Pacific Journal of Mathematics

A BRANCHING LAW FOR THE SYMPLECTIC GROUPS

WILLARD MILLER, JR.

Vol. 16, No. 2 December 1966

A BRANCHING LAW FOR THE SYMPLECTIC GROUPS

WILLARD MILLER, JR.

A "branching law" is derived for the irreducible tensor representations of the symplectic groups, and a relation is given between this law and the representation theory of the general linear groups.

Branching laws for the irreducible tensor representations of the general linear and orthogonal groups are well-known. Furthermore, these laws have a simple form [1]. In the case of the symplectic groups, however, the branching law becomes more complicated and is expressed in terms of a determinant. We derive this result here by brute force applied to the Weyl character formulas, though it could also have been obtained from a more sophisticated treatment of representation theory contained in some unpublished work of Kostant.

The Branching law. Let V^n be an *n*-dimensional vector space over the complex field. The symplectic group in *n* dimensions, $S_p(n/2)$, is the set of all linear transformations $a \in \mathcal{E}(V^n)$, under which a non-degenerate skew-symmetric bilinear form on $V^n \times V^n$ is invariant, [3]. If $\langle \cdot, \cdot \rangle$ is the bilinear form on $V^n \times V^n$ and $a \in \mathcal{E}(V^n)$, then

(1)
$$a \in S_p(n/2)$$
 if and only if $\langle ax, ay \rangle = \langle x, y \rangle$ for all $x, y \in V^n$.

It is well-known that $S_p(n/2)$ can be defined only for even dimensional spaces, $(n=2\mu, \mu \text{ an integer})$. It is always possible to choose a basis $e_i, e'_i, i=1, \dots, \mu$ in V^n such that

$$egin{aligned} \langle e_i,e_j
angle = \langle e_i',e_j'
angle = 0 \ \ 1 \leq i,j \leq \mu \ & \langle e_i,e_j'
angle = \delta_{ij} \ . \end{aligned}$$

We assume that the matrix realization of $S_p(\mu)$ is given with respect to such a basis [3]. The unitary symplectic group, $US_p(\mu)$, is defined by

$$(3) US_p(\mu) = S_p(\mu) \cap U(2\mu)$$

where $U(2\mu)$ is the group of unitary matrices in 2μ dimensions. The irreducible continuous representations of $US_p(\mu)$ can be denoted by ${}^{\mu}\omega_{f_1,\ldots,f_{\mu}}$, where f_1,f_2,\ldots,f_{μ} are integers such that $f_1 \geq f_2 \geq \cdots \geq f_{\mu-1} \geq f_{\mu} \geq 0$.

Received May 8, 1964. This work represents results obtained in part at the Courant Institute of Mathematical Sciences, New York University, under National Science Foundation grant GP-1669.

Let $a \in US_p(\mu)$. The eigenvalues of a are

$$\varepsilon_1, \, \varepsilon_2, \, \cdots, \, \varepsilon_{\mu}, \, \varepsilon_1^{-1}, \, \varepsilon_2^{-1}, \, \cdots, \, \varepsilon_{\mu}^{-1}$$

where $|\varepsilon_1|=1$, $1\leq i\leq \mu$, (see [3]). The character of the representation ${}^{\mu}\omega_{f_1,\dots,f_{\mu}}$ is given by

$$(4) \qquad {}^{\mu}\chi_{f_1,...,f_{\mu}}(a) = \frac{\mid \varepsilon^{l_1} - \varepsilon^{-l_1}, \, \varepsilon^{l_2} - \varepsilon^{-l_2}, \, \cdots, \, \varepsilon^{l\mu} - \varepsilon^{-l\mu} \mid}{\mid \varepsilon^{\mu} - \varepsilon^{-\mu}, \, \varepsilon^{\mu-1} - \varepsilon^{-\mu+1}, \, \cdots, \, \varepsilon - \varepsilon^{-1} \mid}$$

$$l_i = f_i + \mu - i + 1, \ 1 \le i \le \mu.$$

The vertical lines in the numerator and denominator of (4) denote two determinants whose jth rows are obtained by replacing the symbol ε by ε_{j} .

