

# Pacific Journal of Mathematics

**SOME AVERAGES OF CHARACTER SUMS**

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## SOME AVERAGES OF CHARACTER SUMS

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Let  $\chi$  and  $\psi$  be nonprincipal characters mod  $p$ . Let  $f$  be a polynomial mod  $p$  and let  $a_1, \dots, a_p$  be complex constants. We will assume  $a_j = a_k$  for  $j \equiv k(p)$ , and thus have  $a_n$  defined for all  $n$ . Define

$$(1) \quad S = \sum_r a_r \chi(f(r))$$

and

$$(2) \quad J_n(c) = \sum_r \psi(r) \chi(r^n - c),$$

where the variables of summation run through a complete system of residues mod  $p$ .

The averages in question are

$$(3) \quad A_1 = \sum_{a=1}^{p-1} |J_n(a)|^2$$

and

$$(4) \quad A_2 = \sum |S|^2,$$

where the sum in (4) is over the coefficients mod  $p$  of certain fixed powers of the variables in  $f$ . Exact formulae for  $A_1$  will be obtained in all cases, and for  $A_2$  in an extensive class of cases.

Specifically, the following theorems are true.

**THEOREM I.** Let  $f(r) = yr^{m_1} + xr^{m_2} + g(r)$  and assume  $(m_2 - m_1, p - 1) = 1$ . Let the sum in (4) be over all  $x$  and  $y$  mod  $p$ . If  $g$  has a nonzero constant term and neither  $m_1$  nor  $m_2$  is zero, then

$$(5) \quad A_2 = p(p-1) \sum_{r=1}^{p-1} |a_r|^2 + p^2 |a_0|^2.$$

Otherwise,

$$(6) \quad A_2 = p(p-1) \sum_{r=1}^{p-1} |a_r|^2.$$

**THEOREM II.** Let  $d = (n, p-1)$ ,  $\psi(t) = e^{2\pi i(r \text{ ind } (t)/s)}$ , where, naturally,  $s | (p-1)$ ,  $(r, s) = 1$  and  $g^{\text{ind } (t)} \equiv t(p)$  for  $g$  a primitive root mod  $p$ . If  $ds \nmid (p-1)$ , then  $A_1 = 0$ . If  $ds | (p-1)$  and  $\psi\chi^n$  is nonprincipal, then  $A_1 = p(p-1)d$ . If  $ds | (p-1)$  and  $\psi\chi^n$  is principal, then  $A_1 = p(p-1)(d-1) - (p-1)$ .

The following is an immediate consequence of the first theorem.

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**THEOREM III.** *Let  $f$  be as in Theorem I, and assume  $|a_r| = 1$ ,  $r = 1, \dots, p$ . Then there exist  $x_0, y_0, x_1$  and  $y_1$  depending on  $\chi$ , such that the  $S$ , as in (1), for  $x_0$  and  $y_0$  satisfies  $|S| < \sqrt{p}$  and the  $S$ , for  $x_1$  and  $y_1$ , satisfies  $\sqrt{(p-2)} < |S|$ .*

*Proof of Theorem II.* Our principal device is the fact that a function which is periodic mod  $p$  has a unique expansion by means of the characters mod  $p$  [2]. That is if  $h(r) = h(s)$  for  $r \equiv s(p)$ , then for  $n \not\equiv 0(p)$

$$(7) \quad h(n) = \sum_{\theta} b_{\theta} \theta(n),$$

where  $\theta$  runs through the characters mod  $p$ .  $b_{\theta}$  is given by

$$(8) \quad (p-1)b_{\theta} = \sum_r h(r)\bar{\theta}(r).$$

Regarding  $J_n(c)$  as a periodic function mod  $p$  of  $c$ , and expanding  $J_n(c)$  in the form (7), we obtain, by standard methods,

$$(9) \quad J_n(c) = \sum_{\rho^n = \psi\chi^n} \pi(\bar{\rho}, \chi)\rho(c)$$

where  $\pi(\alpha, \beta)$  is a Jacobi sum [1]

$$(10) \quad \pi(\alpha, \beta) = \sum_r \alpha(r)\beta(1-r).$$

The sum in (9) is over all characters  $\rho$  which satisfy the indicated condition.

The expansion (7) has a Parseval identity

$$(11) \quad \sum_{t=1}^{p-1} |h(t)|^2 = (p-1) \sum_{\theta} |a_{\theta}|^2.$$

Thus we can evaluate  $A_1$  by means of (11) and (9) when we know the value of  $|\pi(\alpha, \beta)|^2$ . Now [1]  $|\pi(\alpha, \beta)|^2 = p$  when  $\alpha \neq \varepsilon$ ,  $\beta \neq \varepsilon$  and  $\alpha\beta \neq \varepsilon$ , where  $\varepsilon$  is the principal character. If  $\alpha = \varepsilon$  or  $\beta = \varepsilon$ , then  $|\pi(\alpha, \beta)|^2 = 1$ . If  $\alpha\beta = \varepsilon$  with  $\alpha \neq \varepsilon$  or  $\beta \neq \varepsilon$ , then  $|\pi(\alpha, \beta)|^2 = p$ . By hypothesis,  $\chi$  is nonprincipal. Thus  $|\pi(\bar{\rho}, \chi)|^2$  is  $p$  unless  $\bar{\rho} = \varepsilon$  or  $\bar{\rho}\chi = \varepsilon$ . If  $\bar{\rho} = \varepsilon$ , then  $\bar{\rho} = \varepsilon$  and  $\psi\chi^n$  is principal. If  $\bar{\rho}\chi = \varepsilon$ , then  $\rho = \chi$  and  $\rho^n = \psi\chi^n$  implies  $\psi = \varepsilon$  which is excluded by hypothesis. Let  $N$  be the number of solutions of  $\rho^n = \psi\chi^n$ . If  $\psi\chi^n$  is nonprincipal then  $|\pi(\bar{\rho}, \chi)|^2 = p$  for all  $N$  of the  $\rho$  and  $A_1 = p(p-1)N$ . If  $\psi\chi^n$  is principal, then  $|\pi(\bar{\rho}, \chi)|^2 = p$  for  $N-1$  of the  $\rho$  and  $|\pi(\bar{\rho}, \chi)|^2 = 1$  for  $\rho = \varepsilon$ . Thus, in this case,  $A_1 = (p-1)(p(N-1) + 1) = Np(p-1)^2$ .

