullet)} \in B_1^+(R^1)$$

we see

$$F\left(\sum\limits_{k=1}^n r_k \exp{(i\lambda_k m{\cdot})}
ight) \in B_1^+(R^1)$$

and hence the limit in (4) exists [5, p. 43].

The Kronecker-Weyl theorem [9] next shows that

(5)
$$a_{m}(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^{n} \int_{0}^{2\pi} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} F\left(\sum_{k=1}^{n} r_{k} \exp\left(i\phi_{k}\right)\right) \times \exp\left(-i\sum_{k=1}^{n} m_{k}\phi_{k} - d\phi_{1}d\phi_{2}\cdots d\phi_{n}\right)$$

and hence $a_m(r)$ is independent of the particular $\{\lambda_j\}$ chosen.

A function f defined on the cube $0 \leq x_i < a \ (1 \leq i \leq n)$ is called absolutely monotonic function if

$$\frac{\partial^{j_1+j_2+\cdots+j_n}}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n}}f(x_1, x_2, \cdots, x_n) \ge 0$$

throughout the cube for $j_1, j_2, \dots, j_n = 0, 1, 2, \dots$ Just as in the case of one variable, an absolutely monotonic function admits a power series expansion with nonnegative coefficients.

LEMMA 3. The pointwise limit of absolutely monotonic functions is absolutely monotonic.

Proof. For n = 1 the lemma is well known. We then proceed by induction to n + 1. Suppose

$$\lim_{k \to \infty} f_k(r_1, r_2, \cdots, r_{n+1}) = f(r_1, r_2, \cdots, r_{n+1}) .$$

For fixed r_1, r_2, \dots, r_n we have

$$f_k(r_1, r_2, \cdots, r_{n+1}) = \sum_{j=0}^{\infty} a_{k,j}(r_1, r_2, \cdots, r_n) r_{n+1}^j \rightarrow f(r_1, r_2, \cdots, r_{n+1})$$

and hence

$$f(r_1, r_2, \cdots, r_{n+1}) = \sum_{j=0}^{\infty} a_j(r_1, r_2, \cdots, r_n) r_{n+1}^j$$

with

$$a_j(r_1, r_2, \cdots, r_n) = \underset{k \to \infty}{\operatorname{limit}} a_{k,j}(r_1, r_2, \cdots, r_n)$$
.

Since $a_{k,j}(r_1, r_2, \dots, r_n)$ is an absolutely monotonic function the induction hypothesis implies $a_j(r_1, r_2, \dots, r_n)$ is likewise so and lemma is proved.

LEMMA 4. In the cube $0 \leq r_i < 1/n \ (1 \leq i \leq n)$ (4i) $a_m(r)$ is an absolutely monotonic function

$$(6) a_{m}(r) = \sum_{\substack{0 \le i_{j} \le \infty \\ 1 \le j \le n}} \alpha_{i_{1}, i_{2}, \cdots, i_{n}}(m) r_{1}^{i_{1}} r_{2}^{i_{2}} \cdots r_{n}^{i_{n}}$$

and

(4ii) If $m_i \neq 0$ for every $i \ (1 \leq i \leq n)$ then $\alpha_{i_1,i_2,\ldots,i_n}(m) = 0$ if $i_j = 0$ for some $j \ (1 \leq j \leq n)$.

Proof. 1. Generalizing a result of Rudin [6, p. 618] we will show that if f is continuous in the cube $0 \le x_i < a \ (1 \le i \le n)$ and satisfies

$$(7) \qquad \int_{0}^{2\pi} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} f(a_{1} + b_{1} \cos \theta_{1}, a_{2} + b_{2} \cos \theta_{2}, \cdots, a_{n} + b_{n} \cos \theta_{n})$$
$$\times \prod_{k=1}^{n} \cos j_{k} \theta_{k} d\theta_{k} \ge 0$$

for all integers $j_1, j_2, \dots, j_n = 0, 1, 2, \dots$ whenever $0 \leq b_j \leq a_j, a_j + b_j < a$, then f is absolutely monotonic in the cube $0 \leq x_i < a$ $(1 \leq i \leq n)$.

