

Pacific Journal of Mathematics

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The two sections of this note are unrelated, except that both are concerned with the exposed points of a compact convex subset K of a locally convex space E . In §1 it is proved that if K is of finite dimension d , then the set of all its exposed points can be expressed as the union of a G_δ set, an F_σ set, and $d - 2$ sets each of which is the intersection of a G_δ set with an F_σ set. A sharper assertion is proved for the three-dimensional case, and some related results are obtained for certain infinite-dimensional situations. Section 2 describes a compact convex set in the space \mathbb{R}^c which has no algebraically exposed points. Both sections contain unsolved problems.

When E is finite-dimensional, a point p of K is said to be *exposed* provided $\{p\}$ is the intersection of K with some supporting hyperplane of K , or, equivalently, provided there is a linear form on E whose K -maximum is attained precisely at p . The set of all exposed points of K will be denoted by $\text{exp } K$. It is well known that $\text{exp } K$ is a dense subset of $\text{ext } K$, the set of all extreme points of K . The set $\text{ext } K$ is a G_δ set, and in the two-dimensional case $\text{exp } K$ is also a G_δ set. However, there are three-dimensional sets K for which $\text{exp } K$ is not a G_δ set [1, 4], and in Corson's example [1] the set $\text{exp } K$ is not even the union of a G_δ set and an F_σ set. This suggests the problem of determining the Borel type of $\text{exp } K$, and the answer provided in §1 seems to be fairly complete.

For infinite-dimensional convex sets, the notion of exposed point may be defined in several different ways, all of which are equivalent in the finite-dimensional case. The weakest notion is that of an *algebraically exposed point*, this being a point of K such that $p \in \text{exp}(K \cap P)$ for every two-dimensional flat P through p . The example in §2 provides a negative answer for questions raised by Phelps [7] and Klee [4].

1. The Borel type of the set of exposed points. Let us begin with the main finite-dimensional result.

THEOREM 1.1. *Suppose that K is a closed convex set of finite dimension d , and let $\text{exp } K$ denote the set of all exposed points of K . Then*

- (a) *if $d = 2$, $\text{exp } K$ is a G_δ set;*

Received November 5, 1964. This paper was written at the University of Washington while the first author was a visiting Walker-Ames Professor and the others were supported by a grant from the National Science Foundation (NSF-GP-378).

(b) if $d = 3$, $\text{exp } K$ is the union of a G_δ set and a set which is the intersection of a G_δ set and an F_σ set;

(c) for arbitrary d , $\text{exp } K$ is the union of a G_δ set, an F_σ set, and $d - 2$ sets each of which is the intersection of a G_δ set with an F_σ set.

Proof. For (a), it suffices to note that if $d = 2$, then the set $\text{ext } K \sim \text{exp } K$ is countable. For (b), let us define a *face* of a convex set A as a maximal convex subset of the relative boundary of A . A face of dimension j will be called a *j-face*. Let S denote the union of all 0-faces of K . Let T denote the set of all points x of K such that $\{x\}$ is the intersection of a 1-face or a 2-face of K with a supporting hyperplane of K . Finally, let U denote the set of all endpoints of 1-faces of 2-faces of K . Then S is a G_δ set, T is an F_σ set, and U is countable. Further, $\text{exp } K = S \cup (T \sim V)$ for some $V \subset U$, and the desired conclusion follows.

For an alternative proof, let R denote the set of all points of K at which K admits a unique supporting hyperplane, let $S = R \cap \text{exp } K$, let T be the union of all 1-faces of K which are contained in $K \sim R$, and let U be the set of all endpoints of these 1-faces. Then the above statements about S , T and U and still correct.

For (c), we assume that K is a closed convex subset of Euclidean d -space \mathbb{E}^d . Let X denote the boundary of K and let Y denote the unit sphere $\{y \in \mathbb{E}^d: \|y\| = 1\}$. By means of the usual inner product, Y will be regarded as a set of functions on X . For each point x_0 of X , we define

$$(1) \quad x_0^* = \{y_0 \in Y: y_0(x_0) = \sup y_0 X\},$$

and for each point y_0 of Y we define

$$(2) \quad y_0^\varepsilon = \{x_0 \in X: y_0(x_0) = \sup y_0 X\}.$$

A point x of K is exposed if and only if $x \in X$ and $\{x\} = y^\varepsilon$ for some $y \in Y$.

