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**ON  $n$ -ORDERED SETS AND ORDER COMPLETENESS**

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## ON $n$ -ORDERED SETS AND ORDER COMPLETENESS

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In this paper, the notion of an  $n$ -ordered set is introduced as a natural generalization of that of a totally ordered set (chain). Two axioms suffice to describe an  $n$ -order on a set, which induces three associated structures called respectively: the incidence, the convexity, and the topological structures generated by the order. Some properties of these structures are proved as they are needed for the final theorems. In particular, the existence of natural  $k$ -orders in the "flats" of an  $n$ -ordered set and the fact that (as it happens for chains) the topological structure is Hausdorff.

The idea of Dedekind cut is extended to  $n$ -ordered sets and the notions of strong-completeness, completeness, and conditional completeness are introduced. It is shown that the  $S^n$  sphere is  $s$ -complete when considered as an  $n$ -ordered set. It is also proved that  $E^n$ , the  $n$ -dimensional euclidean space, fails to be  $s$ -complete or complete, but that it is conditionally complete. It is also proved that every  $s$ -complete set is compact in its order topology but that the converse is not true. These results generalize classical ones about the structure of chains and lattices.

II.  $n$ -Ordered sets. An element of the cartesian product  $X^{n+1}$  of a set  $X$  will be called an  $n$ -simplex and denoted by  $\sigma^n = (s_0, s_1, \dots, s_n)$  where  $s_i \in X$  for every  $i$ . The class of even permutations of this sequence is called an oriented  $n$ -simplex and denoted by  $|\sigma^n| = |s_0, s_1, \dots, s_n|$ . The class of odd permutations is another oriented  $n$ -simplex denoted by  $|\sigma^n| = |-(s_0, s_1, \dots, s_n)|$ . The set of all oriented  $n$ -simplexes of  $X$  will be denoted by  $|X^n|$ . In what follows  $n$ -simplex will mean oriented  $n$ -simplex.

The *join* of two simplexes  $|\sigma^h| = |s_0, s_1, \dots, s_h|$  and  $|\tau^k| = |t_0, t_1, \dots, t_k|$  is the  $h + k - 1$ -simplex  $|s_0, s_1, \dots, s_h, t_0, t_1, \dots, t_k|$  and will be denoted by  $|\sigma^h, \tau^k|$ .

An  $n$ -ordered set is a pair  $(X, \varphi_n)$ , where  $X$  is a set and  $\varphi_n$  is a function from  $|X^n|$  to the set  $\{-1, 0, 1\}$  and which satisfies  $A_1$  and  $A_2$ .

$A_1$ .—For every  $|\sigma^n| \in |X^n|$ ;  $\varphi_n | -\sigma^n| = -\varphi_n |\sigma^n|$ .

Before stating  $A_2$  we introduce the following notation:

$$\Phi_i(\sigma^n, \tau^n) = \varphi_n |t_i, s_1, s_2, \dots, s_n| \varphi_n |t_0, t_1, \dots, t_{i-1}, s_0, t_{i+1}, \dots, t_n|$$

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$A_2$ .—If  $\Phi_i(\sigma^n, \tau^n) \geq 0$  for  $i = 0, 1, \dots, n$ ; then  $\varphi_n | \sigma^n | \varphi_n | \tau^n | \geq 0$ .

$D_1$ .—The simplex  $|\pi^{n-1}|$  is said to be an *upper bound* for the set  $\{x_\alpha; \alpha \in I\} \subset X$  if  $\varphi_n | x_\alpha, \pi^{n-1} | \geq 0$  for every  $\alpha \in I$ . If all the relations are strictly  $>$  then  $|\pi^{n-1}|$  is a *proper upper bound*. Similar definitions for *lower bounds* using  $\leq$  and  $<$ .

$D_2$ .—The  $n$ -order  $\varphi_n$  is *open from above* (from below) if every finite subset of  $X$  has a proper upper bound (lower bound).

$T_1$ .—If  $\varphi_n$  is an open from above (or from below)  $n$ -order of  $X$  then the following transitive property holds:

If  $\varphi_n | s_0, s_1, \dots, s_{i-1}, x, s_{i+1}, \dots, s_n | \geq 0$  for all  $i$  and some  $x \in X$  then:  $\varphi_n | \sigma^n | \geq 0$ .

