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RING-LOGICS AND RESIDUE CLASS RINGS

ADIL MOHAMED YAQUB

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Let $(R, \times, +)$ be a commutative ring with unit 1, and let $K = \{\rho_1, \rho_2, \dots\}$ be a transformation group in R . $(R, \times, +)$ is called a ring-logic, mod K essentially if the “+” of R is equationally definable in terms of the “ K -logic” $(R, \times, \rho_1, \rho_2, \dots)$. The Boolean theory results by choosing K to be the group generated by $x^* = 1 - x$ (order 2, $x^{**} = x$). The following result is proved: Let $n = p_1 \cdots p_t$ be square-free, and let R_n be the residue class ring, mod n . Let, $\hat{\cdot}$, be any transitive $0 \rightarrow 1$ permutation of R_{p_i} ($i = 1, \dots, t$). Let, $\bar{\cdot}$, be the induced permutation of R_n defined by $(x_1, \dots, x_t)\bar{\cdot} = (x_1\hat{\cdot}, \dots, x_t\hat{\cdot})$, $x_i \in R_{p_i}$ ($i = 1, \dots, t$), and let K be the transformation group in R_n generated by, $\bar{\cdot}$. Then $(R_n, \times, +)$ is a ring-logic, mod K . An extension of this theorem to the case where n is arbitrary is also considered. The present proofs use the Fermat-Euler Theorem as well as a generalized form of the Chinese Residue Theorem.

The motivation for the study of ring-logics stems from the familiar equational interdefinability of Boolean rings $(R, \times, +)$ and Boolean logics (=Boolean algebras) $(R, \cap, *)$ [5]. In a series of recent publications ([1]–[4]), Foster raised this equational interdefinability, as well as the entire Boolean theory, to a more general level. In particular, Foster showed [2; 3] that any p -ring with unit (and more generally, any p^k -ring with unit) is a ring-logic, modulo certain suitably chosen groups. Furthermore, the author proved [6] that R_n , the residue class ring, mod n , is a ring-logic, modulo the “natural group” (generated by $x\hat{\cdot} = 1 + x$). Our present object is to further extend these results by considering certain transformation groups in R_n of rather general nature, and with respect to which $(R_n, \times, +)$ is a ring-logic (see Theorem 5).

1. The ring of residues mod p^k . Let $(R_{p^k}, \times, +)$ be the residue class ring, mod p^k , where p is prime and $k \geq 1$. Let G denote the group of units in R_{p^k} . Then, as is well known, the order of G is $\varphi(p^k) = p^k - p^{k-1}$, where $\varphi(n)$ is the familiar Euler φ -function (=number of positive integers which do not exceed n and which are relatively prime to n). Let, $\bar{\cdot}$, be a permutation of R_{p^k} . We call, $\bar{\cdot}$, a *transitive* $0 \rightarrow 1$ permutation if (i) $0\bar{\cdot} = 1$, and (ii) for any elements α, β in R_{p^k} , there exists an integer r such that $\alpha\bar{\cdot}^r = \beta$, where $\alpha\bar{\cdot}^r = (\dots((\alpha\bar{\cdot})\bar{\cdot})\bar{\cdot})\bar{\cdot}$ (r -iterations).

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We recall from [4] the *characteristic function* $\delta_\mu(x)$, defined as follows: for any given $\mu \in R_{p^k}$, $\delta_\mu(x) = 1$ if $x = \mu$ and $\delta_\mu(x) = 0$ if $x \neq \mu$. Following [4], we also define: $a \times_{\sim} b = \widehat{(a \times b)}$, where, $\widehat{}$, is the inverse of the $0 \rightarrow 1$ permutation, \sim . One readily verifies that $a \times_{\sim} 0 = 0 \times_{\sim} a = a$. Hence, we have the following “normal expansion formula” [4]:

$$(1.1) \quad f(x, y, \dots) = \sum_{\alpha, \beta, \dots \in R_{p^k}}^{\times_{\sim}} f(\alpha, \beta, \dots) (\delta_\alpha(x) \delta_\beta(y) \dots).$$

In (1.1), α, β, \dots range independently over all the elements of R_{p^k} while x, y, \dots are indeterminates over R_{p^k} . Also, $\sum_{\alpha_i \in R}^{\times_{\sim}} \alpha_i$ denotes $\alpha_1 \times_{\sim} \alpha_2 \times_{\sim} \dots$, where $\alpha_1, \alpha_2, \dots$ are all the elements of R .