In the sphere Y , we define an *open j-ball of radius ε* as a set of the form $N(y, \varepsilon) \cap L \cap Y$, where $y \in Y$, $0 < \varepsilon < \sqrt{2}$, $N(y, \varepsilon)$ is the ε -neighborhood of y in \mathbb{E}^d , and L is a $(j + 1)$ -dimensional linear subspace of \mathbb{E}^d . Let $D(j, \varepsilon)$ denote the set of all points $x \in X$ such that x^ε contains an open j -ball of radius ε . Let C_1, C_2, \dots be a sequence of compact sets whose union is X , and let $Q(j, k, \varepsilon)$ denote the set of all points $x \in X$ for which x^ε contains an open j -ball B of radius ε such that for some $y \in B$, $\text{diam } y^\varepsilon \geq \varepsilon$ and y^ε intersects C_k . It can be verified that each of the sets $D(j, \varepsilon)$ and $Q(j, k, \varepsilon)$ is closed, and that each of the sets $Q(d - 1, k, \varepsilon)$ is empty. The set $\text{exp } K$ is the union of the d sets

$$\begin{aligned} X_0 &= D(0, 1) \sim \bigcup_{k, \varepsilon} Q(0, k, \varepsilon), \\ X_j &= \bigcup_\varepsilon D(j, \varepsilon) \sim \bigcup_{k, \varepsilon} Q(j, k, \varepsilon) \quad (1 \leq j \leq d - 2), \end{aligned}$$

and

$$X_{d-1} = \bigcup_{\varepsilon} D(d-1, \varepsilon).$$

This completes the proof of 1.1.

PROBLEM 1.2. *If K is a three-dimensional compact convex set, must $\exp K$ be the intersection of a G_{δ} set and an F_{σ} set?¹ If the answer is affirmative, what about the general finite-dimensional case?*

In the remainder of this section, X will denote a set and Y will denote a set of real-valued functions on X . The sets x_0^* (for $x_0 \in X$) and y_0^* (for $y_0 \in Y$) are defined as in (1) and (2) above. A point x_0 of X will be called *Y-smooth* provided the set x_0^* consists of a single point of Y ; that is, provided precisely one member of Y attains its maximum at x_0 . And the point x_0 of X will be called *Y-exposed* provided $\{x_0\} = y_0^*$ for some $y_0 \in Y$; that is, provided some member of Y attains its maximum precisely at x_0 . The sets of all *Y-smooth* and *Y-exposed* points of X will be denoted respectively by $\text{sm}_Y X$ and $\text{exp}_Y X$.

The following remarks are elementary but useful. Their proofs are left to the reader.

LEMMA 1.3. *If Y is convex (with respect to the pointwise addition and scalar multiplication of real-valued functions on X), then the set x^* is convex for each $x \in X$.*

LEMMA 1.4. *Suppose that x_0 is a point of X , the functions y_1, y_2, \dots are members of x_0^* , the numbers $\lambda_1, \lambda_2, \dots$ are strictly positive, and the series $\sum_i \lambda_i y_i$ is pointwise convergent to a function $y_0 \in Y$. Then $y_0 \in x_0^*$ and $y_0^* = \bigcap_i y_i^*$.*

LEMMA 1.5. *Suppose that X is a metric space, each member of Y is upper semicontinuous on X , and Y has the topology of uniform convergence on X . Then*

- (a) *if Y is compact, the set-valued transformation $x^* | x \in X$ is upper semicontinuous;*
- (b) *if X is compact, the set-valued transformation $y^* | y \in Y$ is upper semicontinuous.*

When X and Y are as in 1.1, it follows from 1.1 that the set $\text{exp}_Y X$ is both a $G_{\delta\sigma}$ set and an $F_{\sigma\delta}$ set. This may be extended in one direction as follows.

