Pacific Journal of Mathematics

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Vol. 15, No. 4 December 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

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We continue the investigation of the simple anti-commutative algebras obtained from a homogeneous general L.t.s. In particular we consider the algebra which satisfies

$$(1)\qquad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) .$$

The usual process of analyzing a nonassociative algebra is to decompose it relative to elements whose right and left multiplications are diagonalizable linear transformations e.g. idempotents or Cartan subalgebras. In this paper we show that such a process yields only Lie algebras and indicates the difficulty in finding any non-Lie multiplication table for a simple anticommutative algebra satisfying (1).

A general Lie triple system [2] is an extension of a Lie triple system used in differential geometry and Jordan algebras. A general L.t.s. may be regarded as an anti-commutative algebra A with a trilinear operation $[x, y, z]$ so that the mappings $D(x, y): z \rightarrow [x, y, z]$ are derivations of A which generate a Lie algebra, $I(A)$, under commutation satisfying certain natural identities. A homogeneous general L.t.s. is a general L.t.s. for which the operation $[x, y, z]$ is a homogeneous expression in the products of x, y and z; that is, using anti-commutativity, $[x, y, z] = \alpha xy \cdot z + \beta yz \cdot x + \gamma zx \cdot y$ for some fixed α, β, γ in the base field. From [1] we see that if A is a homogeneous general L.t.s. over a field of characteristic zero which is either an irreducible general L.t.s. or $I(A)$-irreducible or a simple algebra, then A is a Lie or Malcev algebra or satisfies

$$(1)\qquad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$$

where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$. The main result of this paper is the following theorem.

THEOREM. *If A is a simple finite dimensional anti-commutative algebra over a field F of characteristic zero which satisfies* (1) *and if A contains a nonzero element u so that right multiplication by u, R_u, is a diagonalizable linear transformation, then A is a Lie algebra.*

Received August 3, 1964. Sponsored in part by NSF Grant GP-1453.

2. Proof of theorem. For any anti-commutative algebra we have the identity

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned}$$

But using (1) we also have

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = -2[J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y)] . \end{aligned}$$

Thus using the two preceding identities we have

$$\begin{aligned} J(w, xy, z) + J(w, yz, x) + J(w, zx, y) \\ = J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned} \tag{2}$$

Now let $u \neq 0$ be an element of A so that $R_u: x \to xu$ is a diagonalizable linear transformation. Then $R_u \neq 0$, for this implies that the one dimensional subspace uF is an ideal of A and therefore equals A. Thus $A^2 = 0$, a contradiction to the simplicity of A. Since R_u acts diagonally in A we may write

$$A = A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$$

where

$$A_\lambda = \{x \in A : x(R_u - \lambda I) = 0\} .$$

We shall now prove

$$A_\alpha A_\beta \subset A_{\alpha+\beta} . \tag{3}$$

For let $x \in A_\alpha$, $y \in A_\beta$, then from (1)

$$\begin{aligned} J(u, x, y)R_u &= J(u, u, xy) + J(u, x, yu) + J(u, y, ux) \\ &= \beta J(u, x, y) - \alpha J(u, y, x) \\ &= (\alpha + \beta)J(u, x, y) . \end{aligned}$$

Thus $J(u, x, y) \in A_{\alpha+\beta}$ and therefore

$$xy(R_u - (\alpha + \beta)I) = xy \cdot u + yu \cdot x + ux \cdot y \in A_{\alpha+\beta} .$$

From this $xy(R_u - (\alpha + \beta)I)^2 = 0$ and setting $xy = \Sigma z_\gamma \in A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$ we see by the diagonal action of R_u that $xy \in A_{\alpha+\beta}$. In particular (3) shows A_0 is a subalgebra of A.

Next we shall show

$$(4) \qquad J(A_\alpha, A_\beta, A_\gamma) = 0 \quad \text{or} \quad \alpha + \beta + \gamma = 0$$

for any characteristic roots α, β, γ of R_u. Let $x \in A_\alpha$, $y \in A_\beta$, $z \in A_\gamma$, then from (3) $J(x, y, z) \in A_{\alpha+\beta+\gamma}$ and therefore

$$\begin{aligned}(\alpha + \beta + \gamma)J(x, y, z) &= J(x, y, z)R_u \\ &= J(u, x, yz) + J(u, y, zx) + J(u, z, xy) \\ &= -\alpha x \cdot yz + (\alpha + \beta + \gamma)x \cdot yz + (\beta + \gamma)yz \cdot x \\ &\quad - \beta y \cdot zx + (\alpha + \beta + \gamma)y \cdot zx + (\alpha + \gamma)zx \cdot y \\ &\quad - \gamma z \cdot xy + (\alpha + \beta + \gamma)z \cdot xy + (\alpha + \beta)xy \cdot z \\ &= 0 .\end{aligned}$$

and this equation proves (4).