2. To see that $a_m(r)$ satisfies (7) (with a = 1/n) we observe that

$$egin{aligned} I &= \left(rac{1}{2\pi}
ight)^n \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} a_{m{m}}(a_1 + b_1 \cos heta_1, \, \cdots, \, a_n + b_n \cos heta_n) \ & imes rac{1}{k=1} \cos j_k heta_k \, d heta_k \ &= \left(rac{1}{2\pi}
ight)^n \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} a_{m{m}}(a_1 + b_1 \cos heta_1, \, \cdots, \, a_n + b_n \cos heta_n) \ & imes \exp -i \sum_{k=1}^n j_k heta_k \, d heta_1 d heta_2 \cdots \, d heta_n \end{aligned}$$

since the integrand in I is an even function of each of the $\{\theta_k\}$. Next, the integral representation of $a_m(r)$ and the Kronecker-Weyl theorem yields

$$egin{aligned} I &= \left(rac{1}{2\pi}
ight)^n \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \ & imes F((a_1+b_1\cos heta_1)\exp(i\phi_1)+\cdots+(a_n+b_n\cos heta_n)\exp(i\phi_n)) \ & imes \exp{-i\sum\limits_{k=1}^n \left(j_k heta_k+m_k\phi_k
ight)d heta_1\cdots d heta_nd\phi_1\cdots d\phi_n} \,. \end{aligned}$$

A final application of the Kronecker-Weyl theorem shows

$$egin{aligned} I &= \liminf_{T o \infty} rac{1}{2T} \int_{-T}^{T} F\!\!\left(\sum\limits_{k=1}^{n} \left(a_k + b_k \cos \zeta_k t
ight) \exp\left(i\lambda_k t
ight)
ight) \ & imes \exp \, -i \sum\limits_{k=1}^{n} \left(j_k \zeta_k + m_k \lambda_k
ight) t \, dt^4 \end{aligned}$$

and this limit is nonnegative because

$$\sum\limits_{k=1}^n \left(a_k + b_k \cos \zeta_k \cdot
ight) \exp \left(i \lambda_k \cdot
ight) \in B_1^+(R^1)$$
 ,

Lemma 1 and [5, p. 43].

3. Suppose first that f satisfies (7) and is of class C^{∞} . To show that

(8)
$$\frac{\partial^{j_1+j_2+\cdots+j_n}}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots \partial x_n^{j_n}}f(x_1, x_2, \cdots, x_n) \ge 0$$

in the cube $0 \leq x_i < a \, (1 \leq i \leq n)$ we let $N = j_1 + j_2 + \cdots + j_n$ and write, by Taylor's theorem,

$$f(a_{1} + b_{1} \cos \theta_{1}, \cdots, a_{n} + b_{n} \cos \theta_{n})$$

$$(9) = \sum_{k=0}^{N} \frac{1}{k!} \left(b_{1} \cos \theta_{1} \frac{\partial}{\partial x_{1}} + \cdots + b_{n} \cos \frac{\partial}{\partial x_{n}} \right)^{k} f \Big|_{\substack{x_{i} = a_{i} \\ 1 \le i \le n}}$$

$$+ \frac{1}{(N+1)!} \left(b_{1} \cos \theta_{1} \frac{\partial}{\partial x_{1}} + \cdots + b_{n} \cos \theta_{n} \frac{\partial}{\partial x_{n}} \right)^{N+1} f \Big|_{\substack{x_{i} = a_{i} + \eta_{i} b_{i} \cos \theta_{i}}}.$$

Multiply (9) by $\prod_{k=1}^{n} \cos j_k \theta_k d\theta_k$ and integrate from 0 to 2π . Set $b_i =$ $b < \min_k a_k$ and let $b \downarrow 0$ to obtain (8).

4. If f is a priori only continuous, we proceed as follows: let $g: R^1 \rightarrow R^1$ satisfy

- (i) $g \in C^{\infty}$
- (ii) g(t) > 0 if 0 < t < 1; g(t) = 0 otherwise (iii) $\int_{0}^{1} g(t) dt = 1.$

If f satisfies (7), then so does

$$egin{aligned} &f_arepsilon(x_1,\,x_2,\,\cdots,\,x_n)=\int_0^1\!\!\int_0^1\cdots\int_0^1\ & imes f(x_1+\delta y_1,\,\cdots,\,x_n+\delta y_n)\prod_{k=1}^ng(y_k)dy_k \end{aligned}$$

⁴ The numbers 2π , λ_1 , \dots , λ_n , ζ_1 , \dots , ζ_n are taken to be rationally independent real numbers.

on the cube $0 \leq x_i < a - \delta$ $(1 \leq i \leq n)$. Now $f_{\delta} \in C^{\infty}$ and the argument in 3. applies to show that f_{δ} is absolutely monotonic. But $f_{\delta} \to f$ (pointwise) in the cube $0 \leq x_i < a$ $(1 \leq i \leq n)$ and Lemma 3 permits us to complete the proof of 4(i).