*Proof.* Apply  $A_2$  to the pair  $|x, \pi^{n-1}|, |\sigma^n|$  where  $|\pi^{n-1}|$  is a proper bound for  $\{s_i\} \cup \{x\}$

**EXAMPLES.**

(a) In the vector space  $V^n$  over the reals define:

$$\varphi_{n-1} | v_0, v_1, \dots, v_{n-1} | = \text{sign of det. } | v_0, v_1, \dots, v_{n-1} |$$

The function  $\varphi_{n-1}$  is an  $n-1$ -order of  $V^n$ .

(b) In the same space define:

$\varphi_n | v_0, v_1, \dots, v_n | = \text{sign of det. } | v_i - v_0 |, i = 1, 2, \dots, n$ .  $\varphi_n$  is an  $n$ -order of  $V^n$ .

(c) The function of example (a) restricted to the sphere  $|V| = 1$  gives an  $n-1$  order of the  $n-1$ -sphere.

(d) Any 1-order satisfying the transitive property of  $T_1$  is equivalent to a chain if we define:  $\varphi_1 | a, b |$  to be  $-1, 0$  or  $1$  according to  $a > b, a = b$  and  $a < b$  respectively.

(e) A field  $G$  is said to be  $n$ -ordered if it is also an  $n$ -ordered set and the mappings:  $f_a: x \rightarrow ax$  and  $g_a: x \rightarrow a + x$  are order-automorphisms for any  $a \neq 0$ .

If we call:  $|\sigma^n| = |S_0, S_1, \dots, S_n|$ ;  $|a\sigma^n| = |aS_0, aS_1 \dots aS_n|$ , and  $|a + \sigma^n| = |a + S_0, a + S_1, \dots, a + S_n|$ , then the definition means exactly that  $\varphi_n | \sigma^n | \varphi_n | a\sigma^n |$  and  $\varphi_n | \sigma^n | \varphi_n | a + \sigma^n |$  depend only on  $a$ . The following examples can be given:

(e<sub>1</sub>) The real numbers field is a 1-ordered (open) field. (This is a well known result).

(e<sub>2</sub>) The complex numbers field is a 2-ordered (open) field if we define for any  $|\sigma^2| = |\alpha_0, \alpha_1, \alpha_2|$ :

$$\varphi_2(\sigma^2) = \frac{i\Delta(\sigma^2)}{|\Delta(\sigma^2)|} \quad \text{where } \Delta(\sigma^2) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \bar{\alpha}_0 & \bar{\alpha}_1 & \bar{\alpha}_2 \end{vmatrix};$$

$\bar{\alpha}$  being the complex conjugate of  $\alpha$ ,  $|\alpha|$  the modulus of  $\alpha$ .

(e<sub>3</sub>) The field of quaternions, considered as a 4-dimensional vector space over  $R$  and with the 4-order of example (b) above becomes a 4-ordered (noncommutative) field.

(f) The  $n$ -order of  $V^n$  given in example (b) makes an  $n$ -ordered vector space out of  $V^n$  in the sense that the mappings  $f_a: x \rightarrow ax$  and  $g_y: x \rightarrow x + y$  are order-isomorphisms for any  $a \in R$ ,  $a \neq 0$  and any  $y \in V^n$ . This example can be generalized as follows:

(g) Let  $V$  be any linear space over the ordered commutative field  $K$ , and  $B \subset V$  any Hamel base for  $V$ . If  $N = \{b_1, b_2, \dots, b_n\}$  is any finite subset of  $B$ : we can make  $V$  into an  $n$ -ordered vector space by defining  $\varphi_n(V_0, V_1, \dots, V_n) = +1, -1$  or  $0$  whenever  $\det (V_i^j - V_0^j)$  is  $>$ ,  $<$  or  $= 0$  in  $K$  ( $V_i^j$  is the coefficient of  $b_j$  in the expression of  $V_i$  in terms of the base  $B$ )

The independence of the axioms follows from the following examples:

In the set  $\{a, b, c\}$  define:  $\varphi_2 | a, b, c | = \varphi_2 | b, a, c | = 1$  and  $\varphi_2 = 0$  elsewhere. This system satisfies  $A_2$  but not  $A_1$ .

