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**We extend several theorems for commutative Banach algebras to topological algebras with a sequence of semi-norms ($F$-algebras). The question of what functions "operate" on an $F$-algebra is considered. It is proven that analytic functions in several complex variables operate by applying a theorem due to Waelbroeck. If all continuous functions operate on an $F$-algebra, then it is an algebra of continuous functions. However, unlike the situation for Banach algebras [6], it is not true that if $\sqrt{\phantom{x}}$ operates the algebra is $C(\Delta)$. This will be shown by an example. A theorem due to Curtis [4], concerning continuity of derivations when the algebra is regular is extended to $F$-algebras. The result is applied to an algebra of Lipschitz functions to show that it has only a trivial derivation.**

Preliminaries. Throughout this paper the letter $A$ will stand for a commutative $F$-algebra. An $F$-algebra is a topological algebra with topology determined by a sequence of algebraic semi-norms. The $n$th semi-norm of an element $x$ in $A$ will be written $\| x \|_n$. We may and shall always assume that for all $x$ in $A$, $\| x \|_n \leqq \| x \|_{n+1}$. $\Delta^+$ will denote the topological space of all continuous multiplicative linear functionals on $A$ with the weak* topology. $\Delta$ will denote $\Delta^+$ minus the zero functional with the relativized topology. For $x$ in $A$, $\hat{x}$ will be the function in $C(\Delta^+)$ (the continuous functions on $\Delta^+$ with the compact-open topology) defined by $\hat{x}(\varphi) = \varphi(x)$. $A$ will be called regular if given $\varphi_0$ in $\Delta$ and $V$ a neighborhood of $\varphi_0$, there is an element $x$ in $A$ such that $\varphi_0(x) = 1$ and $\varphi(x) = 0$ for $\varphi \notin V$. $A$ will be called semi-simple if $\hat{x} = 0$ implies $x = 0$.

A basic device in the study of $F$-algebras is to represent $A$ as the inverse limit of a sequence of Banach algebras $\{A_n\}$ where $A_n$ is the completion of $A/I_n$ with norm $\| x + I_n \| = \| x \|_n$ and $I_n$ is the ideal of all $x$ in $A$ such that $\| x \|_n = 0$. The homomorphism $\pi_{m,n}$: $A_n \longrightarrow A_m$ for $m \leqq n$ is defined as the completion of the mapping $x + I_n \longrightarrow x + I_m$. This representation enables one to construct an element in $A$ by constructing a sequence $\{x_n\}$ such that for each $n$,

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$x_n \in A_n$ and $\pi_{m,n}x_n = x_m$. The homomorphism $\pi_n: A \longrightarrow A_n$ is defined as $x \longrightarrow x + I_n$. Then $\pi_n^*$: (multiplicative linear functionals in $A_n$) $\longrightarrow \Delta^+$ is continuous and one-to-one and so its range, which we shall denote by $\Delta_n^+$ is a compact subset of $\Delta^+$. If $K$ is an arbitrary compact subset of $\Delta^+$, there is an integer $n$ such that $K \subseteqq \Delta_n^+$ [9].

The following theorem, due to Silov, is also valid for $F$-algebras. *If $C$ is a closed and open subset of $\Delta^+$ and the zero homomorphism is not in $C$, then there is an idempotent $e$ in $A$ such that $C = \{\varphi \in \Delta^+ : \varphi(e) = 1\}$.* The extension to $F$-algebras is proven via the device of the previous paragraph. With the aid of Silov's theorem the proof that if $A$ is regular, then $A$ is normal is essentially the same as for Banach algebras.

Since so many of the theorems true for Banach algebras are also true for $F$-algebras with almost the same proofs, it is perhaps appropriate to remark that the difficulties introduced by the sequence of semi-norms are sometimes quite subtle. For example such a seemingly innocuous question as whether a multiplicative linear functional is necessarily continuous is still unanswered.