Let ${}^{\mu}\omega_{f_1,\dots,f_{\mu}}$ and ${}^{\mu-1}\omega_{g_1,\dots,g_{\mu-1}}$ be irreducible representations of the groups $US_p(\mu)$, $US_p(\mu-1)$, respectively. Denote by $R_{g_1}^{f_1,\dots,f_{\mu}}$ the the multiplicity of ${}^{\mu-1}\omega_{g_1,\dots,g_{\mu-1}}$ in the restricted representation ${}^{\mu}\omega_{f_1,\dots,f_{\mu}}/US_p(\mu-1)$ which is obtained by restricting ${}^{\mu}\omega$ to the subgroup of $US_p(\mu)$ consisting of all matrices that leave the basis vectors e_{μ} , e'_{μ} fixed.

Theorem 1. $R_{g_1}^{f_1}::::_{g_{\mu-1}}^{f_{\mu}}=0$ unless $f_i\geq g_i\geq f_{i+2},\ 1\leq i\leq \mu-2$. If these conditions are fulfilled, then

where $D_{i+1,i} = \max[f_{i+1} - g_i, 0], 1 \le i \le \mu - 1$.

Proof. The dependence of the character of the representation ${}^{\mu}\omega_{f_1,...,f_{\mu}}$ on the eigenvalues $\varepsilon_1, \dots, \varepsilon_{\mu}$ of $a \in US_p(\mu)$ can be explicitly exhibited by setting

$$^{\mu}\chi_{f_1,\ldots,f_{\mu}}(a)={}^{\mu}\chi_{f_1,\ldots,f_{\mu}}(\varepsilon_1,\,\cdots,\,\varepsilon_{\mu})$$
.

The restriction of ${}^{\mu}\omega_{f_1,...,f_{\mu}}$ to $US_{p}(\mu-1)$ can be accomplished by requiring that $\varepsilon_{\mu}=1$. It follows that

$$(6) \qquad {}^{\mu}\chi_{f_{1},\dots,f_{\mu}}(\varepsilon_{1},\cdots,\varepsilon_{\mu-1},1) = \sum R_{g_{1}}^{f_{1},\dots,f_{\mu}}\chi_{g_{1},\dots,g_{\mu-1}}(\varepsilon_{1},\cdots,\varepsilon_{\mu-1}) g_{1},\cdots,g_{\mu-1}.$$

We will calculate the constants $R_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu}}$ by carrying out the decomposition in equation (6). If we take the limit as $\varepsilon_{\mu} \to 1$ in the character formula (4), we get

$$(7) \quad {}^{\mu}\chi_{f_{1}, \dots, f_{\mu}}(\varepsilon_{1}, \, \cdots, \, \varepsilon_{\mu-1}, \, 1) \\ \\ = \begin{pmatrix} \varepsilon_{1}^{l_{1}} - \varepsilon_{1}^{-l_{1}} & \varepsilon_{1}^{l_{2}} - \varepsilon_{1}^{-l_{2}} & \cdots & \varepsilon_{1}^{l_{\mu}} - \varepsilon_{1}^{-l_{\mu}} \\ \varepsilon_{2}^{l_{1}} - \varepsilon_{2}^{-l_{1}} & \varepsilon_{2}^{l_{2}} - \varepsilon_{2}^{-l_{2}} & \cdots & \varepsilon_{2}^{l_{\mu}} - \varepsilon_{2}^{-l_{\mu}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{\mu-1}^{l_{1}} - \varepsilon_{\mu-1}^{-l_{1}} & \varepsilon_{\mu-1}^{l_{2}} - \varepsilon_{\mu-1}^{-l_{2}} & \cdots & \varepsilon_{\mu-1}^{l_{\mu}} - \varepsilon_{\mu-1}^{-l_{\mu}} \\ \\ = \begin{pmatrix} \varepsilon_{1}^{l_{1}} - \varepsilon_{1}^{-l_{1}} & \varepsilon_{1}^{l_{2}} - \varepsilon_{1}^{-l_{2}} & \cdots & \varepsilon_{1}^{l_{\mu}} - \varepsilon_{1}^{-l_{\mu}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{\mu-1}^{l_{\mu}} - \varepsilon_{1}^{-l_{\mu}} & \varepsilon_{1}^{l_{\mu-1}} - \varepsilon_{1}^{-l_{\mu+1}} & \cdots & \varepsilon_{1} - \varepsilon_{1}^{-1} \\ \varepsilon_{2}^{l_{\mu}} - \varepsilon_{2}^{-l_{\mu}} & \varepsilon_{2}^{l_{\mu-1}} - \varepsilon_{2}^{-l_{\mu+1}} & \cdots & \varepsilon_{2} - \varepsilon_{2}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{\mu-1}^{l_{\mu}} - \varepsilon_{\mu-1}^{-l_{\mu}} & \varepsilon_{\mu-1}^{l_{\mu-1}} - \varepsilon_{\mu-1}^{-l_{\mu+1}} & \cdots & \varepsilon_{\mu-1} - \varepsilon_{\mu-1}^{-1} \\ \mu & \mu - 1 & \cdots & 1 \end{pmatrix}$$