5. If $m_k \neq 0$ $(1 \leq k \leq n)$ then from (5) we see

$$a_m(0, r_2, \dots, r_n) = a_m(r_1, 0, r_3, \dots, r_n) = \dots$$
$$= a_m(r_1, r_2, \dots, r_{n-1}, 0) = 0$$

and this yields (4)ii.

LEMMA 5. If

(10)
$$a_{k}(r) = \frac{1}{2\pi} \int_{0}^{2\pi} F(r \exp{(i\phi)}) \exp{(-ik\phi)} d\phi$$
$$k = 0, \pm 1, \pm 2, \cdots$$

then

5(i)
$$a_k(-r) = (-1)^k a_k(r)$$

and

5(ii)
$$a_k(r) = \sum_{j=0}^{\infty} a_{k,j} r^j$$
 $-1 \leq r \leq 1$

with

$$a_{k,j} \geqq 0 \quad \sum\limits_{j=0}^{\infty} a_{k,j} < \infty$$
 .

Thus

$$a_k(r) = egin{cases} \sum\limits_{j=0}^\infty a_{k,2j} r^{2j} & ext{if } k ext{ is an even integer} \ \sum\limits_{j=0}^\infty a_{k,2j+1} r^{2i+1} & ext{if } k ext{ is an odd integer} . \end{cases}$$

Proof. For 5(i) note

$$a_k(-r) = rac{1}{2\pi} \int_0^{2\pi} F(r \exp i(\phi + \pi)) \exp \left(-ik\phi\right) d\phi = (-1)^k a_k(r) \; .$$

Proceeding as in the proof of Lemma 4, we show that

so that $a_k(\cos \cdot) \in B^+(R^1)$. It follows from [4, p. 202] that

$$a_{\scriptscriptstyle k}(\cos heta) = \sum\limits_{\scriptscriptstyle j=0}^\infty b_{\scriptscriptstyle k,j} \cos j heta$$

with

$$b_{\scriptscriptstyle k,j} \geqq 0 \sum\limits_{\scriptscriptstyle j=0}^\infty b_{\scriptscriptstyle k,j} < \infty$$
 .

If T_j denotes the *j*th Tchebychev polynomial then

(11)
$$a_k(x) = \sum_{j=0}^{\infty} b_{k,j} T_j(x) \quad -1 \leq x \leq 1.$$

But for $0 \leq x \leq 1$, Lemma 4 yields the representation

$$a_{\scriptscriptstyle k}(x) = \sum\limits_{\scriptscriptstyle j=0}^\infty a_{\scriptscriptstyle k,\,j} x^j$$

with

$$a_{\scriptscriptstyle k,j} \geqq 0 \sum\limits_{\scriptscriptstyle j=0}^\infty a_{\scriptscriptstyle k,j} < \infty$$
 .

Using elementary properties of the Tchebychev polynomials and the fact that the Fourier series of a C^{∞} function may be differentiated term-by-term, 5(i) and (11) imply that the equality

$$\sum\limits_{j=0}^\infty a_{k,j} x^j = \sum\limits_{j=0}^\infty b_{k,j} T_j(x)$$

extends to $-1 \leq x \leq 1$, and this proves 5(ii).

3. Proof of Theorem 1 with hypothesis of continuity. $F(r \exp(i\phi))$ is a continuous, periodic, nonnegative definite function. We can therefore write

(12)
$$F(r \exp (i\phi)) = \sum_{k=-\infty}^{\infty} a_k(r) \exp (ik\phi)$$
$$0 \le r \le 1 \qquad 0 \le \phi \le 2\pi$$

with

$$a_k(r) \geq 0 \ (k=0, \ \pm 1, \ \pm 2, \ \cdots) \sum_{k=-\infty}^\infty a_k(r) = F(r)$$
 .

