

# Pacific Journal of Mathematics

## **SOME RESULTS IN THE LOCATION OF ZEROS OF POLYNOMIALS**

ZALMAN RUBINSTEIN

## SOME RESULTS IN THE LOCATION OF ZEROS OF POLYNOMIALS

ZALMAN RUBINSTEIN

Three out of the four theorems proved in this paper deal with the location of the zeros of a polynomial  $P(z)$  whose zeros  $z_i, i = 1, 2, \dots, n$  satisfy the conditions  $|z_i| \leq 1$ , and  $\sum_{i=1}^n z_i^p = 0$  for  $p = 1, 2, \dots, l$ . One of those estimates is

$$\left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| < \frac{l+1}{|z|(|z|^{l+1} - 1)}$$

for  $|z| > 1$ .

The fourth result is of a different nature. It refines, in particular, a theorem due to Eneström and Takeya. It is shown that no zero of the polynomial  $h(z) = \sum_{k=0}^n b_k z^k$  lies in the disk

$$\left| z - \frac{\beta e^{-i\theta}}{\beta + 1} \right| < \frac{1}{\beta + 1},$$

where  $\beta = \max_{|z|=1} |h'(z)| / \max_{|z|=1} |h(z)|$ , and  $\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$ .

We generalize and strengthen certain well-known results due to Biernacki [1], Dieudonné [3, 5], and Takeya [8].

We use repeatedly a recent result due to Walsh which is a generalized form of an earlier theorem of his [10]. It concerns the case in which all the zeros of a polynomial lie within a certain distance of their centroid.

**THEOREM 1.** Let  $h(z) = \sum_{k=0}^n b_k z^k$  ( $b_k$  complex),

$$\beta = \frac{\max_{|z|=1} |h'(z)|}{\max_{|z|=1} |h(z)|},$$

$\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$ , and let  $C_\beta$  be the disc  $|z - \beta e^{-i\theta}/(\beta + 1)| < 1/(\beta + 1)$ , then no zero of  $h$  lies in  $C_\beta$ .

*Proof.* Consider the function  $F(z) = e^{-i\varphi} h(ze^{i\theta})/m$ , where  $h(e^{i\theta}) = me^{i\varphi}$ . Then  $F$  satisfies the conditions,  $|F(z)| < 1$  in  $|z| < 1$ ,  $F(1) = 1$ . Let  $x_n \rightarrow 1$  as  $n \rightarrow \infty$ ,  $0 < x_n < 1$ , and let  $\alpha = \lim_{n \rightarrow \infty} [(1 - |F(x_n)|)/(1 - x_n)]$ . Then  $\alpha \leq |F'(1)|$ . It follows readily (see [2] p. 57) that

$$\lim_{n \rightarrow \infty} [(1 - |F(x_n)|)/(1 - x_n)] = F'(1) = e^{i(\theta - \varphi)} h'(e^{i\theta})/m = |h'(e^{i\theta})|/m.$$

---

Received June 3, 1964. This research was sponsored by the Air Force Office of Scientific Research.

We apply now the following result due to Julia [2]: If a function  $f$  is regular in the unit disc and  $|f(z)| < 1$  for  $|z| < 1$ , and there exists a sequence of number  $z_1, \dots, z_n, \dots$  such that  $\lim_{n \rightarrow \infty} z_n = 1, \lim_{n \rightarrow \infty} f(z_n) = 1, \lim_{n \rightarrow \infty} [(1 - |f(z_n)|)/(1 - |z_n|)] = \alpha$  then

$$(1) \quad \frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leq \alpha \frac{|1 - z|^2}{1 - |z|^2} \quad \text{for } |z| < 1.$$

In (1), set  $f(z) = F(z), \alpha = |h'(e^{i\theta})|/m$ . If  $F(z_0) = 0$  and  $|z_0| < 1$ , then  $(1 - |z_0|^2)/|1 - z_0|^2 \leq \alpha$ , which is equivalent to  $e^{-i\theta}z_0 \notin C_\alpha$ . Since  $\alpha \leq \beta$ , it follows that  $C_\beta \subset C_\alpha$ ; hence  $e^{-i\theta}z_0 \notin C_\beta$ , which concludes the proof.