¹ If K has no 2-faces, $\exp K$ is the union of a G_{δ} set and an F_{σ} set, but the example of [1] shows that this is not true in general.

THEOREM 1.6. *Suppose that X is a compact metric space and Y is a convex set of upper semicontinuous real-valued functions on X . Suppose that Y is compact in the topology of uniform convergence. Then the set of all Y -exposed points of X is an $F_{\sigma\delta}$ set.*

Proof. Define

$$M = \bigcup_{x \in X} \{x\} \times x^s \subset X \times Y,$$

and for each $\varepsilon > 0$ define

$$W_\varepsilon = \{y \in Y: \text{diam } y^\varepsilon < \varepsilon\}.$$

It follows from 1.5 (a) that M is closed and from 1.5 (b) that W_ε is open. Hence the set $M \cap (X \times W_\varepsilon)$ is an F_σ subset of the compact set $X \times Y$, and its projection A_ε on X must also be an F_σ . To complete the proof it suffices to show that

$$\text{exp}_Y X = \bigcap_{\varepsilon > 0} A_\varepsilon.$$

Inclusion in one direction is obvious. For the other, we consider an arbitrary point $x_0 \in \bigcap_{\varepsilon > 0} A_\varepsilon$ and want to show that $x_0 \in \text{exp}_Y X$. From the definition of A_ε it follows that for each $\varepsilon > 0$ there exists $y(\varepsilon) \in x_0^s$ such that $\text{diam } y(\varepsilon)^\varepsilon < \varepsilon$. Since the set x_0^s is convex by 1.3 and compact by 1.5 (a), it must include the function $y_0 = \sum_i 2^{-i} y(2^{-i})$. Then 1.4 implies that $\text{diam } y_0^\varepsilon = 0$, and the desired conclusion follows.

The remaining theorems of the present section are proved by refinements of the reasoning of 1.1 and 1.6. Indeed, 1.1 (c) could be derived as a corollary of 1.7 below², and 1.6 as a corollary of 1.9. However, the simpler arguments were given first as an aid to clarity.

When j is a nonnegative integer and Z is a subset of a linear space E , the j -interior of Z ($\text{int}_j Z$) is defined as the set of all points $z \in Z$ such that for some j -dimensional flat F through z , z is interior to the set $Z \cap F$ with respect to the natural topology of F .

THEOREM 1.7. *Suppose that X and Y are as in 1.6. For $j = 0, 1, \dots$, let X_j denote the set of all points $x \in X$ such that $\dim x^s \geq j$ and $y^\varepsilon = \{x\}$ for all $y \in \text{int}_j x^s$. Then X_0 is a G_δ set and each of the sets X_1, X_2, \dots is the intersection of a G_δ set and an F_σ set.*

Proof. For the sake of simplicity, we assume at first that every

² Of course, the Euclidean sphere Y of 1.1 is not convex. However, it can be replaced in the proof of 1.1 by the boundary of a cube, which is the union of a finite number of convex sets. Another technical complication in deriving 1.1 directly from 1.7 would result from the fact that the set X in 1.1 need not be compact but only σ -compact.

member of Y is bounded below as well as (automatically) above. Then Y is a compact convex subset of the Banach space E of all bounded real-valued functions on X . For $\varepsilon > 0$ and for $j = 0, 1, 2, \dots$, let $D(j, \varepsilon)$ denote the set of all points $x \in X$ such that x^s contains an open j -ball of radius ε .³ Let $Q(j, \varepsilon)$ denote the set of all points $x \in X$ such that x^s contains an open j -ball B of radius ε with $\text{diam } y^e \geq \varepsilon$ for some $y \in B$. It is evident that

$$X_j = \bigcup_{\varepsilon > 0} D(j, \varepsilon) \sim \bigcup_{\varepsilon > 0} Q(j, \varepsilon).$$

Further, $D(j, \varepsilon)$ and $Q(j, \varepsilon)$ are both antitone functions of ε , and $D(0, \varepsilon) = D(0, 1)$ for all $\varepsilon > 0$. Thus, it suffices to show that each of the sets $D(j, \varepsilon)$ and $Q(j, \varepsilon)$ is closed. This follows from 1.5, but the case in which $j \geq 1$ may require some explanation.