In the set  $\{a, b, c, d, e\}$  define:

$$\begin{aligned} \varphi_2 | e, c, d | &= \varphi_2 | e, c, a | = \varphi_2 | e, c, b | = \varphi_2 | d, a, b | \\ &= \varphi_2 | d, b, c | = \varphi_2 | d, c, a | = \varphi_2 | a, c, b | = 1 \end{aligned}$$

and define  $\varphi_2$  on the remaining simplexes according to  $A_1$ . This system satisfies  $A_1$  but not  $A_2$ .

### III. Consequences of the Axioms.

$D_3$ .—Two elements  $x, y$  of  $W$  are said to be *equivalent* if for every  $|\pi^{n-1}| \in |X^{n-1}|$  we have:  $\varphi_n | x, \pi^{n-1} | = \varphi_n | y, \pi^{n-1} |$ . They are *conjugate* if  $\varphi_n | x, \pi^{n-1} | = -\varphi_n | y, \pi^{n-1} |$ . The relation between equivalent elements is an equivalence relation and the set of equivalence classes can be  $n$ -ordered in the usual way. For this set the following axiom holds.

$A_3$ .—*There are no distinct equivalent points.*

From now on we assume  $(X, \varphi_n)$  satisfies  $A_1, A_2$  and  $A_3$  and call  $(X, \varphi_n)$  a reduced  $n$ -ordered system. An easy consequence of  $A_3$  is:

$C_3$ .—An element  $x \in X$  has at most one conjugate  $x^*$ .

$D_4$ .—A simplex  $|\sigma^k|$ ,  $k \leq n$ , is said to be *singular* if for every  $|\pi^{n-k-1}|$  we have:  $\varphi_n | \sigma^k, \pi^{n-k-1} | = 0$ . In particular  $|\sigma^n|$  is singular if:  $\varphi_n | \sigma^n | = 0$ .

The following theorems follow easily and are stated without proof:

$T_2$ .— $x^*$  nonsingular, is the conjugate of  $x$ , if and only if  $|x, x^*|$  is singular.

$T_3$ .—Any simplex with repeated elements is singular.

$T_4$ .—There is at most one singular 0-simplex.

$T_5$ .—If  $x \neq y$ , for some  $|\pi^{n-1}|: \varphi_n | x, \pi^{n-1} | \neq \varphi_n | y, \pi^{n-1} |$ .

We have also:

$T_6$ .—If  $\Phi_i(\sigma^n, \tau^n) \leq 0$  for  $i = 0, 1, 2, \dots, n$  then  $\varphi_n | \sigma^n | \varphi_n | \tau^n | \leq 0$ . (Compare  $A_2$ )

$C_6$ .—If  $\Phi_i(\sigma^n, \tau^n) = 0$  for  $i = 0, 1, 2, \dots, n$  then  $\varphi_n | \sigma^n | \varphi_n | \tau^n | = 0$ .

$T_7$ .—If  $\Phi_i(\sigma^n, \tau^n) \geq 0$  for  $i = 0, 1, 2, \dots, n$  and  $\varphi_n | \sigma^n | \varphi_n | \tau^n | = 0$  then:  $\Phi_i(\sigma^n, \tau^n) = 0$  for every  $i$ .

IV. Flats and relative orders.

$D_5$ .—Given a nonsingular  $k$ -simplex  $|\pi^k|$ ,  $k < n$ , the set  $F | \pi^k | = \{x; |x, \pi^k| \text{ is singular}\}$  will be called the flat determined by  $\pi^k$

$T_8$ .—If  $s_i \in F | \pi^{n-1} |$ ,  $i = 0, 1 \dots n$  then  $|\sigma^n|$  is singular.

*Proof.* Apply  $C_6$  to the pair  $|\sigma^n|, |x, \pi^{n-1}|$  where the last simplex is nonsingular (Such an  $x$  exists by  $D_4$ )

$C_8$ .—If  $|\sigma^n|$  and  $|\tau^n|$  are both nonsingular, then for some  $i: |t_i, s_1, s_2, \dots, s_n|$  is not singular.