In (12) we set $z = r \exp(i\phi)$ and use Lemma 5 to conclude that

(13)
$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \overline{z}^m + \sum_{1 \le m \le n < \infty} (d_{n,m} z^n / \overline{z}^m + e_{n,m} \overline{z}^n / z^m)$$

with

$$egin{aligned} & c_{n,m} \geqq 0 \ (n,\,m=0,\,1,\,2,\,\cdots) \ & d_{n,m} \geqq 0 \ e_{n,m} \geqq 0 \ (1 \leqq m \leqq n < \infty) \ & \sum_{n,m=0}^{\infty} c_{n,m} + \sum_{1 \leqq m \leqq n < \infty} (d_{n,m} + e_{n,m}) = 1 \ . \end{aligned}$$

We will now show that $d_{n_0,m_0} = 0$. Let $2\pi, \lambda_1, \dots, \lambda_{n_0}, \lambda$ be rationally independent real numbers and set

(14)
$$z = r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t)$$

in (13) where

$$0 \leq r < 2/3$$
 $r_{\scriptscriptstyle k} = r/2n_{\scriptscriptstyle 0}$ $(1 \leq k \leq n_{\scriptscriptstyle 0})$.

Let $m = (m_0, \underbrace{1, 1, \cdots, 1}_{n_0})$ and note by Lemma 4

(15)
$$a_m(r, r_1, r_2, \cdots, r_{n_0}) = C_m r r_1 r_2 \cdots r_{n_0} + o(r r_1 r_2 \cdots r_{n_0})$$
$$= C_m \left(\frac{1}{2n_0}\right)^{n_0} r^{n_0+1} + o(r^{n_0+1}).$$

Examing the term $z^{\alpha}/\overline{z}^{\beta}$ with z as in (14) we obtain

$$(16) \qquad \frac{\left(r\exp\left(i\lambda t\right) + \sum_{k=1}^{n_0} r_k \exp\left(i\lambda_k t\right)\right)^{\alpha}}{\left(r\exp\left(-i\lambda t\right) + \sum_{k=1}^{n_0} r_k \exp\left(-i\lambda_k t\right)\right)^{\beta}} \\ = r^{\alpha-\beta} \left(\exp\left(i\lambda t\right) + \frac{1}{2n_0} \sum_{k=1}^{n_0} \exp\left(i\lambda_k t\right)\right)^{\alpha} \exp\left(i\beta\lambda t\right) \\ \times \sum_{p=0}^{\infty} b_p \left\{\frac{1}{2n_0} \sum_{k=1}^{n_0} \exp\left(-i(\lambda_k - \lambda)t\right)\right\}^{p} \qquad (b = 1)$$

so that only the terms $z^{\alpha}/\overline{z}^{\beta}$ with $\beta = m_0 - j$, $\alpha = n_0 + j$ $(0 \le j \le m_0 - 1)$ yield a contribution to $a_m(r, r_1, r_2, \cdots, r_{n_0})$. But with z as in (14)

with $D_{j,\neq} = 0$ for j = 0. Thus (15) implies that $d_{n_0,m_0} = 0$. A similar argument shows $e_{n_0,m_0} = 0$ and the theorem is proved with the hypothesis of continuity.

4. The continuity of F^{5} . We begin with an interpolation lemma.

LEMMA 6. Let $z_n \rightarrow z_0$ ($|z_n| < 1, n = 0, 1, 2, \cdots$). There exists a ch.f. ϕ , a sequence (of real numbers) $t_k \rightarrow 1$ and a sequence (of integers) $\{n_k\}$ such that $\phi(t_k) = z_{n_k}$.

Proof. Let $\tau_n = 1 - (2/3)9^{-n}$; then $(9^n/2)\tau_n \equiv (1/6) \pmod{1}$ while $(9^{n+m}/2)\tau_n \equiv (1/2) \pmod{1}$ for m > 0. Hence

 $^{^5}$ We wish to acknowledge our thanks to the referee for the statement and proof of Lemma 6.

$$\cosrac{\pi}{2}9^n au_n=rac{\sqrt{3}}{2}$$
 , $\cosrac{\pi}{2}9^{n+m} au_n=0~(m>0)$

and $\cos{(\pi/2)9^n} = 0$. Let $\{\eta_n\}$ be a sequence of positive numbers such that

$$|z_{\scriptscriptstyle 0}| + \sum\limits_{n=1}^\infty \eta_n < 1$$
 .