Set
$$s_i(j)=(arepsilon_i)^j-(arepsilon_i)^{-j},\ 1\leqq i\leqq \mu-1$$
 $d_i=arepsilon_i+arepsilon_i^{-1}-2,\ 1\leqq i\leqq \mu-1$.

It is easy to verify the formula

(8)
$$s_i(n-1)d_i = s_i(n) - 2s_i(n-1) + s_i(n-2).$$

Also, the relation

(9)
$$s_i(n) = d_i[s_i(n-1) + 2s_i(n-2) + \cdots + ks_i(n-k) + \cdots + (n-1)s_i(1)] + ns_i(1)$$

can be established by induction on (8).

Consider the determinant in the denominator of equation (7). Using obvious abbreviations, we have

$$\begin{vmatrix} s(\mu), s(\mu-1), \cdots s(2), s(1) \\ \mu, \mu-1, \cdots 2, 1 \end{vmatrix} =$$

$$= \begin{vmatrix} s(\mu) - s(\mu - 1), s(\mu - 1) - s(\mu - 2), \cdots, s(2) - s(1), s(1) \\ 1 & , & 1 & , \cdots, & 1 & , 1 \end{vmatrix}$$

$$= \begin{vmatrix} s(\mu) - 2s(\mu - 1) + s(\mu - 2), s(\mu - 1) - 2s(\mu - 2) + s(\mu - 3), & \cdots, s(2) - 2s(1), s(1) \\ 0 & , & 0 & \cdots, & 0 & , 1 \end{vmatrix}$$

$$= \begin{vmatrix} s(\mu) - 2s(\mu - 1) + s(\mu - 2), s(\mu - 1) - 2s(\mu - 2) + s(\mu - 3), \cdots, s(2) - 2s(1) \end{vmatrix} '$$

$$= \prod_{i=1}^{\mu - 1} d_i \mid s(\mu - 1), s(\mu - 2), \cdots, s(2), s(1) \mid '.$$

Equation (8) was used in the last step of (10). The quantity $|\cdot|'$ stands for a determinant of order $\mu - 1$.

Now, consider the numerator of (7).

We have

$$\begin{array}{c} (11) \\ \left| \begin{array}{c} s(l_{1}), \, (s(l_{2}), \, \cdots, s(l_{\mu}) \, \middle| \, = \, (\text{using } (9)) \\ l_{1} \, , \, l_{2} \, , \, \cdots, \, l_{\mu} \, \middle| \, \\ \\ = \left| \begin{array}{c} d[s(l_{1}-1)+\cdots+(l_{1}-1)s(1)]+l_{1}s(1), \, \cdots, d[s(l_{\mu}-1)+\cdots+(l_{\mu}-1)s(1)] \\ l_{1} \, , \, \cdots, \, l_{\mu} \, \\ \\ = l_{\mu} \left| d\Big\{ [s(l_{1}-1)+\cdots+(l_{1}-1)s(1)]-\frac{l_{1}}{l_{\mu}}[s(l_{\mu}-1)+\cdots+(l_{\mu}-1)s(1)] \Big\}, \, \cdots \\ \\ \cdots, d\Big\{ [s(l_{\mu-1}-1)+\cdots+(l_{\mu-1}-1)s(1)] \\ & - \frac{l_{\mu-1}}{l_{\mu}}[s(l_{\mu}-1)+\cdots+(l_{\mu}-1)s(1)] \Big\} \right|' \, . \end{array}$$