$T_9$ .—If  $|\mu^h, \pi^k|$ ,  $h + k = n - 1$ , is nonsingular, the function  $\varphi_h | \sigma^h | = \varphi_n | \sigma^h, \pi^k |$  is a reduced  $h$ -order defined on the  $h$ -simplexes  $|\sigma^h|$  of the set  $F | \mu^h |$ ,  $\varphi_h$  is called the order of  $F | \mu^h |$  relative to  $|\pi^k|$ . The proof is straightforward.

$T_{10}$ .—(Invariance of the relative order).—If  $\varphi_h$  and  $\psi_h$  are the relative orders of  $F | \mu^h |$  by  $|\pi^k|$  and  $|\tau^k|$  respectively, then:

$$\varphi_h | \sigma^h | = \psi_h | \mu^h | \varphi_h | \mu^h | \psi_h | \sigma^h | \text{ for any } |\sigma^h| \subset F | \mu^h |.$$

*Proof.* We consider first the case where  $|\pi^k|$  and  $|\tau^k|$  differ by only one element. Let  $|\pi^k| = |a, \xi^{k-1}|$  and  $|\tau^k| = |\pm(b, \xi^{k-1})|$  and apply  $A_2$  to the pair:  $|b, \xi^{k-1}, \mu^h|$  and  $|a, \xi^{k-1}, \sigma^h|$ . It is easily seen that the only  $\Phi_i$  different from 0 is:

$$\varphi_n | a, \xi^{k-1}, \mu^h | \varphi_n | b, \xi^{k-1}, \sigma^h |.$$

Hence:  $\varphi_n | \tau^k, \mu^h | \varphi_n | \pi^k, \sigma^h | = \varphi_n | \pi^k, \mu^h | \varphi_n | \tau^k, \sigma^h |$  and the theorem follows since:  $\varphi_n | \tau^k, \mu^h | \neq 0$ .

For the general case we construct inductively, using  $C_8$ , the sequence:  $|\pi_{-1}^k| = |\pi^k|$ ;  $|\pi_j^k| = |t_{i_0}, t_{i_1}, \dots, t_{i_j}, p_{j+1}, p_{j+2}, \dots, p_n|$  where the  $t_i$  are elements of  $|\tau^k|$  and apply the previous result several times to the  $h$ -orders relative to  $|\pi_j^k|$  and  $|\pi_{j+1}^k|$  for  $j = -1, 0, 1, \dots, n$ .

Since the previous result is independent of  $|\tau^k|$  and  $|\pi^k|$  we have:

$C_{10}$ .—The orders induced in  $F|\mu^h|$  by  $|\pi^k|$  and  $|\tau^k|$  are either identical or opposite and we may speak of the two “natural” orders in any flat  $F|\mu^h|$ .

V. Convexity theorems.

$D_6$ .—The element  $x$  is said to be contained in the nonsingular simplex  $|\pi^h|$  if for some natural order of  $F|\pi^h|$  we have:

$$a_i = \varphi_h | p_0, p_1, \dots, p_{i-1}, x, p_{i+1}, \dots, p_n | \varphi_h | \pi^h | \geq 0$$

for every  $0 \leq i \leq h$ .

If every  $a_i > 0$  we say that  $x$  is interior to  $|\pi^h|$ .

$D_7$ .—The segment  $(\bar{a}, \bar{b})$  is the set of interior points of the nonsingular 1-simplex  $|a, b|$ .

$D_8$ .—A set  $C \subset X$  is said to be convex if for every  $a, b \in C$ , such that  $|a, b|$  is not singular we have:  $(\bar{a}, \bar{b}) \subset C$ .

From the definitions follows:

$T_{11}$ .—If  $x$  is contained in (interior to)  $|\sigma^h|$  it is also contained in (interior to)  $|\sigma^h|$ .

$T_{12}$ .—If  $x$  is contained in (interior to)  $|\sigma^n|$  and every  $s_i$  satisfies:  $\varphi_n | s_i, \pi^{n-1} | \geq 0$  for some  $|\pi^{n-1}|$ , then  $\varphi_n | x, \pi^{n-1} | \geq 0 (> 0)$ .