**COROLLARY 1.** *Let  $h(z) = \sum_{k=0}^n b_k z^k, b_k > 0$ . Then  $\beta = \sum_{k=1}^n kb_k / \sum_{k=0}^n b_k$ , and no zero is in the disc*

$$\left| z - \frac{\sum_{k=0}^n kb_k}{\sum_{k=0}^n (k+1)b_k} \right| < \frac{\sum_{k=0}^n b_k}{\sum_{k=0}^n (k+1)b_k}.$$

*In particular, if  $b_k$  is a strictly increasing sequence, then all the zeros of  $h(z)$  lie in the complement of  $C_\beta$  with respect to the unit disc. This makes more precise the theorem of Eneström andakeya [8].*

In a recent paper, Tchakaloff [9] (see also [7]) has proved that if all the zeros of the polynomials

$$(2) \quad P_k(z) = a_n^{(k)} z^n + \dots + a_0^{(k)} (a_n^{(k)} > 0, k = 1, \dots, m)$$

lie in the unit disc and if  $A_k > 0 (k = 1, \dots, m)$ , then all the zeros of the polynomial  $\sum_{k=1}^m A_k P_k(z)$  lie in the disc  $|z| \leq 1/\sin(\pi/2n)$ , and that this is the best possible result. We prove a more precise result in the case where there is more information about the zeros of  $P_k(z)$ .

**THEOREM 2.** *Let the polynomials  $P_k(z) (k = 1, \dots, m)$  of the form (2) have all their zeros  $z_{ik} (i = 1, \dots, n; k = 1, \dots, m)$  in the unit disc and let  $A_k > 0 (k = 1, \dots, m)$ . Suppose that  $\sum_{i=1}^n z_{ik}^p = 0$  for  $p = 1, \dots, l (k = 1, \dots, m)$ . Then all the zeros of the polynomial  $\sum_{k=1}^m A_k P_k(z)$  lie in the disc  $|z| \leq (\sin \pi/2n)^{-1/(l+1)}$ . For values of the form  $n = (l+1)r$ , the exact bound does not exceed  $(\sin(\pi(l+1))/2n)^{-1/(l+1)}$ .*

*Proof.* Without loss of generality we may assume that  $a_n^{(k)} = 1$ . By a recent result due to Walsh [11] the polynomials  $P_k$  satisfy the equality  $P_k(z) = (z - \varphi_k(z))^n$ , where  $|\varphi_k(z)| < |z|^{-l}$  for  $|z| > 1$ . Let  $\zeta$  be a point outside the unit disc at which the circle  $|z| = |\zeta|^{-l}$

subtends an angle  $\Psi$ . On the circle  $|z| = |\zeta|^{-l}$  there exists a point  $a$ , such that  $0 \leq \arg((\zeta - \varphi_k)/(\zeta - a)) \leq \Psi$ , and

$$(3) \quad \sum_{k=1}^m A_k P_k(\zeta) = (\zeta - a)^n \sum_{k=1}^m A_k \left( \frac{\zeta - \varphi_k}{\zeta - a} \right)^n.$$

One deduces from equation (3) that

$$\sum_{k=1}^m A_k P_k(\zeta) \neq 0 \text{ if } \Psi < \frac{\pi}{n}.$$

For  $\Psi = \pi/n$ ,  $\sin(\pi/2n) = |\zeta|^{-(l+1)}$ . This proves the first part of the theorem. The example  $A_1 = A_2 = 1$ ,  $m = 2$ ,  $P_1(z) = (z^{l+1} + \mu)^r$ ,  $P_2(z) = (z^{l+1} + \bar{\mu})^r$ , where  $\mu = i \exp(i\pi/2n)$ , proves the second part of the theorem, since in this case the polynomial  $P_1(z) + P_2(z)$  has the zero

$$z = \left[ \sin \frac{\pi(l+1)}{2n} \right]^{-1/(l+1)}.$$

Dieudonné has proved [3], (for a different proof see [4]), that if the polynomial  $P$  has all its zeros in the closed unit disc, then

$$(4) \quad \left| \frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)} \right| \leq \frac{1}{|z| - 1}, \quad \text{for } |z| > 1.$$

We give a short proof of (4), which at the same time yields a stronger inequality in the case where the centroid of the zeros of  $P$  is at the origin.