We now have the following

LEMMA 1. *Let, \sim , be any transitive permutation of R_{p^k} , and let K be the transformation group in R_{p^k} generated by, \sim . Then all the elements of R_{p^k} are equationally definable in terms of the K -logic (R_{p^k}, \times, \sim) .*

Proof. Since, \sim , is a *transitive* permutation of R_{p^k} , therefore, $R_{p^k} = \{0, 0^{\sim}, 0^{\sim^2}, \dots, 0^{\sim^{p^k-1}}\}$. Similarly, we have, $xx^{\sim}x^{\sim^2} \dots x^{\sim^{p^k-1}} = 0$, for all x in R_{p^k} . The last equation shows that 0 (and with it $0^{\sim}, 0^{\sim^2}, \dots, 0^{\sim^{p^k-1}}$) is expressible in terms of the K -logic, and the lemma is proved.

LEMMA 2. *Let $G = \{1, \zeta_2, \zeta_3, \dots, \zeta_\varphi\}$ be the group of units in the residue class ring $(R_{p^k}, \times, +)$. Let, \sim , be a transitive $0 \rightarrow 1$ permutation of R_{p^k} satisfying $1^{\sim} = \zeta_2, \zeta_2^{\sim} = \zeta_3, \dots, \zeta_{\varphi-1}^{\sim} = \zeta_\varphi$, but otherwise, \sim , is entirely arbitrary. Let K be the transformation group in R_{p^k} generated by, \sim . Then each characteristic function $\delta_\mu(x)$, $\mu \in R_{p^k}$, is equationally definable in terms of the K -logic (R_{p^k}, \times, \sim) .*

Proof. Since, \sim , is *transitive*, therefore, there exists an integer r such that $\mu^{\sim^r} = 0$. Now, one readily verifies that

$$\delta_\mu(x) = (x^{\sim^{r+1}}x^{\sim^{r+2}}x^{\sim^{r+3}} \dots x^{\sim^{r+\varphi}})^{p^k - p^{k-1}},$$

since, by the Fermat-Euler Theorem, $a^{p^k - p^{k-1}} = 1$ for all a in G . This proves the lemma.

THEOREM 3. *Let K, \sim , be as in Lemma 2. Then the residue class ring $(R_{p^k}, \times, +)$ is a ring-logic, mod K .*

Proof. By (1.1), $x + y = \sum_{\alpha, \beta \in R_{p^k}}^{\times_{\sim}} (\alpha + \beta) (\delta_\alpha(x) \delta_\beta(y))$. By Lemma 1 and Lemma 2, each of $\alpha + \beta$, $\delta_\alpha(x)$, and $\delta_\beta(y)$, is expressible in terms

of the K -logic. Hence, the “+” of R_{p^k} is equationally definable in terms of the K -logic. Next, we show that $(R_{p^k}, \times, +)$ is *fixed* by its K -logic. Suppose that $(R_{p^k}, \times, +')$ is another ring with the same class of elements R_{p^k} and the same “ \times ” as $(R_{p^k}, \times, +)$ and which has the *same logic* as $(R_{p^k}, \times, +)$. To prove that $+ = +'$. But this follows, since, up to isomorphism, there is only one cyclic group of order p^k .

2. The general case. In attempting to generalize Theorem 3 to the residue class ring $(R_n, \times, +)$, n arbitrary, we need the following concept of independence, introduced by Foster [4].