from (1) and (3) we have

$$J(A_0, A_0, A_0)A_0 \subset J(A_0, A_0, A_0)$$

and for $\alpha \neq 0$ we have from (1), (3) and (4),

$$\begin{aligned}J(A_0, A_0, A_0)A_\alpha &\subset J(A_\alpha, A_0, A_0) \\ &= 0 .\end{aligned}$$

Thus $J(A_0, A_0, A_0)A \subset J(A_0, A_0, A_0)$ and therefore $J(A_0, A_0, A_0)$ is an ideal of A whicn is contained in $A_0 \neq A$. Since A is a simple algebra this yields

$$(5) \qquad J(A_0, A_0, A_0) = 0 .$$

Next we shall prove that if α is a nonzero characteristic root so that $-\alpha$ is also a characteristic root, then

$$(6) \qquad J(A_\alpha, A_{-\alpha}, A_0) = 0 .$$

For using (1), (3) and (5) we obtain

$$J(A_\alpha, A_{-\alpha}, A_0)A_0 \subset J(A_\alpha, A_{-\alpha}, A_0)$$

and for any $\beta \neq 0$ we also obtain

$$\begin{aligned}J(A_\alpha, A_{-\alpha}, A_0)A_\beta &\subset J(A_\beta, A_\alpha, A_{-\alpha}A_0) \\ &\quad + J(A_\beta, A_{-\alpha}, A_0A_\alpha) \\ &\quad + J(A_\beta, A_0, A_\alpha A_{-\alpha}) \\ &\subset J(A_\beta, A_\alpha, A_{-\alpha}) + J(A_\beta, A_0, A_0) \\ &= 0 ,\end{aligned}$$

also using (4). Thus as in the proof of (5), $J(A_\alpha, A_{-\alpha}, A_0)$ is an ideal of A which must be zero. Adopting the usual convention that if α is a characteristic root but $-\alpha$ is not, then $A_{-\alpha} = 0$ we see that (6) holds for any characteristic root α.

Next let

$$B = \sum_{\alpha \neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha \neq 0} A_\alpha ,$$

then if $\beta \neq 0$ we see from (3) that $BA_\beta \subset B$. If $\beta = 0$, then from (6) we obtain $(A_\alpha A_{-\alpha})A_0 \subset A_\alpha A_{-\alpha}$ and therefore $BA_0 \subset B$. Thus B is an ideal of A and therefore $B = 0$ or $B = A$. If $B = 0$, then $R_u = 0$, a contradiction. Therefore we have

$$(7) \qquad A = \sum_{\alpha \neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha \neq 0} A_\alpha .$$

Now from (4) and (6) we have for any characteristic roots β and $\alpha \neq 0$, $J(A_\alpha, A_{-\alpha}, A_\beta) = 0$ and therefore

$$(8) \qquad J(A_\alpha, A_{-\alpha}, A) = 0 \qquad (\alpha \neq 0) .$$

We shall use (7) and (8) together with the following lemma to prove A is a Lie algebra.

LEMMA. *Let* $N = \{x \in A : J(x, A, A) = 0\}$, *then*
(i) $J(a, b, A) = 0$ *implies* $ab \in N$;
(ii) N *is an ideal of* A *which is a Lie algebra.*

Proof. Clearly (ii) follows from (i). So let $a, b \in A$ be such that $J(a, b, A) = 0$ and let $w, z \in A$. Then from (1) and (2) we have

$$\begin{aligned} 0 &= wJ(a, b, z) \\ (9) \qquad &= J(w, ab, z) + J(w, bz, a) + J(w, za, b) \\ &= J(wa, b, z) + J(wb, z, a), \text{ using (2)} . \end{aligned}$$

Now interchanging z and w in this last equation we obtain $0 = J(za, b, w) + J(zb, w, a) = J(w, bz, a) + J(w, za, b)$ and using this in (9) yields $J(ab, w, z) = 0$; that is, $ab \in N$.

To show that A is a Lie algebra, suppose it is not. Then from the lemma $N = 0$ and from (8) $A_\alpha A_{-\alpha} \subset N = 0$. Thus from (7) $A = \sum_{\alpha \neq 0} A_\alpha$ and therefore $A_0 = 0$; this contradicts $0 \neq u \in A_0$.

BIBLIOGRAPHY

1. A. Sagle, *On anti-commutative algebras and general Lie triple systems*, to appear in Pacific J. Math.
2. K. Yamaguti, *On the Lie triple system and its generalization*, J. Sci. Hiroshima University, **21** (1958), 155-160.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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* Basil Gordon, Acting Managing Editor until February 1, 1966.

Pacific Journal of Mathematics

Vol. 15, No. 4 December, 1965