**Functions that operate on a commutative semi-simple $F$-algebra.** A function $f: D \subseteqq \mathbf{C} \longrightarrow \mathbf{C}$ is said to "operate" on an $F$-algebra $A$ if $f \circ \hat{x} \in \hat{A}$ whenever $x \in A$ and the range $\hat{x} \subseteqq D$. It is not difficult to adapt Katznelson's proof in [5] to show that if every continuous function operates on $A$, then $A = C(\Delta)$. However another theorem due to Katznelson which states: *If $A$ is a self-adjoint Banach algebra and $\sqrt{\phantom{x}}$ operates on the positive functions in $\hat{A}$, then $A = C(\Delta)$* is no longer true for $F$-algebras; as the following example shows.

Let $H$ be the subalgebra of $l^\infty$ consisting of those sequences $\{a_n\}$ for which there is a number, $a$ such that $|a_n - a|^{1/n} \longrightarrow 0$. Let $H'$ be the subalgebra of $H$ consisting of those sequences for which $a = 0$. Let $\tau$ be the linear transformation from $H'$ to the entire functions defined by $\tau(\{a_n\})(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$. For each integer $N$ and for $\{a_n\} \in H'$ defined $\|\{a_n\}\|_N = \sup[|\tau(\{a_n\})(\lambda)| : |\lambda| \leqq N]$. $\|-\|_N$ is evidently a vector space norm. It is also algebraic; for suppose $\{a_n\}$ and $\{b_n\} \in H'$, $f = \tau(\{a_n\})$, $g = \tau(\{b_n\})$ and $F = \tau(\{a_nb_n\})$. Then

$$F(\lambda) = (1/2\pi i)\int_{|w|=M} f(w)g(\lambda/w)dw/w \,.$$

$H'$ is a complete $F$-algebra under the sequence of norms defined above and $H$ is the $F$-algebra obtained by adjoining a unit to $H'$.

For $n = 0, 1, 2, \cdots$, define $z_n$ as the sequence which is 1 in the $n$th coordinate and 0 in all the other coordinates. These elements generate $H'$ (since the polynomials are dense in the entire functions) and together with the unit of $H$ generate $H$. $\Delta(H)$ is homeomorphic to the one-point compactification of the integers, the point corresponding to the integer $n$ being the functional sending $z_n$ into 1.

It is evident that $\hat{H}$ is a self-adjoint subalgebra of $C(\Delta(H))$, and that $H$ is semi-simple and regular. Yet, although $\sqrt{\phantom{x}}$ operates on the nonnegative elements of $\hat{H}$, $H \neq C(\Delta(H))$.

For $U$ an open subset of $\mathbf{C}^n$ let $H(U)$ be the $F$-algebra of all holomorphic functions on $U$ with the compact-open topology. For $\sigma$ an arbitrary subset of $\mathbf{C}^n$, let $H(\sigma)$ be the direct limit of the $F$-algebras $H(U)$ for $U$ ranging over open sets containing $\sigma$ directed as follows: $H(U) \geqq H(V)$ if $U \subseteqq V$.

Let $a_1, \cdots, a_n$ be elements of a commutative $F$-algebra, say $A$, with unit. For $\varphi \in \Delta = \Delta(A)$, let $\sigma(\varphi)$ be the point in $\mathbf{C}^n$ $(\varphi(a_1), \cdots, \varphi(a_n))$ and let $\sigma = \{\sigma(\varphi) : \varphi \in \Delta\}$.

THEOREM. *There is a continuous homomorphism $\tau$ from $H(\sigma)$ to $A$ such that $\varphi(\tau f) = f(\sigma(\varphi))$ for every $\varphi$ in $\Delta$ and every $f$ in $H(\sigma)$ and $\tau(z_i) = a_i$, $i = 1, \cdots, n$. (Evidently $f \in H(\sigma)$ defines a function on $\sigma$.)*

*Proof.* Waelbroeck, in [11], proved that such a continuous homomorphism exists for even more general topological algebras providing the elements, $a_1, \cdots, a_n$ are regular, i.e. have compact spectrum. An element of an $F$-algebra needn't be regular, but an element of a Banach algebra is of course regular. We will apply Waelbroeck's theorem to each of the Banach algebras $A_s$ where $A$ is the inverse limit of $\{A_s\}$.