DEFINITION. Let $\{U_1, \dots, U_t\}$ be a finite set of algebras of the same species S . We say that the algebras U_1, \dots, U_t are *independent* or satisfy the *Chinese Residue Theorem*, if, corresponding to each set $\{\Psi_i\}$ of expressions of species S , there exists a single expression X such that $\Psi_i = X \pmod{U_i}$ ($i = 1, \dots, t$). By an *expression* we mean some composition of one or more indeterminate-symbols x, \dots in terms of the primitive operations of U_1, \dots, U_t ; $\Psi_i = X \pmod{U_i}$ means that this is an identity of the algebra U_i .

As usual, we shall use the *same* symbols to denote the operation symbols of the algebras U_1, \dots, U_t when these algebras are of the same species. We now have the following

LEMMA 4. Let p_1, \dots, p_t be distinct primes. Let, \sim , be any transitive $0 \rightarrow 1$ permutation of $R_{p_i^{k_i}}$, and let K_i be the transformation group in $R_{p_i^{k_i}}$ generated by, \sim , ($i = 1, \dots, t$). Then the K_i -logics $(R_{p_i^{k_i}}, \times, \sim)$ ($i = 1, \dots, t$) are independent.

Proof. Let $n = p_1^{k_1} \dots p_t^{k_t}$ and let $E = x \sim x^{-2} \dots x^{-n-1}$. Let $p_i^{k_i} n_i = n$. Since $(p_i^{k_i}, n_i) = 1$, therefore, there exist integers r_i, s_i such that $r_i n_i - s_i p_i^{k_i} = 1$. Now, one readily verifies that

$$\omega_i = \text{def} = E^{\sim r_i n_i} = \begin{cases} 1 \pmod{R_{p_i^{k_i}}} , \\ 0 \pmod{R_{p_j^{k_j}}} \end{cases} \quad (j \neq i) .$$

To prove the independence of the logics $(R_{p_i^{k_i}}, \times, \sim)$, let $\{\Psi_i\}$ be a set of t expressions of species \times, \sim ; i.e., primitive composition of indeterminate-symbols in terms of the operations \times, \sim . Define

$$X = \Psi_1 \omega_1 \times \sim \dots \times \sim \Psi_t \omega_t .$$

It is readily verified that $\Psi_i = X \pmod{R_{p_i^{k_i}}}$ ($i = 1, \dots, t$), since $a \times \sim 0 = 0 \times \sim a = a$. This proves the lemma.

We are now in a position to consider $(R_n, \times, +)$ in regard to the concept of ring-logic. Indeed, let $n = p_1^{k_1} \cdots p_t^{k_t}$, where the p_i are distinct primes ($i = 1, \dots, t$), and let $G_i = \{1, \zeta_{i2}, \zeta_{i3}, \dots, \zeta_{i\varphi_i}\}$ be the group of units in the residue class ring $(R_{p_i^{k_i}}, \times, +)$. For each i , define, $\widehat{\cdot}$, to be a transitive $0 \rightarrow 1$ permutation of $R_{p_i^{k_i}}$ satisfying $1^{\widehat{\cdot}} = \zeta_{i2}, \zeta_{i2}^{\widehat{\cdot}} = \zeta_{i3}, \dots, (\zeta_{i, \varphi_i - 1})^{\widehat{\cdot}} = \zeta_{i\varphi_i}$, but otherwise, $\widehat{\cdot}$, is entirely arbitrary, and let K_i be the transformation group in $R_{p_i^{k_i}}$ generated by, $\widehat{\cdot}$. Now, it is well known that the residue class ring R_n is isomorphic to the direct product of $R_{p_1^{k_1}}, \dots, R_{p_t^{k_t}}$:

$$R_n \cong R_{p_1^{k_1}} \times \cdots \times R_{p_t^{k_t}} \text{ (direct product), } n = p_1^{k_1} \cdots p_t^{k_t} .$$

Furthermore, it is easily seen that by defining $(x_1, \dots, x_t)^{\widehat{\cdot}} = (x_1^{\widehat{\cdot}}, \dots, x_t^{\widehat{\cdot}})$, $(x_1, \dots, x_t) \in R_n$, we obtain a transitive $0 \rightarrow 1$ permutation of R_n . Let K be the transformation group in R_n generated by the above permutation, $\widehat{\cdot}$. We now have the following