Set $q_j(i) = s_j(l_i - 1) + 2s_j(l_i - 2) + \cdots + (l_i - 1)s_j(1)$, $1 \le i \le \mu$, $1 \le j \le \mu - 1$. Then, we find that the numerator of (7) is equal to

$$\begin{aligned} & (12) \quad l_{\mu} \prod_{i=1}^{\mu-1} d_{i} \Big| q(1) - \frac{l_{1}}{l_{\mu}} q(\mu), \ q(2) - \frac{l_{2}}{l_{\mu}} q(\mu), \ \cdots, \ q(\mu-1) - \frac{l_{\mu-1}}{l_{\mu}} q(\mu) \Big|' \\ & = \prod_{i=1}^{\mu-1} d_{i} \{ l_{\mu} \mid q(1), q(2), \cdots, q(\mu-1) \mid' \\ & - l_{1} \mid q(\mu), q(2), q(3), \cdots, q(\mu-1) \mid' \\ & - l_{2} \mid q(1), q(\mu), q(3), \cdots, q(\mu-1) \mid' \\ & - \cdots - l_{\mu-1} \mid q(1), q(2), \cdots, q(\mu-2), q(\mu) \mid' \} \ . \end{aligned}$$

Dividing the last expression in (12) by the last expression in (10), we cancel the factor $\prod_{i=1}^{\mu-1} d_i$. Thus, to calculate $R_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu}}$ it only remains to expand the determinants in (12) as linear combinations of determinants of the form

(13)
$$|s(h_1), s(h_2), \dots, s(h_{\mu-1})|', h_1 > \dots > h_{\mu-1} > 0$$
.

Set $p_i=g_i+\mu-i$, $1\leq i\leq \mu-1$. Then $R_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu}}$ will be the

coefficient of the determinant

$$|s(p_1), s(p_2), \dots, s(p_{\mu-1})|'$$

in the expansion of (12). It is straightforward to show that

$$(14) \quad R_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu}} = \sum_{\sigma} sgn\sigma \langle l_{\sigma^{(1)}} - p_1 \rangle \langle l_{\sigma^{(2)}} - p_2 \rangle \cdots \langle l_{\sigma^{(\mu-1)}} - p_{\mu-1} \rangle l_{\sigma^{(\mu)}}$$

where the sum is taken over all permutations σ of the integers $1, 2, \dots, \mu$. The quantity

(15)
$$\langle l_i - p_j \rangle = \begin{cases} l_i - p_j & \text{if } l_i - p_j \ge 0 \\ 0 & \text{if } l_i - p_j < 0. \end{cases}$$

Thus,

$$(16) \qquad R_{g_{1},\ldots,g_{\mu-1}}^{f_{1},\ldots,f_{\mu}} = \begin{vmatrix} \langle l_{1}-p_{1}\rangle, & \langle l_{1}-p_{2}\rangle \cdots \langle l_{1}-p_{\mu-1}\rangle, & l_{1} \\ \langle l_{2}-p_{1}\rangle, & \langle l_{2}-p_{2}\rangle \cdots \langle l_{2}-p_{\mu-1}\rangle, & l_{2} \\ \vdots & \vdots & \vdots \\ \langle l_{\mu}-p_{1}\rangle, & \langle l_{\mu}-p_{2}\rangle \cdots \langle l_{\mu}-p_{\mu-1}\rangle, & l_{\mu} \end{vmatrix}$$

An analysis of expression (16) yields the theorem.

Corollary.
$$R_{g_1,\ldots,g_{\mu-2},0}^{f_1,\ldots,f_{\mu-1},0}=R_{g_1,\ldots,g_{\mu-2}}^{f_1,\ldots,f_{\mu-1}}$$

Proof. Direct verification from expression (5).

It is well-known that the continuous irreducible representations of the $n \times n$ unitary group U(n) can be denoted by ${}^n\nu_{f_1,\dots,f_n}$ where the integers f_1, f_2, \dots, f_n can take on all values consistent with $f_1 \ge f_2 \ge \dots f_n$, [3]. We make the assumption that $f_n \ge 0$.