*Proof.* We assume  $\varphi_n | \sigma^n | > 0$  and apply  $A_2$  to the pair:

$$|x, \pi^{n-1}|, |\sigma^n| \text{ to get } \varphi_n | x, \varphi^{n-1} | \geq 0.$$

Now if  $x$  is interior to  $|\sigma^n|$ ,  $\varphi_n | x, \pi^{n-1} |$  cannot be 0, otherwise by  $C_6$  and  $T_8$  we would have  $\varphi_n | \sigma^n | = 0$  which contradicts our assumption.

$D_9$ .—We say that  $|\sigma^h|$  is contained in (interior to)  $|\pi^h|$  if every  $s_i$  is contained in (interior to)  $|\pi^h|$ .

Using the previous theorem we now can prove:

$T_{13}$ .—If  $x$  is contained in  $|\sigma^n|$  and  $|\sigma^n|$  is contained in (interior to)  $|\pi^n|$  then  $x$  is contained in (interior to)  $|\pi^n|$ .

This theorem can be extended in a natural way to the case of two simplexes  $|\sigma^h|$  and  $|\pi^k|$  where  $h$  and  $k$  can be different from  $n$ . We omit the details. As a corollary of these theorems we have:

$T_{14}$ .—The sets  $Ct|\sigma^h|$  and  $Int|\sigma^h|$  formed by the elements which are contained in and interior to  $|\sigma^h|$  respectively, are convex.

VI. The induced structures. Given an  $n$ -ordered set  $(X, \varphi_n)$  the following structures are said to be induced by the order:

(a) The incidence structure,  $(X, \mathcal{F})$  where  $\mathcal{F}$  is the family of flats of  $(X, \varphi_n)$ .

(b) The *convexity structure*  $(X, \mathcal{C})$  where  $\mathcal{C}$  is the family of convex subsets of  $(X, \varphi_n)$ .

(c) The *topological structure*  $(X, \mathcal{F})$  where  $\mathcal{F}$  is the family of closed sets generated by the sub-base  $\mathcal{B}$ . The elements of  $\mathcal{B}$  are the sets  $\bar{B}_{\pi^{n-1}}^+ = \{x; \varphi_n | x, \pi^{n-1} | \geq 0\}$  for any nonsingular  $|\pi^{n-1}|$ , together with the  $\bar{B}_{\pi^{n-1}}^- = \{x, \varphi_n | x, \pi^{n-1} | \leq 0\}$ . We prove the following theorem concerning the topological structure  $(X, \mathcal{F})$

*T<sub>15</sub>.*—The topological space  $(X, \mathcal{F})$  is Hausdorff, provided  $(X, \varphi_n)$  contains no singular point.

*Proof.* If  $|x, y|$  is singular then by  $T_2$ ,  $x = y^*$ . Since  $x$  is not singular, for some  $|\pi^{n-1}|$  we have  $\varphi_n | x, \pi^{n-1} | > 0$ , and therefore  $\varphi_n | y, \pi^{n-1} | < 0$ . The sets  $B_{\pi^{n-1}}^+ = \{z; \varphi_n | z, \pi^{n-1} | > 0\}$  and  $B_{\pi^{n-1}}^- = \{z; \varphi_n | z, \pi^{n-1} | < 0\}$  are disjoint (open) neighborhoods of  $x$  and  $y$  respectively.

If  $|x, y|$  is not singular, for some  $\pi^{n-2}$ ,  $\varphi_n | x, y, \pi^{n-2} | > 0$ . Assume first that for some  $z$ , we have:  $0 \neq \varphi_n | z, x, \pi^{n-2} | \neq \varphi_n | z, y, \pi^{n-2} | \neq 0$ . To be precise let  $\varphi_n | z, x, \pi^{n-2} | < 0$  and  $\varphi_n | z, y, \pi^{n-2} | > 0$  and call  $|\pi^{n-1}| = |z, \pi^{n-2}|$ . Then  $B_{\pi^{n-1}}^+$  and  $B_{\pi^{n-1}}^-$  are the required neighborhoods. If such a  $z$  does not exist, call  $|\tau^{n-1}| = |x, \pi^{n-2}|$  and  $|\sigma^{n-1}| = |y, \pi^{n-2}|$ . It is easily verified that  $B_{\tau^{n-1}}^-$  and  $B_{\sigma^{n-1}}^+$  satisfy the requirement. The above theorem is an extension of a well known result in the topology of chains. (See [1] p. 39)

The following result is important and will be needed in the sequel:

*T<sub>16</sub>.*—If  $x, y$  are contained in  $|\sigma^n|$  then  $x$  is contained in some  $|\sigma_i^n| = |s_0, s_1, \dots, s_{i-1}, y, s_{i+1}, \dots, s_n|$ .