THEOREM 5. *The residue class ring $(R_n, \times, +)$, n arbitrary, is a ring-logic, mod K , where K is the transformation group in R_n above.*

Proof. Let $n = p_1^{k_1} \cdots p_t^{k_t}$, where the p_i are distinct primes ($i = 1, \dots, t$). By Theorem 3, each $(R_{p_i^{k_i}}, \times, +)$ is a ring-logic, mod K_i , where K_i is as defined above ($i = 1, \dots, t$). Hence, for each i , there exists an expression Ψ_i such that

$$x_i + y_i = \Psi_i(x_i, y_i; \times, \widehat{\cdot}), \text{ for all } x_i, y_i \text{ in } R_{p_i^{k_i}} .$$

But, by Lemma 4, the K_i -logics $(R_{p_i^{k_i}}, \times, \widehat{\cdot})$ are independent ($i = 1, \dots, t$), and hence there exists a single expression X such that $X = \Psi_i \pmod{R_{p_i^{k_i}}}$ ($i = 1, \dots, t$). Now, let $x = (x_1, \dots, x_t), y = (y_1, \dots, y_t)$ be any elements of $R_n (\cong R_{p_1^{k_1}} \times \cdots \times R_{p_t^{k_t}})$. Since the operations are component-wise in this direct product, therefore,

$$\begin{aligned} X(x, y; \times, \widehat{\cdot}) &= X((x_1, \dots, x_t), (y_1, \dots, y_t); \times, \widehat{\cdot}) \\ &= (X(x_1, y_1; \times, \widehat{\cdot}), \dots, X(x_t, y_t; \times, \widehat{\cdot})) \\ &= (\Psi_1(x_1, y_1; \times, \widehat{\cdot}), \dots, \Psi_t(x_t, y_t; \times, \widehat{\cdot})) \\ &= (x_1 + y_1, \dots, x_t + y_t) \\ &= x + y . \end{aligned}$$

Hence, the “+” of R_n is *equationally* definable in terms of the K -logic $(R_n, \times, \widehat{\cdot})$. The proof that $(R_n, \times, +)$ is fixed by its K -logic follows as in the “fixed” part of the proof of Theorem 3, since again, up to isomorphism, there is only one cyclic group of order n . This completes the proof of the theorem.

We shall now take a closer look at the case where $n = p_1 \cdots p_t$ is square-free. In this case the group G_i of units in R_{p_i} (=field) is precisely the set of all nonzero elements of R_{p_i} ($i = 1, \dots, t$), and the $\hat{\cdot}$, described above (see paragraph preceding Theorem 5) for R_{p_i} is now simply any transitive $0 \rightarrow 1$ permutation of R_{p_i} . Hence, we have the following

COROLLARY 6. *Let $n = p_1 \cdots p_t$ be square-free, and let, $\hat{\cdot}$, be any transitive $0 \rightarrow 1$ permutation of R_{p_i} ($i = 1, \dots, t$). Let, $\hat{\cdot}$ be the induced permutation of R_n defined by $(x_1, \dots, x_t)\hat{\cdot} = (x_1\hat{\cdot}, \dots, x_t\hat{\cdot})$, $x_i \in R_{p_i}$ ($i = 1, \dots, t$), and let K be the transformation group in R_n generated by, $\hat{\cdot}$. Then $(R_n, \times, +)$ is a ring-logic, mod K .*

Thus, if, in particular, we choose $x\hat{\cdot} = 1 + x$ in the above Corollary, we obtain the following (compare with [6]).

COROLLARY 7. *Let n be square-free, and let N be the "natural group", generated by $x\hat{\cdot} = 1 + x$. Then $(R_n, \times, +)$ is a ring-logic, mod N .*

Upon choosing, $\hat{\cdot}$, in Theorem 5 in all of the various available ways, we obtain the corresponding transformation groups K with respect to which $(R_n, \times, +)$ is a ring-logic.

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