$N$ , the number of solutions of  $\rho^n = \psi\chi^n$ , is the number of solutions of  $\sigma^n = \psi$ . It is a standard lemma from the theory of cyclic groups of order  $k$  that  $a^n = b$  has  $(n, k)$  or 0 solutions according to whether

or not order  $b \mid k/(n, k)$ . Also,  $N$  is the number of solutions of  $x^n = \psi(g)$ , for  $x$ , in  $(p - 1) - st$  roots of unity. From either description of  $N$ , it follows that  $N = d$  or  $N = 0$  according as  $ds \mid (p - 1)$  or  $ds \nmid (p - 1)$ , and the theorem follows.

*Proof of Theorem I.* Referring to the hypotheses of Theorem I,

$$|S|^2 = \sum_{r,s} a_r \bar{a}_s \chi(yr^{m_1} + xr^{m_2} + g(r)) \bar{\chi}(ys^{m_1} + xs^{m_2} + g(s))$$

and thus,

$$(12) \quad A_2 = \sum a_r \bar{a}_s \sum \chi(yr^{m_1} + xr^{m_2} + g(r)) \chi(ys^{m_1} + xs^{m_2} + g(s)) = T_1 + T_2.$$

$T_1$  is the sum of the terms in (12) such that  $r \not\equiv 0$  and  $s \not\equiv 0$ .  $T_2$  is the sum of the terms in (12) such that  $r \equiv 0$  or  $s \equiv 0$ .  $T_1$  can be written

$$(13) \quad T_1 = \sum_{r \not\equiv 0, s} a_r \bar{a}_s \chi^{m_1}(r/s) A(r^{m_2-m_1}, r^{-m_1}g(r); s^{m_2-m_1}, s^{-m_1}g(s))$$

where

$$A(a, b; c, d) = \sum_{y+cx+d \not\equiv 0} \chi\left(\frac{y+ax+b}{y+cx+d}\right).$$

Now,

$$A(a, b; c, d) = \sum_x \sum_{y \not\equiv 0} \chi\left(\frac{y+x(a-c)+(b-d)}{y}\right).$$

Except when  $(a-c)x + (b-d) \equiv 0(p)$ ,

$$\sum_{y \not\equiv 0} \chi\left(\frac{y+(a-c)x+(b-d)}{y}\right) = -1.$$

Also,  $(a-c)x + (b-d) \equiv 0(p)$  when  $x \equiv ((b-d)/(a-c))(p)$  or when  $a \equiv c$  and  $b \equiv d$ . Thus, if  $a \not\equiv c$  or  $b \not\equiv d$ , then

$$A(a, b; c, d) = -(p-1) + p - 1 = 0.$$

If  $a \equiv c$  and  $b \equiv d$ , then

$$A(a, b; c, d) = p(p-1).$$

In view of this (13) becomes the sum over all  $r$  and  $s$  such that  $r \not\equiv 0 \not\equiv s$  and  $r^{m_2-m_1} = s^{m_2-m_1}$ ,  $r^{-m_1}g(r) = s^{-m_1}g(s)$ . Since  $(m_2 - m_1, p - 1) = 1$ , we have  $r \equiv s$ . Thus the sum in (13) is over those  $r$  and  $s$  such that  $r \not\equiv 0 \not\equiv s$  and  $r \equiv s$ . Thus

$$T_1 = p(p-1) \sum_{r=1}^{p-1} |a_r|^2.$$

Now

$$(14) \quad T_2 = \sum_{r \neq 0} a_r \bar{a}_0 \sum_{x,y} \chi(yr^{m_1} + xr^{m_2} + g(r)) \bar{\chi}(g(0)) \\ + \sum_{s \neq 0} a_0 \bar{a}_s \sum_{x,y} \chi(g(0)) \bar{\chi}(ys^{m_1} + xs^{m_2} + g(s)) \\ + |a_0|^2 \sum_{x,y} \chi(g(0)) \bar{\chi}(g(0)) = p^2 |a_0|^2 |\chi(g(0))|^2,$$

except when  $m_1 = 0$  or  $m_2 = 0$ .

Thus, if  $g(0) \equiv 0$ ,

$$A_2 = p(p-1) \sum_{r \neq 0} |a_r|^2$$

and if  $g(0) \not\equiv 0$ , then

$$A_2 = p(p-1) \sum_{r \neq 0} |a_r|^2 + p^2 |a_0|^2,$$

when  $m_1 = 0$  or  $m_2 = 0$ , then  $\chi(g(0))$  in (14) must be changed to  $\chi(y + g(0))$  or  $\chi(x + g(0))$ , and  $A_2$  is given by (6).

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