**THEOREM 3.** *If all the zeros  $z_i (i = 1, \dots, n)$  of the polynomial  $P(z)$  lie in the closed unit disc and if  $\sum_{i=1}^n z_i^k = 0 (k = 1, \dots, l)$ , then for  $|z| > 1$  the following sharp estimate holds*

$$(5) \quad \left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| \leq \frac{l+1}{|z|(|z|^{l+1} - 1)}.$$

*Inequality (5) holds also for  $l = 0$ , in which case the second condition imposed on the  $z_i$  is to be omitted.*

*Proof.* By a recent result due to Walsh [12], there exists a function  $\varphi(z)$ ,  $|\varphi(z)| < |z|^{-l}$ , such that for  $|z| > 1$

$$(6) \quad \frac{P'(z)}{P(z)} = \frac{n}{z - \varphi(z)}.$$

An estimate due to Goluzin [6], applied to  $\varphi$  yields the inequality

$$(7) \quad |\varphi'(z)| \leq \frac{l|z|^{l-1}}{|z|^{2l} - 1} (1 - |\varphi(z)|^2),$$

for  $|z| > 1$ . Since by (6)

$$(8) \quad \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} = \frac{\varphi(z) - z\varphi'(z)}{z(z - \varphi(z))}$$

is follows, using (7), that

$$\left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| \leq \frac{1}{|z|} \left[ \frac{|\varphi(z)|}{|z| - |\varphi(z)|} + \frac{l|z|^l}{|z|^{2l} - 1} \frac{1 - |\varphi(z)|^2}{|z| - |\varphi(z)|} \right]$$

It remains to prove the inequality

$$(9) \quad \frac{x}{a - x} + \frac{la^l}{a^{2l} - 1} \frac{1 - x^2}{a - x} \leq \frac{l + 1}{a^{l+1} - 1}$$

for all  $0 \leq x \leq a^{-l}$ , and  $a > 1$ .

If we denote the left hand side of (9) by  $f(x)$ , then  $f(a^{-l}) = (l + 1)/(a^{l+1} - 1)$ , and  $f'(x) \geq 0$  provided the function  $g(x) = a^{2l+1} - a + la^l(x^2 - 2ax + 1)$  is nonnegative. Since  $g'(x) \leq 0$  it is enough to show that  $h(a) = g(a^{-l})$  is nonnegative. Indeed one verifies that  $h(1) = 0$  and  $h'(a) > 0$  for all  $a > 1$ .

The particular case  $P(z) = z^n - 1$ ,  $l = n - 1$ , shows that the bound (5) cannot, in general, be improved.

The result due to Dieudonné follows from (7) and (8).

Finally, we discuss a problem raised by Biernacki [1], which was also treated by Dieudonné [5], namely that of determining a region containing all but, possibly, one zero of the polynomial  $aP(z) + P'(z)$  for all complex  $a$ . Each of the above authors has proved that if all the zeros of  $P$  lie in the unit disc, then the concentric disc of radius  $2^{1/2}$  is the smallest concentric disc that has the above mentioned property. Assuming additional information about the zeros of  $P$ , we obtain a smaller disc for all but possibly  $l + 1$  zeros of the polynomial  $z^l P(z) + aP'(z)$ .

**THEOREM 4.** *If all the zeros  $z_i (i = 1, \dots, n)$  of the polynomial  $P(z)$  lie in the closed unit disc and if  $\sum_{i=1}^n z_i^k = 0 (k = 1, \dots, l)$ , then for all complex  $a$  at least  $n - 1$  zeros of the polynomial  $z^l P(z) + aP'(z)$  lie in the disc  $|z| \leq 2^{1/(2(l+1))}$ .*

*Proof.* Proceeding as in the proof of Theorem 3, we have

$$\frac{P'(z)}{P(z)} = -\frac{z^l}{a} = \frac{n}{z - \varphi(z)},$$

satisfied by any zero of the polynomial  $z^l P + aP'$  which exceeds 1 in modulus. Set  $g(z) = z^{-l}\varphi(1/z)$ ,  $w = z^{l+1}$  and  $h(w) = g(z)$ . Then  $|g(z)| < 1$  if  $|z| < 1$  and

$$(10) \quad g(z) = \frac{1}{z^{l+1}} + an$$

$$(11) \quad h(w) = \frac{1}{w} + an.$$

If for some  $a$  the polynomial  $z^l P + aP'$  has at most  $n - 2$  zeros in the disc  $|z| \leq 2^{1/(2(l+1))}$ , then equation (10) has at least  $l + 2$  roots in the disc  $|z| < 2^{-1/(2(l+1))}$ , and hence equation (11) has at least two roots in the disc  $|w| < 2^{-1/2}$ . This was proved to be impossible in [5]