Consider a sequence x_α of points of X , converging to a point $x_0 \in X$. Suppose that $\{x_1, x_2, \dots\} \subset D(j, \varepsilon)$. Then for each i there are a point $z_i \in E$ and a j -dimensional linear subspace L_i of E such that

$$z_i + N(0, \varepsilon) \cap L_i \subset x_i^s.$$

For each i , L_i admits a basis $\{b_i^1, \dots, b_i^j\}$ consisting of points of norm ε such that for $1 < h \leq j$, the point b_i^h is at distance ε from the linear hull of the set $\{b_i^1, \dots, b_i^{h-1}\}$.⁴ The set x_0^s is compact and (by 1.5 (a)) every neighborhood of x_0^s contains all but finitely many of the sets x_i^s . Thus, we may assume without loss of generality that each of the sequences z_α and b_α^h is convergent; say $z_\alpha \rightarrow z_0$ and $b_\alpha^h \rightarrow b_0^h$. We claim that the set $\{b_0^1, \dots, b_0^j\}$ is linearly independent. Indeed, suppose the contrary, whence for some $h > 1$ we have $b_0^h = \sum_{1 \leq r < h} \lambda_r b_0^r$. But then for each i ,

$$\begin{aligned} \|b_i^h - \sum_{1 \leq r < h} \lambda_r b_i^r\| &\leq \|b_i^h - b_0^h\| + \|\sum_{1 \leq r < h} \lambda_r (b_0^r - b_i^r)\| \\ &\leq (1 + \sum_{1 \leq r < h} |\lambda_r|) \max_{1 \leq r \leq h} \|b_i^r - b_0^r\|, \end{aligned}$$

and for all sufficiently large i this is inconsistent with the way in which the basis $\{b_i^1, \dots, b_i^j\}$ was chosen. Thus the set $\{b_0^1, \dots, b_0^j\}$ is linearly independent and its linear hull L_0 is a j -dimensional linear subspace of E . It is easily verified that

$$z_0 + N(0, \varepsilon) \cap L_0 \subset x_0^s,$$

whence $x_0 \in D(j, \varepsilon)$ and it follows that the set $D(j, \varepsilon)$ is closed.

Now suppose in addition that $\{x_1, x_2, \dots\} \subset Q(j, \varepsilon)$. Then the points z_i and linear subspaces L_i can be chosen so that $\text{diam } y^e \geq \varepsilon$ for some

³ Here a j -ball of radius ε is a set of the form $N(y, \varepsilon) \cap F$, where F is a j -dimensional flat through y .

⁴ This is an easy application of the Hahn-Banach theorem.

$y \in z_i + N(0, \varepsilon) \cap L_i$. But then $\text{diam } z_i^e \geq \varepsilon$, for it follows from 1.4 that y^e is constant on every open segment contained in a set x_i^e and hence on every open j -ball contained in x_i^e . With $\text{diam } z_i^e \geq \varepsilon$ and $z_\alpha \rightarrow z_0$, it follows from 1.5 (b) that $\text{diam } z_0^e \geq \varepsilon$ and consequently the set $Q(j, \varepsilon)$ is closed.

Now to complete the proof of 1.7, we abandon the assumption that all of the members of Y are bounded. Let S denote the linear space of all real-valued functions on X , and for $s_1, s_2 \in S$ let

$$\rho(s_1, s_2) = \sup_{x \in X} |s_1(x) - s_2(x)|.$$

The function ρ satisfies all of the requirements for a metric except that it may have the value $+\infty$. The ρ -topology for S is the topology of uniform convergence. Let

$$R = \{(s_1, s_2) : \rho(s_1, s_2) < \infty\} \subset S \times S$$

Then R is an equivalence relation on S , and each equivalence class is both open and closed. Note that if $(s_1, s_2) \notin R$, then no two points of the segment $[s_1, s_2]$ are in the same equivalence class. Choose $y_0 \in Y$ and let $\zeta(s) = s - y_0$ for all $s \in S$. Then ζ is an affine isometry of S onto S , and ζ carries Y into the Banach space E of all bounded real-valued functions on X . From this point on, the proof is merely a paraphrase of the one already given when $Y \subset E$.