*Proof.* Call  $P_i = \varphi^n | \sigma_i^n |$  and  $P_{ij} = \varphi^n | \sigma_{ij}^n | = \varphi^n | s_0, s_1, \dots, s_{j-1}, x, s_{j+1}, \dots, s_{i-1}, y, s_{i+1}, \dots, s_n |$  for  $i \neq j$ . Clearly  $P_{ij} = -P_{ji}$ . We put  $P_{ii} = \varphi^n | s_0, s_1, \dots, s_{i-1}, x, s_{i+1}, \dots, s_n |$ .

Applying  $A_2$  to the pair:

$$|\sigma_i^n| \text{ and } |\sigma_{jk}^n| \text{ we get:}$$

If  $P_j P_{ki}$  and  $P_k P_{ij}$  are both  $\geq 0$  then  $P_i P_{kj} \geq 0$ . We may assume  $\varphi^n | \sigma^n | > 0$ . Then by  $D_6$  all  $P_r$  are  $\geq 0$ . Hence we have transitively:  $P_{ki} \geq 0$  and  $P_{ij} \geq 0$  imply  $P_{kj} \geq 0$ . Using this, we can prove easily, by induction on  $n$ , that for a certain value of  $K$ , say  $k = k_0$  all  $P_{k_0 j} \geq 0$ ,  $j = 0, 1, 2, \dots, n$ , and this means that  $x$  is contained in  $|\sigma_{k_0}^n|$ .

**VI.  $n$ -Order completeness.** In the theory of ordered sets a lattice is said to be complete if every subset of it has a L.U.B. and a G.L.B. This notion is equivalent to that of compactness of the associated topological space (interval topology) when applied to chains. (See [3]) In this sense the lattice of real numbers fails to be complete.

(See [3], p. 51) On the other hand it is *conditionally complete* because every *bounded* subset has a L.U.B. and a G.L.B. This property is equivalent to the fact that every Dedekind cut has a separation element. We proceed to extend these ideas to  $n$ -ordered sets. Let  $(x, \varphi_n)$  be a reduced  $n$ -ordered set. Every element  $x \in X$  determines an  $n-1$ -order in  $X$  by defining:  $\varphi_{n-1}|\pi^{n-1}| = \varphi_n|x, \pi^{n-1}|$ . Consider now the subsets of  $|X^{n-1}|$  defined by:  $C_x^+ = \{|\pi^{n-1}|; \varphi_{n-1}|\pi^{n-1}| \geq 0\}$  and  $C_x^- = \{|\pi^{n-1}|; \varphi_{n-1}|\pi^{n-1}| \leq 0\}$  It is clear that  $C_x^+ \cup C_x^- = |X^{n-1}|$  and  $x$  is called the separation element of the pair  $(C_x^+, C_x^-)$ . We also have for every nonsingular  $|\pi^{n-1}|: |\pi^{n-1}| \in C_x^+ \cap C_x^-$  if and only if:  $x \in F|\pi^{n-1}|$ . We now extend the notion of "cut" to  $n$ -ordered sets. Let  $C^+$  and  $C^-$  be two subsets of  $|X^{n-1}|$  such that  $C^+ \cup C^- = |X^{n-1}|$  and  $\gamma$  any object not in  $X$ . Let  $X^*$  be the set  $X \cup \{\gamma\}$ . We extend the function  $\varphi_n$  to the set  $|X^{*n}|$  by defining:  $\varphi_n^*|\gamma, \pi^{n-1}| = +1, -1$  or  $0$  whenever  $|\pi^{n-1}|$  is in  $C^+ - C^-, C^- - C^+$  or in  $C^+ \cap C^-$ , resp. Then  $\varphi_n^*|\pi^n| = \varphi_n|\pi^n|$  for  $|\pi^n| \in |X^n|$ . We call  $\gamma$  the ideal element defined by  $(C^+, C^-)$ .