Theorem 4 is sharp for all  $l$  and  $n$  of the form  $n = 2k(l + 1)$ ,  $k = 1, 2, \dots$ . The upper limit is attained by the zeros of the polynomial

$$P(z) = (z^{2l+2} - 2^{1/2}z^{l+1} + 1)^{n/(2(l+1))}.$$

#### REFERENCES

1. M. Biernacki, *Sur les équations algébriques contenant des paramètres arbitraires*, Bull. Acad. Polon. Sci. (1927), 541-685.
2. C. Carathéodory, *Conformal Representation*, Cambridge University Press (1958).
3. J. Dieudonné, *Sur les polynômes dont toutes les racines sont intérieures au cercle unité*, Bull. Sci. Math. **56** (1932), 176-178.
4. ———, *Sur quelques applications de la théorie des fonctions bornées aux polynômes dont toutes les racines sont dans un domaine circulaire donné*, Actualités Sci. Ind. no. 114, Paris (1934).
5. ———, *Sur quelques points de la théorie des zéros des polynômes*, Bull. Sci. Math. **58** (1934), 273-296.
6. G. M. Golusin, *Geometrische Funktionentheorie*, Deutscher Verlag der Wissenschaften (1957).
7. M. Marden, *On the zeros of certain rational functions*, Trans. Amer. Math. Soc. **32** (1930), 658-668.
8. ———, *The geometry of the Zeros*, Math. Surveys no. 3, New York (1949).
9. L. Tchakaloff, *Sur la distribution des zéros d'une classe des polynômes algébriques*, C. R. Acad. Bulgare Sci. **13** (1960), 249-251.
10. J. L. Walsh, *On the location of the roots of certain types of polynomials*, Trans. Amer. Math. Soc. **24** (1922), 163-180.
11. ———, *Grace's theorem on the zeros of polynomials revisited*, Proc. Amer. Math. Soc. **15** (1964), 354-360.
12. ———, *The location of the zeros of the derivative of a rational function, revisited*, J. Math. Pures Appl. **43** (1964), 353-370.

HARVARD UNIVERSITY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON

Stanford University  
Stanford, California

R. M. BLUMENTHAL

University of Washington  
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California  
Los Angeles, California 90007

\*RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

\* Basil Gordon, Acting Managing Editor until February 1, 1966.



Robert James Blattner, <i>Group extension representations and the structure space</i> .....	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i> .....	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i> .....	1119
Hsin Chu, <i>Fixed points in a transformation group</i> .....	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i> .....	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i> .....	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i> .....	1153
Michel Mendès France, <i>A set of nonnormal numbers</i> .....	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i> .....	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i> .....	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i> .....	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i> .....	1239
James Patrick Jans, <i>Some aspects of torsion</i> .....	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i> .....	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i> .....	1279
Ronald John Larsen, <i>Almost invariant measures</i> .....	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i> .....	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i> .....	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i> .....	1319
Mitsuru Nakai, <i><math>\Phi</math>-bounded harmonic functions and classification of Riemann surfaces</i> .....	1329
L. G. Nova, <i>On <math>n</math>-ordered sets and order completeness</i> .....	1337
Fredos Papan gelou, <i>Some considerations on convergence in abelian lattice-groups</i> .....	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i> .....	1365
John R. Ringrose, <i>On sub-algebras of a <math>C^*</math>-algebra</i> .....	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i> .....	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i> .....	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i> .....	1397
Hans Samelson, <i>On small maps of manifolds</i> .....	1401
Annette Sinclair, <i><math> \varepsilon(z) </math>-closeness of approximation</i> .....	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i> .....	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i> .....	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i> .....	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i> .....	1435
Horace C. Wiser, <i>Embedding a circle of trees in the plane</i> .....	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i> .....	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i> .....	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i> .....	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in <math>S^n</math> be cellular</i> .....	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i> .....	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i> .....	1474