We define inductively a sequence $\{\phi_n\}$ of positive-definite functions as follows; let

$$\phi_{\scriptscriptstyle 0}(t) = |\, z_{\scriptscriptstyle 0}^{\,}|\, e^{i\, (rg z_{\scriptscriptstyle 0})\,t}$$
 .

Assume that $\phi_0, \phi_1, \dots, \phi_p$ have been defined such that $\phi_j(1) = 0$ for j > 0. Choose integers m_{p+1} and n_{p+1} such that

$$r_{p+1} = \left|\sum\limits_{j=0}^{p} \phi_{j}({ au_{m_{p+1}}}) - z_{n_{p+1}}
ight| < rac{\eta_{p+1}}{2}$$

and define

$$\phi_{p+1}(t)=2r_{p+1}(\cosarepsilon_{p+1}t)\Bigl(\cosrac{\pi}{2}9^{m_{p+1}t}\Bigr)e^{i\lambda_{p+1}t}$$

where ε_{p+1} and λ_{p+1} are chosen such that

$$\phi_{p+1}(\tau_{m_{p+1}}) = z_{n_{p+1}} - \sum_{j=0}^{p} \phi_j(\tau_{m_{p+1}})$$
 .

We shall assume that the sequence $\{m_k\}$ is strictly increasing. If we set $t_k = \tau_{m_k}$ and

$$\phi(t) = \sum_{j=0}^\infty \phi_j(t) + arepsilon arepsilon(t)$$

where $\Delta(x) = \max(0, 1-2 | x |)$ and $\varepsilon > 0$ is such that $\phi(0) = 1$ then $\phi(t_k) = z_{n_k} (k = 1, 2, \cdots)$ and $\phi \in \mathcal{O}(\mathbb{R}^1)$.

LEMMA 7. F is continuous in the open unit disk $\{z: |z| < 1\}$.

Proof. Suppose not; then there would exist a z_0 , $|z_0| < 1$ and a sequence $\{z_n\} (|z_n| < 1)$ such that $z_n \to z_0$ and $F(z_n) \not\to F(z_0)$. By passing to a subsequence if necessary we can assume that $\{F(z_n)\}$ converges. By Lemma 6 there is a ch.f. ϕ and a sequence (of real numbers) $\{t_k\}$ with limit one such that $\phi(t_k) = z_{n_k}$. But then

$$F(z_{\scriptscriptstyle 0}) = F(\phi(1)) = \liminf_{k o \infty} \, F(\phi(t_k)) = \liminf_{k o \infty} \, F(z_{\scriptscriptstyle n_k})$$

which is a contradiction.

REMARK. For future reference let us note that Lemma 1 now shows that F operates on $B_1^+(R^1) \cup \varPhi(R^1)$.

LEMMA 8. F is continuous on $-1 \leq x \leq 1$.

Proof. By observing that $F(\cos \cdot) \in \Phi(R^1)$, we obtain, just as in Lemma 5

$$F(x) = \sum_{n=0}^{\infty} p_n T_n(x)$$

where $p_n \ge 0$ and

$$\sum\limits_{n=0}^{\infty} p_n = 1$$
 .

Since $|T_n(x)| \leq 1$ on $-1 \leq x \leq 1$, F is continuous there.

THEOREM 2. F is continuous on Δ .

Proof. As we have already remarked, F operates on $B_1^+(R^1) \cup \Phi(R^1)$. Now Lemmata 2-5 carry over mutatis mutandis to prove that

(20)
$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \overline{z}^m$$
$$|z| < 1$$

where $c_{n,m} \ge 0$. Setting z = x in (20) and using Lemma 8 we see that

$$\lim_{x\uparrow 1}\sum\limits_{k=0}^{\infty}\sum\limits_{\substack{n,m\geq 0\\n+m=k}}^{\infty}c_{n,m}x^{k}=F(1)=1$$
 .

But the $\{c_{n,m}\}$ are nonnegative and hence

$$\sum_{n,m=0}^{\infty} c_{n,m} = 1$$
 .

