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A result announced by R. F. Brown in 1963, and completed by Brown and Fadell, generalizing classical results of H. Hopf for differentiable manifolds, is the following:

THEOREM: **Let M be a compact connected topological manifold; then**

(a) M admits arbitrarily small maps with a single fixed point;

(b) If the Euler characteristic χ_M of M is zero, then M admits arbitrarily small maps without fixed points (and conversely). Here a map is small if it is close to the identity map. We propose to give a short proof of this theorem.

We will use the recent result of J. Kister (also Mazur and Stallings) that any microbundle over a complex is a bundle [4]. We note that according to [2] the result (b) holds also for manifolds with boundary.

2. Characteristic class. We consider the tangent microbundle $\tau_M: M \xrightarrow{d} M \times M \xrightarrow{p_1}$; here d is the diagonal map, and p_1 the first projection (cf. [5]). Attached to τ_M is the Thom class u, a well-defined element of $H^n(M \times M, M \times M - d(M))$ (here $n = \dim M$); the coefficients used are the integers $\mathbf{Z}$, if M is orientable, and twisted integers, determined by the orientations of the horizontal factor M at the points of $M \times M$, in the nonorientable case. (Cf. [6] for details in the orientable case.) We write $\tilde{u}$ for the image of u in the absolute group $H^n(M \times M)$; the Euler class e_M is the image of $\tilde{u}$ in $H^n(M)$ under the diagonal map d^* (twisted coefficients in the nonorientable case). Furthermore, M has a fundamental cycle μ (again twisted coefficients for nonorientable M). It is a well-known fact that the value $\langle e_M, \mu\rangle$ of e_M on μ equals the Euler-Poincaré characteristic χ_M of M.

[Since this is not easy to find in the literature, we sketch a proof: First assume M orientable. Let $\{x_i\}$ be a basis for $H^*(M)$ modulo torsion, and let $\{\alpha_i\}$ be the basis of $H_*(M)$ modulo torsion, dual to $\{x_i\}$ under $\langle\ ,\ \rangle$; put $r_i = \dim \alpha_i$. Define $\{x'_j\}$ by $\mathfrak{d} x'_i = \alpha_i$, where $\mathfrak{d}$ is the Poincaré duality operator $\mathfrak{d} x = x \cap \mu$; then $\{x'_j\}$ is again a basis for $H^*(M)$ modulo torsion. Finally let $\{\alpha'_j\}$ be dual to $\{x'_j\}$ under $\langle\ ,\ \rangle$. One verifies that $d_*\mu = \Sigma \alpha_i \times \alpha'_i$ modulo torsion (use $\langle x \times y, d_*\mu\rangle = \langle x \cup y, \mu\rangle$). Now $\tilde{u}$ satisfies the relation $\langle x, \alpha\rangle = (-1)^{n-r}\langle \tilde{u}, \mathfrak{d} x \times \alpha\rangle$ for $x \in H^r(M)$ (cf. [6]). Therefore we have $\langle e_M, \mu\rangle = \langle \tilde{u}, d_*\mu\rangle =$

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$\langle \tilde{u}, \Sigma \alpha_i \times \alpha_i' \rangle = \Sigma(-1)^{r_i} \langle x_i', \alpha_i' \rangle = \Sigma(-1)^{r_i} = \chi_M$. For nonorientable M let $\hat{M}$ be the orientable double covering, and use the facts that the Thom class is preserved under the covering map, that the fundamental cycle of $\hat{M}$ maps onto twice the (twisted) fundamental cycle of M, and that $\chi_{\hat{M}} = 2\chi_M$ (as one can see, e.g., from the Smith sequence).]

In particular, if $\chi_M = 0$, then also the Euler class e_M vanishes. Furthermore, in all this discussion we may, by Kister's result, replace the tangent microbundle by an actual bundle (in the local product sense) whose fibre is $\mathbf{R}^n$ with a well-defined origin and which therefore has a well-defined 0-section. We denote this bundle by $\bar{\tau}_M$.

3. **Proof of theorem.** We begin with part (b); thus assume $\chi_M = 0$. Embed M in a number space $\mathbf{R}^k$ with $k \geqq 2n+1$, and let V be a (closed) polyhedral neighborhood of M that retracts onto M, via the map r. We consider the bundle $r^*\bar{\tau}_M$, induced from the bundle $\bar{\tau}_M$ (see end of § 2) by r. By naturality the Euler class of $r^*\bar{\tau}_M$ vanishes. Therefore, if K is any polyhedron of dimension $\leqq n$ contained in V, the restriction of $r^*\bar{\tau}_M$ to K admits a nonvanishing section (i.e., one that does not meet the 0-section of $r^*\bar{\tau}_M$); to prove this one uses the interpretation of the Euler class as obstruction. Let $\mathscr{A}$ be a finite, open covering of M, of dimension n, such that (a) the nerve $N_{\mathscr{A}}$ can be realized in V and (b) an associated barycentric map $f\colon M \to N_{\mathscr{A}}$ (cf. [3], p. 69) is homotopic to the identity 1_M of M in V; this exists of course. Let s be a nonvanishing section of $r^*\bar{\tau}_M \mid N_{\mathscr{A}}$. Applying the covering homotopy theorem to the map $s \circ f$ of M into the bundle formed by the complement of the 0-section of $r^*\bar{\tau}_M$ and to the homotopy between f and 1_M, one gets a nonvanishing section of $r^*\bar{\tau}_M \mid M$, i.e. of $\bar{\tau}_M$. This section amounts of course to a fixed-point-free map of M into itself. Again according to Kister, $\bar{\tau}_M$ can be assumed to lie in any preassigned neighborhood of the diagonal of $M \times M$, which means that the map can be constructed as close to the identity as one pleases.

The converse is classical (Lefschetz fixed point theorem).

4. **Proof of theorem continued.** We come to part (a). As before we imbed M in a Euclidean space $\mathbf{R}^k$, and r is a retraction of some neighborhood of M onto M. Let A be a coordinate system in M (i.e., an open subset homeomorphic to R^n), and let B, respectively C, be the subsets of A corresponding to the set of points in R^n of norm <1, respectively $<\frac{1}{2}$. There exists a polyhedral neighborhood W of $M - B$ in R^k, whose r-image lies in $M - C$. Since $H^n(M - C)$ (twisted coefficients if needed) vanishes ($M - C$ being a manifold with

nonempty boundary), the characteristic class of $r^*\bar{\tau}_M \mid W$ is zero. By the same argument as before, the bundle $\bar{\tau}_M \mid M - B$ has a nonvanishing section, which can be interpreted as a map f of $M - B$ into M, without fixed points. We may assume that the f-image of the bounday of $M - B$ lies in A (by taking $\bar{\tau}_M$ small enough), and it is then clear, using $A \approx \boldsymbol{R}^n$, how to extend f to a map of M into itself whose only fixed point is the point of A corresponding to the origin of $\boldsymbol{R}^n$.

If f is homotopic to the identity map of M (as it will be if it is small enough: apply r to the linear homotopy in $\boldsymbol{R}^k$), then the index of the fixed point is χ_M: the index equals $\pm$ the intersection number of the graph of f in $M \times M$ and the diagonal, and it is well known that this is χ_M under the present circumstances. In fact, this last remark yields another version of the proof of (a): if $\chi_M = 0$, one can extend f over B without any fixed point.

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