U(n) contains a subgroup $G(n-2)=U(n-2)\dotplus E_2$ where E_2 is the 2×2 unit matrix, which is obviously isomorphic to U(n-2). (see [1], page 16 for the notation). We identify G(n-2) and U(n-2) by this isomorphism. Thus the irreducible continuous representations of G(n-2) will be denoted by ${}^{n-2}\nu_{g_1},...,g_{n-2}$.

Denote by $M_{g_1,\ldots,g_{n-2}}^{f_1,\ldots,f_n}$ the multiplicity of $n^{-2}\nu_{g_1,\ldots,g_{n-2}}$ in the restricted representation ${}^n\nu_{f_1,\ldots,f_n}/G(n-2)$. The quantity $M_{g_1,\ldots,g_{n-2}}^{f_1,\ldots,f_n}$ can be computed from the Weyl character formula for the irreducible representations of U(n) in the same way as we have done for the irreducible representations of $US_p(\mu)$. We give only the results of this computation.

Theorem 2. Let $M_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu+1}}$ be the multiplicity of $^{\mu-1}\nu_{g_1,\ldots,g_{\mu-1}}$ in $^{\mu+1}\nu_{f_1,\ldots,f_{\mu+1}}/U(\mu-1)$ as defined above. Then

$$M_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu,0}}=R_{g_1,\ldots,g_{\mu-1}}^{f_1,\ldots,f_{\mu}}$$
 .

Corollary.
$$M_{g_1,\ldots,g_{\mu-2}}^{f_1,\ldots,f_{\mu}}=R_{g_1,\ldots,\mu_{\mu-2},0}^{f_1,\ldots,f_{\mu}}$$

REFERENCES

- 1. H. Boerner, Representations of Groups, North Holland, Amsterdam. 1963.
- 2. W. Miller, On a class of vector-valued functions covariant under the classical groups with applications to physics, National Science Foundation Technical Report. University of California, Berkeley. 1963.
- 3. H. Weyl, The Classical Groups. Princeton University Press, Princeton. 1946.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES NEW YORK UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University Stanford, California

R. M. BLUMENTHAL

University of Washington Seattle, Washington 98105 *I. Dugundji

University of Southern California Los Angeles, California 90007

RICHARD ARENS

University of California Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. Wolf

K. Yosida

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

AMERICAN MATHEMATICAL SOCIETY CHEVRON RESEARCH CORPORATION TRW SYSTEMS

NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Paul A. White, Acting Editor until J. Dugundji returns.

Pacific Journal of Mathematics

Vol. 16, No. 2 December, 1966

Loren N. Argabright, <i>Invariant means on topological semigroups</i>	193
William Arveson, A theorem on the action of abelian unitary groups	205
John Spurgeon Bradley, Adjoint quasi-differential operators of Euler	
type	213
Don Deckard and Lincoln Kearney Durst, <i>Unique factorization in power</i>	
series rings and semigroups	239
Allen Devinatz, The deficiency index of ordinary self-adjoint differential	
operators	243
Robert E. Edwards, <i>Operators commuting with translations</i>	259
Avner Friedman, Differentiability of solutions of ordinary differential	
equations in Hilbert space	267
Boris Garfinkel and Gregory Thomas McAllister, Jr., Singularities in a	
variational problem with an inequality	273
Seymour Ginsburg and Edwin Spanier, Semigroups, Presburger formulas,	
and languages	285
Burrell Washington Helton, Integral equations and product integrals	297
Edgar J. Howard, First and second category Abelian groups with the n-adic	
topology	323
Arthur H. Kruse and Paul William Liebnitz, Jr., An application of a family	
homotopy extension theorem to ANR spaces	331
Albert Marden, I. Richards and Burton Rodin, <i>On the regions bounded by</i>	
homotopic curves	337
Willard Miller, Jr., A branching law for the symplectic groups	341
Marc Aristide Rieffel, A characterization of the group algebras of the finite	
groups	347
P. P. Saworotnow, <i>On two-sided H*-algebras</i>	365
John Griggs Thompson, Factorizations of p-solvable groups	371
Shih-hsiung Tung, Harnack's inequalities on the classical Cartan	
domains	373
Adil Mohamed Yaqub, <i>Primal clusters</i>	379