For every integer $k$ let $\sigma_k$ be defined as above for $\pi_k a_1, \cdots, \pi_k a_n$, let $\tau_k$ be the continuous homomorphism from $H(\sigma_k)$ to $A_k$. $\forall k: \sigma_k \subseteqq \sigma$ and there is a continuous homomorphism $\nu_k: H(\sigma) \longrightarrow H(\sigma_k)$. The essence of the proof is that the sequence $\{f_k\}$ where $f_k \in A_k$ is defined as $\tau_k \circ \nu_k(f\}$ satisfies $\pi_{s,t} f_t = f_s$ for $s \leqq t$. For then the sequence $\{f_k\}$ defines an element $\tau f$ in $A$.

If each $A_k$ were semi-simple, then it would follow that $\pi_{s,t} f_t = f_s$ for $s \leqq t$. For Waelbroeck's theorem implies that $(\pi_{s,t} f_t)^{\wedge} = \hat{f}_s$. However, even if $A$ is semi-simple, it does not follow that each $A_k$ is semi-simple.

Let $s$ and $t$ be two fixed integers with $s \leqq t$. We shall examine the construction of $f_s$. Let $b_i = \pi_s a_i$ for $i = 1, \cdots, n$. $f \in H(\sigma)$ may be considered as a function holomorphic in a neighborhood, say $W$, of $\sigma$ and, therefore, of $\sigma_s$. The following assertions are proven in [11].

(1) $\sigma_s$ is convex in the following sense. There is a finite set of polynomials in $n$ variables, say $p_1, \cdots, p_r$ and neighborhoods $D_1, \cdots, D_n$ of the spectrum of $b_1, \cdots, b_n$ respectively and neighborhoods $D_{n+1}, \cdots, D_{n+r}$

of the spectrum of $b_{n+1} = p_1(b_1, \cdots, b_n)$, $\cdots$, $b_{n+r} = p_r(b_1, \cdots, b_n)$ respectively such that the following two facts are true:

(a). $\sigma_s \subseteqq D \subseteqq W$ where $D = \{\lambda \in D_1 \times \cdots \times D_n : p_i(\lambda) \in D_{n+i}$ for $i = 1, \cdots r\}$.

(b). If $E = D_1 \times \cdots \times D_n \times \cdots \times D_{n+r}$ and $X = \{(\lambda, p_1(\lambda), \cdots, p_r(\lambda)) : \lambda \in D\}$, then the restriction mapping, $\rho$, from $E$ to $X$ is a continuous open homomorphism of $H(E)$ *onto* $H(X)$ with kernel the ideal generated by $\{z_{n+k} - p_k(z_1, \cdots, z_n) : k = 1, \cdots, r\}$. By (a), $f$ is a holomorphic function on $D$ and determines a function $F \in H(X)$ where $F(\lambda, p(\lambda))$ is defined to be $f(\lambda)$ (i.e. $F$ depends only on the first $n$ coordinates). By (b), $F = \rho(G)$ where $G \in H(E)$.

(2) Define $\alpha : H(E) \to A_s$ by

$$\alpha(H) = (1/2\pi i)^{n+r} \int_{\Gamma_1} \cdots \int_{\Gamma_{n+r}} H(\lambda_1, \cdots, \lambda_{n+r})(\lambda^1 - b_1)^{-1} \cdots (\lambda_{n+r} - b_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r}$$

where $\Gamma_i$ is a rectifiable curve in $D_i$ including in its interior the spectrum of $b_i$ for $i = 1, \cdots, n + r$. $\alpha$ is a continuous homomorphism and $\alpha(z_i) = b_i$ for $i = 1, \cdots, n + r$. Thus, by (b), if $\rho(G_1) = \rho(G) = F$, then $\alpha(G_1) = \alpha(G)$. $f_s$ is defined as $\alpha(G)$.

(3) If the system of polynomials $p_1, \cdots, p_r$ and the neighborhoods $D_1, \cdots, D_{n+r}$ are replaced by another system which meets the condition $\sigma_s \subseteqq D \subseteqq W$, then the same element $f_s \in A_s$ arises.