$D_{10}$ .—A pair  $(C^+, C^-)$  of subsets of  $|X^{n-1}|$  is said to be a *cut* if the following properties are satisfied:

- (a)  $C^+ \cup C^- = |X^{n-1}|$
- (b)  $(X^*, \varphi_n^*)$  is an  $n$ -ordered set. (Satisfies  $A_1$  and  $A_2$ )

$D_{11}$ .—A cut  $(C^+, C^-)$  is said to be *interior* or a *Dedekind cut* if the ideal element  $\gamma$  defined by the cut is interior to some  $|\sigma^n|$  of  $X$ . This means that for some  $|\sigma^n|$  and every  $i$  we have:

$$\varphi_n^*|s_0, s_1, \dots, s_{i-1}, \gamma, s_i, \dots, s_n|\varphi_n^*|\sigma^n| > 0.$$

$D_{12}$ .—An  $n$ -ordered set  $(X, \varphi_n)$  is said to be *strongly complete* ( $s$ -complete) if every cut has a separation element in  $X$ . It is *conditionally s-complete* if every interior cut has a separation element. It is *order complete* if the topological space  $(X, \mathcal{F})$  is compact.

$T_{17}$ .—If  $(X, \varphi_n)$  is  $s$ -complete, then every element has a conjugate.

*Proof.* For every  $x \in X$  the sets  $C_x^+$  and  $C_x^-$  obviously form a cut  $(C_x^+, C_x^-)$ . It is also clear that the pair  $(C_x^-, C_x^+)$  is also a cut defining  $x^*$ .

$T_{18}$ .—The  $S^n$  sphere with the  $n$ -order defined in II, example  $c$ , is *strongly complete*.

We give only an idea of the proof: For any nonsingular  $n$ -simplex  $|\pi^n|$  in  $S^n$  and taking antipodal points, we have a decomposition of  $S^n$  into  $2^{n+1}$  simplexes. Given a cut  $(C^+, C^-)$ , the ideal element,  $\gamma$  is order-contained in one of them say  $|\pi_0^n|$ . The repeated barycentric subdivisions of  $|\pi_0^n|$  furnish, (because of  $T_{16}$ ) a sequence of simplexes



$|\pi_i^n|$ ,  $i = 0, 1, 2 \dots$  such that  $\gamma$  is interior to all of them and their diameters tend to 0. There is also a unique point  $p$  of  $S^n$  common to all the  $|\pi_i^n|$ . It is easily shown that  $p$  is the separation element of the cut.

It follows from  $T_{17}$  that  $E^n$ , the euclidean  $n$ -space with the  $n$ -order of example II(b) is not  $s$ -complete and from  $D_{11}$  that it is not order complete. This is not surprising if we recall the initial remark of this section. But we can prove:

$T_{19}$ .— $E^n$  with the  $n$ -order of example II(b) is conditionally  $s$ -complete.

We omit the proof since it is entirely similar to that of  $T_{18}$ . The relationship between order-completeness,  $s$ -completeness, and compactness is established in the following theorem which is similar to the classical result for partially ordered sets and chains. (See [3] and [2])

$T_{20}$ .—If the ordered set  $(X, \varphi_n)$  is  $s$ -complete, then it is order complete i.e. the space  $(X, \mathcal{F})$  is compact.

*Proof.* Let  $\mathcal{G}$  be a collection of closed sets of  $(X, \mathcal{F})$  with the finite intersection property. It follows from a well known theorem of Alexander that we may restrict ourselves to the case where  $\mathcal{G}$  consists of elements from the sub-base  $\mathcal{B}$ . (See VI a) Let  $\mathcal{M}$  be a maximal extension of  $\mathcal{G}$  in  $\mathcal{F}$  with respect to the property. Then an element of  $\mathcal{F}$  belongs to  $\mathcal{M}$  if and only if it meets every element of  $\mathcal{M}$ . (See [4])