Thus our series in (20) extends to a continuous function on \varDelta . We assert that F is equal to this extension. For let $\phi \in \varPhi(R^1) t_k \to t_0$ with $0 < |\phi(t_k)| < 1, |\phi(t_0)| = 1$. Then $F(\phi)$ is a continuous function and thus limit $F(\phi(t_k)) = F(\phi(t_0))$. But

$$\begin{split} \text{limit } F(\phi(t_k)) &= \text{limit } \sum_{n,m=0}^{\infty} c_{n,m}(\phi(t_k))^n (\overline{\phi(t_k)})^m \\ &= \sum_{n,m=0}^{\infty} c_{n,m}(\phi(t_0))^n (\overline{\phi(t_0)})^m \end{split}$$

and thus

$$F(\phi(t_0)) = \sum\limits_{n,m=0}^{\infty} c_{n,m}(\phi(t_0))^n (\overline{\phi(t_0)})^m$$

5. Concluding remarks. In order to obtain the general theorem we require two propositions due to Herz [2 p. 165, p. 167].

PROPOSITION 1. If a locally compact abelian group H has elements of arbitrarily high order then every F which operates on $(B_1^+(H), B^+(H))$ is continuous.

PROPOSITION 2. If a locally compact abelian group H has elements of arbitrarily high order, then every F which operates on $(B_1^+(H), B^+(H))$ operates on $(B_1^+(Z), B^+(Z))$.

REMARKS. 1. In Propositions 1 and 2 it is assumed that F is defined on $\{z: |z| < 1\}$.

2. Proposition 1 does not include our Lemma 7 since we assume merely that F operates on $\Phi(R^1)$, not on $(B_1^+(R^1), B^+(R^1))$.

THEOREM 2. If a locally compact abelian group H has elements of arbitrarily high order, then F operates on $(B_1^+(H), B^+(H))$ if and only if

$$F(z)=\sum\limits_{n,m=0}^{\infty}c_{n,m}z^n\overline{z}^m$$
 , $(\mid z\mid <1)$

where $c_{n,m} \geq 0$.

Proof. By Propositions 1 and 2 we may assume that H = Z and that F is continuous. It suffices, by the proof of Theorem 1, to show that F operates on $(B_1^+(R^1), B^+(R^1))$. Suppose $\lambda \in B_1^+(R^1)$ and set $\phi = F(\lambda)$. Since ϕ is continuous all that must be verified is that ϕ is a nonnegative-definite function. For any $\delta > 0$, the sequence $\{\lambda_n = \lambda(n\delta)\}$ is nonnegative definite and therefore by the hypothesis $\{\phi(n\delta)\}$ is a nonnegative definite sequence for any $\delta > 0$. Since ϕ is continuous

$$\int_{0}^{4} \int_{0}^{4} \phi(u-v) \exp (ix(u-v)) du dv$$
$$= \liminf_{\delta \downarrow 0} \sum_{n,m=1}^{4/\delta} \phi((n-m)\delta) \exp ix\delta(n-m) \delta^{2}$$

But since $\{\phi(n\delta)\}$ is a nonnegative-definite sequence for each $\delta > 0$

$$\sum_{n,m=1}^{A/\delta} \phi((n-m)\delta) \exp{ix\delta(n-m)} \quad \delta^2 \ge 0$$

and hence by Cramer's criterion ϕ is nonnegative definite.

We conclude with a few remarks.

1. There is a formal relation between the result of [1] and our Theorem 1. Every real-entire function F can be written in the form

$$F = (F_1 - F_2) + i(F_3 - F_4)$$

where F_1 , F_2 , F_3 and F_4 satisfy (*). On the other hand every $\hat{\mu} \in B(\hat{G})$ is of the form

$$\widehat{\mu}=(\widehat{\mu}_{\scriptscriptstyle 1}-\widehat{\mu}_{\scriptscriptstyle 2})+i(\widehat{\mu}_{\scriptscriptstyle 3}-\widehat{\mu}_{\scriptscriptstyle 4})$$

where $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$ and $\hat{\mu}_4$ are in $B^+(\hat{G})$. A direct proof of our theorem starting from this observation would be desirable.