1.8. COROLLARY. *Suppose that X and Y are as in 1.6. Then the set $\exp_Y X \cap \text{sm}_Y X$ is a G_δ set, and the set $\{x \in \exp_Y X : \text{diam } x^s < \infty\}$ is a $G_{\delta\sigma}$ set.*

Proof. Note that the first set is equal to $X_0 \sim \bigcup_{\varepsilon < 0} D(1, \varepsilon)$ (in the notation of 1.7), while the second is equal to

$$\bigcup_{j=0}^{\infty} \left(X_j \sim \bigcup_{\varepsilon > 0} D(j+1, \varepsilon) \right).$$

For a shorter proof that $\exp_Y X \cap \text{sm}_Y X$ is a G_δ set, let M be as in 1.6 and let $N = \{(x, y) \in M : \text{diam } x^s = 0 = \text{diam } y^e\}$. Then N is a G_δ set and the projection of N onto X is a homeomorphism.

Our final aim in this section is to extend Theorem 1.6 and to apply the result thus obtained. For these purposes, we require some additional terminology. Suppose that X is a set, J_α is a sequence of classes of subsets of X , and Y is a set of real-valued functions on X . A point x_0 of X will be called (Y, J_α) -exposed provided there exists $y_0 \in Y$ such that for each i it is true that $\sup y_0 J_i < y_0(x_0)$ for some $J_i \in J_i$. The

set of all such points will be denoted by $\exp_{(Y, \mathcal{J}_\alpha)} X$. If X is a metric space, ϵ_α is a sequence of positive numbers converging to zero, and \mathcal{J}_i is the set of all complements of open ϵ_i -neighborhoods of points of X , then a point x_0 is (Y, \mathcal{J}_α) -exposed if and only if there exists $y_0 \in Y$ such that the sets $\{x \in X: y_0(x) > y_0(x_0) - \epsilon\}$ ($\epsilon > 0$) form a base of neighborhoods of x_0 in X . Such a point x_0 will be called *strongly Y -exposed* and the set of all such points will be denoted by $\text{sexp}_Y X$. It is evident that $\text{sexp}_Y X \subset \exp_Y X$, with equality when X is compact and the members of Y are all upper semicontinuous. Lindenstrauss [5] has an example in which $\text{sexp}_Y X \neq \exp_Y X$ even though X is a weakly compact convex subset of a Banach space E and Y is the conjugate space of E .

THEOREM 1.9. *Suppose that X is a metric space, \mathcal{J}_α is a sequence of classes of subsets of X , F is a complete metric linear space⁵ of real-valued functions on X , and Y is a closed convex subset of F . Suppose that every member of Y is upper semicontinuous on X , and that convergence in Y implies uniform convergence on every member of $\bigcup_{i=1}^\infty \mathcal{J}_i$ as well as on every compact subset of X . Then*

- (a) *if Y is σ -compact, the set of all (Y, \mathcal{J}_α) -exposed points is an $F_{\sigma\delta}$ set in X ;*
- (b) *if X is an analytic set and Y is separable, the set of all its (Y, \mathcal{J}_α) -exposed is an analytic set.⁶*

Proof. For each i , let A_i denote the union of all sets y^e such that for some set $J \in \mathcal{J}_i$, $\sup yJ_i < \sup yX$. We claim that $\exp_{(Y, \mathcal{J}_\alpha)} X = \bigcap_{i=1}^\infty A_i$, where inclusion is obvious in one direction. For the reverse direction, let us consider an arbitrary point $x_0 \in \bigcap_{i=1}^\infty A_i$. For each i , there exist $y_i \in Y$ and $J_i \in \mathcal{J}_i$ such that

$$\sup y_i J_i < \sup y_i X = y_i(x_0).$$

We assume without loss of generality that the space E is topologized by means of a metric ρ which is not only complete but also translation-invariant [3]. Let the sequence of numbers $\lambda_1, \lambda_2, \dots$ be such that always $0 < \lambda_i < 2^{-i} > \rho(0, \lambda_i y_i)$, whence the two series $\sum \lambda_n$ and $\sum \lambda_n y_n$ converge respectively to a number $\lambda \in [0, 1]$ and a function $z \in F$. With $y_0 = \lambda^{-1}z$, we have $y_0 \in Y$, and since ρ -convergence implies pointwise convergence on X it is evident that for each i ,

$$\sup y_0 J_i < \sup y_0 X = y_0(x_0).$$

Hence $x_0 \in \exp_{(Y, \mathcal{J}_\alpha)} X$.