Using the notation of  $T_{15}$  we now define  $(C^+, C^-)$ :

$$|\pi^{n-1}| \in C^+ \text{ if } \bar{B}_{\pi^{n-1}}^+ \in \mathcal{M} \quad \text{and} \quad |\pi^{n-1}| \in C^- \text{ if } \bar{B}_{\pi^{n-1}}^- \in \mathcal{M} .$$

We shall prove that  $(C^+, C^-)$  satisfies  $D_{10}$  and is therefore a cut. If  $|\pi^{n-1}|$  is not in  $C^+$  for some  $M_0 \in \mathcal{M}$  we have:

$$M_0 \subset X - \bar{B}_{\pi^{n-1}}^+ = B_{\pi^{n-1}}^- \subset \bar{B}_{\pi^{n-1}}^- .$$

It follows that every  $M \in \mathcal{M}$  meets  $\bar{B}_{\pi^{n-1}}^-$  since it meets  $M_0$ . Or  $|\pi^{n-1}| \in C^-$ . Therefore  $C^+ \cup C^- = |X^{n-1}|$ . In order to show that  $D_{10}$ (b) holds, we first prove the following result:

If  $\gamma$  is the ideal element defined by  $(C^+, C^-)$  and  $\gamma$  satisfies a finite system of equalities:  $\varphi_n | \gamma, \sigma_i^{n-1} | = e_i$ ;  $i = 1, 2, \dots, l$ , then there is some  $z \in X$  which, when substituted for  $\gamma$ , also satisfies the equalities.

*Proof.* If  $e_i = 1$ , then  $|\sigma_i^{n-1}|$  is in  $C^+$  but not in  $C^-$  and therefore  $\bar{B}_{\sigma_i^{n-1}}^-$  fails to meet at least one element of  $\mathcal{M}$ . Denote it by  $M_i$ . Similarly if  $e_j = -1$ ,  $\bar{B}_{\sigma_j^{n-1}}^+$  does not meet  $M_j \in \mathcal{M}$ . And if  $e_k = 0$ , both  $\bar{B}_{\sigma_k^{n-1}}^+$  and  $\bar{B}_{\sigma_k^{n-1}}^-$  belong to  $\mathcal{M}$ . We call  $M_k = \bar{B}_{\sigma_k^{n-1}}^+ \cap \bar{B}_{\sigma_k^{n-1}}^-$ ;

clearly  $M_k \in \mathcal{M}$ .

We consider now  $I = \bigcap_r M_r$ ,  $r = 1, 2, \dots, l$ . Since  $I$  is not empty, we take any  $z \in I$ . It can be readily seen that  $z$  satisfies all the equalities. To show now that  $(C^+, C^-)$  is a cut, it suffices to check  $A_2$  since  $A_1$  is obviously satisfied. But if some pair  $|\sigma^n|, |\tau^n|$  of  $n$ -simplexes of  $X^*$  fails to satisfy  $A_2$ , by the previous result the same is true when we put  $z \in X$  instead of  $\gamma$ , and this leads to a contradiction. Let  $s$  be the separation element of the cut  $(C^+, C^-)$  and  $G$  any element of  $\mathcal{G}$ . Since  $G$  belongs to the sub-base  $\mathcal{B}$  and  $T_{17}$  holds, it can be written  $G = \bar{B}_{\tau^{n-1}}^+$  for some  $|\tau^{n-1}|$ : This means  $|\tau^{n-1}| \in C^+$  and  $\varphi_n |s, \tau^{n-1}| \geq 0$ , or equivalently,  $s \in \bar{B}_{\tau^{n-1}}^+ = G$ . This completes the proof.

That the converse of the above theorem is not true, can be seen by means of the following example:

Let  $(S^2, \varphi_2)$  be the 2-sphere with the 2-order of Example II(c) and  $K$  the finite subset of six elements  $(\pm i, \pm j, \pm k)$ , 2-ordered by the restriction of  $\varphi_2$  to  $K$ . Then  $K$  is compact in the induced topology but the cut generated by the elements  $\pm(1/3)(i + j + k)$  of  $S^2$  in  $(S^2, \varphi_2)$ , restricted to  $(K, \varphi_2)$ , have no separation elements in  $(K, \varphi_2)$  and therefore it is not  $s$ -complete.

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