2. The proof given here of Theorem 1 demonstrates in one stroke that F is real-analytic in \varDelta and if it is expressed as a power series in z and \overline{z} it has nonnegative coefficients. If one could prove directly that F operates on all Fourier transforms assuming values in \varDelta then proof of the theorem could be completed in two steps:

(A) F is real-analytic [7, Chapter VI] and thus

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \overline{z}^m$$

(B) $c_{n,m} \ge 0$ $(n, m = 0, 1, 2, \cdots)$ The second step is a consequence of the explicit representation

$$egin{aligned} c_{n,m} &= \liminf_{r \downarrow 0} \liminf_{T o \infty} rac{1}{r^{n+m}} rac{1}{2T} \int_{-T}^{T} Fig(\sum_{k=1}^{n+m} r_k \exp{(i\lambda_k t)}ig) \ & imes \exp{ig(\sum_{k=1}^n \lambda_k t - i\sum_{k=1}^m \lambda_{n+k} tig)} \, dt^6 \end{aligned}$$

where the inner limit exists and is positive by virtue of Lemma 1 and [5, p. 43] and the outer limit exists by (A) above.

3. For nondiscrete G with elements of arbitrarily high order one can show by using the methods used in the proof of Theorem 1, that F operates on $\mathcal{O}(G)$ if and only if F satisfies (*). If G is discrete this needn't be the case, and F needn't even be continuous as, F(z) = 0 (|Z| < 1), =1 (|z| = 1), which operates on $\mathcal{O}(Z)$ already shows. For such discrete groups we don't know if it is true that F operates on $\mathcal{O}(G)$ implies that F must operate on $B_1^+(G)$. If it were true then at least the structure of F for |z| < 1 could be determined.

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Pacific Journal of Mathematics Vol. 15, No. 4 December, 1965

Robert James Blattner, Group extension representations and the structure space	1101
Glen Eugene Bredon, On the continuous image of a singular chain complex	1115
David Hilding Carlson, On real eigenvalues of complex matrices	1119
Hsin Chu, Fixed points in a transformation group	1131
Howard Benton Curtis, Jr., The uniformizing function for certain simply connected Riemann	
surfaces	1137
George Wesley Day, Free complete extensions of Boolean algebras	1145
Edward George Effros, The Borel space of von Neumann algebras on a separable Hilbert	
space	1153
Michel Mendès France, A set of nonnormal numbers	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i>	1171
Frederick Paul Greenleaf, Norm decreasing homomorphisms of group algebras	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i>	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i>	1239
James Patrick Jans, <i>Some aspects of torsion</i>	1249
Laura Ketchum Kodama, Boundary measures of analytic differentials and uniform	
approximation on a Riemann surface	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic</i>	
functions	1279
Ronald John Larsen, Almost invariant measures	1295
You-Feng Lin, Generalized character semigroups: The Schwarz decomposition	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i>	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i>	1319
Mitsuru Nakai, Φ -bounded harmonic functions and classification of Riemann surfaces	1329
L. G. Novoa, On n-ordered sets and order completeness	1337
Fredos Papangelou, Some considerations on convergence in abelian lattice-groups	1347
Frank Albert Raymond, Some remarks on the coefficients used in the theory of homology	
manifolds	1365
John R. Ringrose, <i>On sub-algebras of a C[*]-algebra</i>	1377
Jack Max Robertson, Some topological properties of certain spaces of differentiable	
homeomorphisms of disks and spheres	1383
Zalman Rubinstein, Some results in the location of zeros of polynomials	1391
Arthur Argyle Sagle, On simple algebras obtained from homogeneous general Lie triple	
systems	1397
Hans Samelson, On small maps of manifolds	1401
Annette Sinclair, $ \varepsilon(z) $ -closeness of approximation	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional</i>	
curvature	1415
Earl J. Tatt, Invariant splitting in Jordan and alternative algebras	1421
L. E. Ward, On a conjecture of R. J. Koch	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i>	1435
Horace C. Wiser, <i>Embedding a circle of trees in the plane</i>	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i>	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their</i> application to learning theory	1471
Charles Vernon Coffman, Correction to: Non-linear differential equations on cones in Banach	
spaces	1472
P. H. Doyle, III, Correction to: A sufficient condition that an arc in S^n be cellular	1474
P. P. Saworotnow, Correction to: On continuity of multiplication in a complemented	
algebra	1474