Let $\{p_1, \cdots, p_r, D_1, \cdots, D_{n+r}\}$ be a system used to to define $f_t$. Suppose $c_i = \pi_t a_i$ for $i = 1, \cdots, n$ and $c_{n+k} = p_k(c_1, \cdots, c_n)$ for $k = 1, \cdots, r$. Then

$$\pi_{s,t} f_t = \pi_{s,t}(1/2\pi i)^{n+r} \int \cdots \int G(\lambda)(\lambda_1 - c_1)^{-1} \cdots (\lambda_{n+r} - c_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r} = (1/2\pi i)^{n+r} \int \cdots \int G(\lambda)(\lambda_1 - b_1)^{-1} \cdots (\lambda_{n+r} - b_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r} = f_s .$$

For the system $\{p_1, \cdots, p_n, D_1, \cdots, D_{n+r}\}$ may be used to define $f_s$: $sp(b_i) \subseteqq sp(c_i) \subseteqq D_i$ for $i = 1, \cdots, n + r$ and $\sigma_s \subseteqq \sigma_t \subseteqq D \subseteqq W$. Thus $\tau f$ is well defined.

If $\varphi \in \Delta$, then $\varphi \in \Delta_k$ for some integer $k$, say $\varphi = \pi_k^* \psi$ for $\psi \in \Delta(A_k)$, then $f(\sigma(\varphi)) = f(\sigma_k(\psi)) = \psi(f_k) = \varphi(\tau f)$. $\tau z_i = a_i$, since $(z_i)_s = \pi_s a_i$ for every integer $s$, for $i = 1, \cdots, n$. $\tau$ is continuous, since $f_\alpha \to f_0 \Rightarrow$ for all $k$ $\nu_k f_\alpha \to \nu_k f_0 \Rightarrow$ for all $k$ $\tau_k \circ \nu_k f_\alpha \to \tau_k \circ \nu_k f_0$ (i.e. for all $k$ $(f_\alpha)_k \to (f_0)_k) \Rightarrow \tau f_\alpha \to \tau f_0$.

This theorem, except for continuity of the operational calculus, is also proven in [1] via the Arens-Calderon theorem [2].

**Continuity of derivations.** A derivation on an algebra $A$ is a linear operator $D$ satisfying $D(xy) = xDy + (Dx)y$ for every $x$ and $y$ in $A$. If $A$ is a commutative $F$-algebra, a linear transformation $D: A \to C(\Delta)$ satisfying $D(xy) = \hat{x}Dy + (Dx)\hat{y}$ will be called a derivation into $C(\Delta)$. It is conjectured that a derivation on a Banach algebra must be continuous. Curtis [4] proved that if a Banach algebra is regular, then any derivation is continuous, in fact any derivation from the algebra to $C(\Delta)$ is continuous. This theorem will be extended to allow the algebra to be an $F$-algebra. It will then be applied to some $F$-algebras to determine all derivations in these algebras.

The following lemma is a modification of one in [3] and its proof is essentially the same.

LEMMA. *Let $t$ be an algebraic homomorphism from a commutative $F$-algebra $A$ to a semi-normed algebra $B$. Let $\{g_k\}$ and $\{h_k\}$ be two sequences of elements in $A$ such that for all $n$: $g_n h_n = g_n$ and if $m \neq n$, then $h_n h_m = 0$. Then it is not possible that for all $n$ $\| tg_n \| > n \| g_n \|_n \| h_n \|_n$.*

COROLLARY. *If $D$ is a derivation from a regular commutative semi-simple $F$-algebra $A$ to $C(\Delta)$, then $D$ is continuous.*

*Proof.* Let $\{A_k\}$ and $\{\Delta_k\}$ be defined as in the preliminaries. Since every compact subset of $\Delta$ is contained in some $\Delta_N$, it suffices to prove that if $x_n \to 0$, then $Dx_n \to 0$ uniformly on each $\Delta_N$. The procedure will be to show:

(1) for all $N$ there is an at most finite set $F_N \subseteqq \Delta_N$ such that $Dx_n \to 0$ uniformly on the closure of $[\Delta_N \backslash F_N]$;

(2) if $\varphi$ is isolated in $\Delta$, then $Dx(\varphi) = 0$ for every $x$ in $A$; and

