

# Pacific Journal of Mathematics

**EMBEDDING A CIRCLE OF TREES IN THE PLANE**

HORACE C. WISER

## EMBEDDING A CIRCLE OF TREES IN THE PLANE

H. C. WISER

Concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold, it is shown here that there is an embedding if the continuum is decomposable and the manifold is orientable. The embedding is obtained by constructing an annulus on the manifold containing the continuum; in the nonorientable case an annulus or a Möbius strip containing the continuum may be found. Similar results are obtained for continua on a 2-manifold which have a decomposition into continua with zero 1-dimensional Betti numbers such that the decomposition space is a finite planar graph.

This extends the results of [2] concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold. Definitions and a summary of other results may be found in [2].

**THEOREM 1.** *A decomposable homogeneous proper subcontinuum of an orientable 2-manifold can be embedded in the plane.*

*Proof.* Let  $X$  be a decomposable proper subcontinuum of an orientable 2-manifold  $M$ . By Theorem 11 of [2] there is a continuous collection  $G$  of disjoint continua filling  $X$  such that the decomposition space  $X'$  is a simple closed curve and the elements of  $G$  are mutually homeomorphic, homogeneous, and treelike. Consider the upper semi-continuous decomposition of  $M$  whose nondegenerate elements are the elements of  $G$ . By Theorem 1 of [1], the decomposition space  $M'$  is homeomorphic to  $M$ . Let  $A'$  be a closed annular neighborhood of the simple closed curve  $X'$  on  $M'$ . Let  $f$  be the projection map from  $M$  to  $M'$  and  $A$  be  $f^{-1}(A')$ . Consider  $A'$  to be filled by a continuous collection of simple closed curves  $\{J_\alpha\}$ ,  $\alpha \in [0, 1]$ , where  $J_{1/2} = X'$ . Then  $f$  is one-to-one on  $A-X$  and  $f^{-1}(J_\alpha)$  must be compact since  $J_\alpha$  is compact; thus  $f/f^{-1}(J_\alpha)$  is a continuous one-to-one map of a compact set for  $\alpha \neq 1/2$  and therefore a homeomorphism. For a closed subinterval  $I$  of  $[0, 1]$  not containing  $1/2$ ,  $f^{-1}(\Sigma_{\alpha \in I} J_\alpha)$  is a closed annulus. If  $p \in A - f^{-1}(J_0 + J_1)$  then  $f(p)$  is in the interior of  $A'$  and thus  $p$  is interior to  $A$ . The continuum  $A$  must then be a 2-manifold with boundary consisting of  $f^{-1}(J_0) + f^{-1}(J_1)$ . Fit 2-cells  $C_0$  and  $C_1$  to the boundary curves of  $A$  to make a 2-manifold  $M_1$  without boundary. Considering an upper semi-continuous decomposition of  $M_1$  with the elements of  $G$  as the nondegenerate elements, we have  $M'_1$  as merely

$A'$  with 2-cells  $C'_0$  and  $C'_1$  fitted to  $J_0$  and  $J_1$ ; i.e.,  $M'_1$  is a 2-sphere. Using Theorem 1 of [1] again,  $M_1$  must be homeomorphic to  $M'_1$  and  $A$  must be a closed annulus;  $X$  is thus planar.

In the same manner we have the following results:

**THEOREM 2.** *A decomposable homogeneous proper subcontinuum of a nonorientable 2-manifold is contained in an open annulus or open Möbius strip on the manifold.*

**THEOREM 3.** *If a subcontinuum of an orientable 2-manifold has an upper semi-continuous decomposition into continua with zero mod 2 1-dimensional Betti numbers<sup>1</sup> such that the decomposition space is a finite planar graph then the continuum can be embedded in the plane.*

In view of Theorem 1, a nonplanar homogeneous subcontinuum of an orientable 2-manifold would have to be in the class of nontreelike indecomposable continua. No planar homogeneous continuum in this class is known, although the pseudo-circle is a candidate. It would be nice to eliminate the condition of orientability in Theorem 1.

#### REFERENCES

1. J. H. Roberts and N. E. Steenrod, *Monotone transformations of two-dimensional manifolds*, Ann. of Math. **39** (1938), 851-863.
2. H. C. Wisner, *Decomposition and homogeneity of continua on a 2-manifold*, Pacific J. Math. **12** (1962), 1145-1162.

UNIVERSITY OF UTAH  
AND  
WASHINGTON STATE UNIVERSITY

---

<sup>1</sup> As in [1], the statement "the continuum  $X$  on the 2-manifold  $M$  has a zero mod 2, 1-dimensional Betti number" means that for any complex  $K$  of  $M$  containing  $X$  there is a smaller complex  $L$  of  $M$  containing  $X$  such that each of its 1-cycles bounds in  $K$ .

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON

Stanford University  
Stanford, California

R. M. BLUMENTHAL

University of Washington  
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California  
Los Angeles, California 90007

\*RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

\* Basil Gordon, Acting Managing Editor until February 1, 1966.

Robert James Blattner, <i>Group extension representations and the structure space</i> .....	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i> .....	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i> .....	1119
Hsin Chu, <i>Fixed points in a transformation group</i> .....	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i> .....	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i> .....	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i> .....	1153
Michel Mendès France, <i>A set of nonnormal numbers</i> .....	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i> .....	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i> .....	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i> .....	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i> .....	1239
James Patrick Jans, <i>Some aspects of torsion</i> .....	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i> .....	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i> .....	1279
Ronald John Larsen, <i>Almost invariant measures</i> .....	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i> .....	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i> .....	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i> .....	1319
Mitsuru Nakai, <i><math>\Phi</math>-bounded harmonic functions and classification of Riemann surfaces</i> .....	1329
L. G. Nova, <i>On <math>n</math>-ordered sets and order completeness</i> .....	1337
Fredos Papan gelou, <i>Some considerations on convergence in abelian lattice-groups</i> .....	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i> .....	1365
John R. Ringrose, <i>On sub-algebras of a <math>C^*</math>-algebra</i> .....	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i> .....	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i> .....	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i> .....	1397
Hans Samelson, <i>On small maps of manifolds</i> .....	1401
Annette Sinclair, <i><math> \varepsilon(z) </math>-closeness of approximation</i> .....	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i> .....	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i> .....	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i> .....	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i> .....	1435
Horace C. Wiser, <i>Embedding a circle of trees in the plane</i> .....	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i> .....	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i> .....	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i> .....	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in <math>S^n</math> be cellular</i> .....	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i> .....	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i> .....	1474