(3) if $\varphi \in \Delta_N$ is isolated in $\Delta_m$ for every $m \geqq N$, then $\varphi$ is isolated in $\Delta$. (1), (2), and (3) imply that $Dx_n(\varphi) \to 0$ for every $\varphi$ and this together with (1) implies that $Dx_n \to 0$ uniformly on $\Delta_N$. This is basically the some proof as in [4]. The third step is the only novel point in the proof. It does not follow from the fact that every compact set is contained in some $\Delta_N$. The example of Arens' ([7] problem $2E$) shows this. (3) may be proven as follows: Suppose $\varphi \in \Delta_N$ is isolated in $\Delta_m$ for all $m \geqq N$. By Silov's theorem, for each $m \geqq N$, there is an idempotent $e_m \in A_m$ such that $\varphi(e_m) = 1$ and $\varphi'(e_m) = 0$ if $\varphi' \in \Delta_m$ and $\varphi' \neq \varphi$ (identifying $\Delta_m$ with $\Delta(A_m)$). Then, because each $e_m$ is an idempotent and $(\pi_{r,s}e_s)^\wedge = \hat{e}_r$ for $N \leqq r \leqq s$, $\pi_{r,s}e_s = e_r$ for $N \leqq r \leqq s$ (two idempotents in $A_r$ equal modulo the radical are iden-

tical). Thus $\{e_m\}$ defines an idempotent $e$ in $A$ such that $\varphi(e) = 1$ and $\varphi'(e) = 0$ for $\varphi' \neq \varphi$ and $\varphi' \in \Delta$

Steps (1) and (2) will be sketched. Proof of (1): Let $B$ be the semi-normed algebra which as an algebra is $A$, but with semi-norm $\| x \| = \| x \|_N + \| Dx \|_N$. Let $F = \{\varphi \in \Delta_N : x \to Dx(\varphi)$ is not a continuous linear functional$\}$. Since $A$ is an $F$-space, the principle of uniform boundedness applies. Since for each $x$ in $A$ $\{Dx(\varphi) : \varphi \in \Delta_N \backslash F\}$ is bounded (by $\| Dx \|_N$), $Dx_n \to 0$ uniformly on $\Delta_N \backslash F$. $F$ is a finite set. If not, then there is an infinite sequence $\{\varphi_n\} \subseteqq F$ with mutually disjoint neighborhoods. Since the algebra is by hypothesis regular, there are sequences $\{y_n\}, \{z_z\}$ such that $\hat{y}_n(\varphi_n) = 1$, $y_n z_n = y_n$ and $z_n z_m = 0$ if $m \neq n$. Then since $\varphi_n \in F$, there is an $x_n$ in $A$ such that $| Dx_n(\varphi_n) | > n \| x_n \|_n \cdot \| y_n \|_n \cdot \| z_n \|_n$. Thus letting $g_n = x_n y_n$ and $h_n = z_n$, we have $\| g_n \| \geqq \| Dg_n \|_N > n \| g_n \|_n \cdot \| h_n \|_n$ and this contradicts the previous lemma. Thus we may let $F$ be $F_N$. Proof of (2): Let $\varphi \in \Delta$ be isolated. Choose, by Silov's theorem an idempotent $e$ such that $\varphi(e) = 1$ and $\varphi'(e) = 0$ for $\varphi' = \varphi$. Then $De(\varphi) = 0$ and, by semi-simplicity, $ex = \varphi(x)e$ for any $x$ in $A$. Hence

$$0 = D(ex)(\varphi) = x(\varphi)De(\varphi) + Dx(\varphi) = Dx(\varphi)$$

for any $x$ in $A$.

By the closed graph theorem and the previous corollary, if $D$ is a derivation on a regular commutative semi-simple $F$-algebra, then $D$ is continuous.

Let $C^\infty(R)$ be the algebra of infinitely differentiable functions on the real line. For $f$ in $C^\infty(R)$, let

$$\| f \|_n = \textstyle\sum_{k=0}^n \sup [| f^{(k)}(t) | : -n \leqq t \leqq n]/k! \,.$$

$C^\infty(R)$ is a regular semi-simple $F$-algebra. If $D$ is a derivation on $C^\infty(R)$ and $x$ is the function mapping $t$ into $t$, then for any polynomial $p$ in $x$, $Dp(x) = p'(x)Dx$. Since the polynomials in $x$ are dense in $C^\infty(R)$ and since $D$ is continuous, $Df = f'Dx$ for any $f$ in $C^\infty(R)$.

As a second application of the previous corollary, we show that the following algebra of Lipschitz functions has no nontrivial derivations.

Let $\alpha \leqq 1$. Let $L_\alpha$ be the subalgebra of $C(R)$ consisting of functions of period 1 with finite norm $\| - \|_\alpha$ where $\| f \|_\alpha$ is defined to be

$$\sup [| f(t) | : t \in R] + \sup [| f(s+h) - f(s)|/| h |^\alpha : s \in R, h \neq 0] \,.$$

Let $1_\alpha = \{ f \in L_\alpha : \overline{\lim} [| f(s+h) - f(s) |/| h |^\alpha \to 0 : h \to 0]$ for $s \in R\}$. For $\alpha < 1$, $L_\alpha$ is a Banach space, $1_\alpha$ a closed subspace, and $L_\alpha$ is isomorphic to $1_\alpha^{**}$ [8]. Let $\alpha_n = 1 - 1/n$ and $L$ be $\cap L_{\alpha_n}$ with the sequence of algebraic norms $\{\| - \|_{\alpha_n}\}$. $L$ may also be defined as the inverse limit of $\{L_{\alpha_n}\}$. $L_{\alpha_{n+1}} \subseteqq 1_{\alpha_n} \subseteqq L_{\alpha_n}$ and so $L$ is also the inverse limit of $\{1_{\alpha_n}\}$. This implies that $L = L^{**}$, however even more is true: A bounded subset of $L$ must have compact closure, i.e., $L$ is a Montel space. For let $S$ be a bounded set in $L \subseteqq 1_{\alpha_n}$. $1_{\alpha_n}$ is isometrically isomorphic as a Banach space with a subspace of $C(W^*)$ where $W^*$ is a compact set obtained as follows: Let $U = \{t \in R: 0 \leqq t \leqq 1\}$, $V = \{(r, s): 0 \leqq r \leqq 1, 0 < r - s \leqq 1/2\}$ and $W = U \cup V$, then $W$ is a locally compact space and $W^*$ is its one-point compactification. The isomorphism $f \rightarrow \tilde{f}$ is defined by $\tilde{f}(\infty) = 0$, $\tilde{f}(t) = f(t)$, and

$$\tilde{f}(r, s) = [f(r) - f(s)]/(r - s)^{\alpha_n} .$$

To see that $S$ is precompact in $L$ it suffices to show that $S$ is precompact in each $1_{\alpha_n}$ or, equivalently, that $\tilde{S}$ is equicontinuous. This follows from the fact that there is a number $K$ such that

$$f \in S \Rightarrow \| f \|_{\alpha_{n+1}} \leqq K .$$

The representation of $1_{\alpha_n}$ as $C(W^*)$ is due to DeLeeuw [8].

A derivation $D$ on $L$ must map every element into 0. For $L$ is a regular, commutative, semi-simple $F$-algebra and so it suffices to show that if $f \in L$, then $\varphi(Df) = 0$ for any $\varphi \in \Delta(L)$. $D(f - \varphi(f)) = Df$ and $f - \varphi(f)$ is in the kernel, $M$, of $\varphi$. So it suffices to show that $D[M] \subseteqq M$. Since $M$ is an ideal, $D[M^2] \subseteqq M$. $M^2 \neq M$, but $M^2$ is dense in $M$ and so, since $D$ must be continuous, $D[M] \subseteqq M$. (Any maximal ideal $M$ must be the set of all functions in $L$ vanishing at some $t_0$ where $0 \leqq t_0 < 1$. The function $\sin([t - t_0]/2\pi)$ is in $M$ but not in $M^2$. Sherbert [10] proved that $M^2$ is dense in $M$ for the Banach algebra $1_\alpha$, in fact for algebras of Lipschitz functions on more general spaces than the unit interval. His proof works as well for $L$.)

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