Now let

⁵ Addition and scalar multiplication in F are assumed to be jointly continuous.

⁶ An *analytic set* is a continuous image of a Borelian subset of the Hilbert cube.

$$M = \bigcup_{x \in X} \{x\} \times x^s \subset X \times Y,$$

and for each i let W_i denote the set of all points $y \in Y$ such that

$$\sup yJ_i < \sup yX$$

for some $J_i \in \mathcal{J}_i$. Since convergence in Y implies pointwise convergence on X as well as uniform convergence on every member of $\bigcup_{i=1}^{\infty} J_i$, each set W_i is open. The set M is closed, for if $(x_i, y_i), (x_2, y_2), \dots$ is a sequence in M with $x_\alpha \rightarrow x_0$ and $y_\alpha \rightarrow y_0$, then

$$\begin{aligned} y_0(x_0) &\stackrel{(1)}{\geq} \limsup_{i \rightarrow \infty} y_0(x_i) = \limsup_{i \rightarrow \infty} y_i(x_i) \\ &\stackrel{(3)}{=} \limsup_{i \rightarrow \infty} (\sup y_i X) \stackrel{(4)}{\geq} \sup y_0 X. \end{aligned}$$

Here (1) is a consequence of the upper semicontinuity of y_0 , (2) of the uniform convergence of y_α to y_0 on the compact set $\{x_0, x_1, x_2, \dots\}$, (3) of the fact that $(x_i, y_i) \in M$, and (4) of the pointwise convergence of y_α to y_0 on X .

Since M is closed and W_i is open, the set $M \cap (X \times W_i)$ is an F_σ subset of $X \times Y$. The projection of $M \cap (X \times W_i)$ on X is exactly A_i . If X is analytic and Y is separable, then $X \times Y$ is analytic, whence each A_i is an analytic set and thus the same is true of the set $\exp_{(Y, \mathcal{J}_\omega)} X$. If Y is σ -compact then each set A_i is an F_σ set (and hence $\exp_{(Y, \mathcal{J}_\omega)} X$ is an F_{σ^8} set in X), for when Z is a compact subset of Y the projection on X of a closed subset of $X \times Z$ must be closed in X . This completes the proof of 1.9.

COROLLARY 1.10. *Suppose that X is a metric space, $C_B(X)$ is the Banach space of all bounded continuous real-valued functions on X (in the topology of uniform convergence), and Y is a closed convex subset of $C_B(X)$. Then the set $\text{sexp}_Y X$ of all strongly Y -exposed points of X is*

- an analytic set if X is analytic and Y is separable;*
- an F_{σ^8} subset of X if Y is σ -compact.*

Proof. Apply 1.9, taking as \mathcal{J}_i the set of all complements of open 2^{-i} -neighborhoods of points of X .

COROLLARY 1.11. *Suppose that X is a locally compact separable metric space, $C(X)$ is the space of all continuous real-valued functions on X (in the topology of uniform convergence on compact sets), and Y is a closed convex subset of $C(X)$. Then the set $\text{exp}_Y X$ of all Y -exposed points of X is*

*an analytic set if Y is separable;
an $F_{\sigma\delta}$ subset of X if Y is σ -compact.*

Proof. Let X be metrized by means of a metric η such that all η -bounded sets have compact closure. Then apply 1.9, taking as J_i the family of all sets of the form $\{x: 2^{-i} \leq \eta(x, x_0) \leq 2^i\}$ for $x_0 \in X$.

COROLLARY 1.12. *Suppose that E is a Banach space whose conjugate space E' is separable. If X is a bounded analytic set in E , then $\text{sexp}_{E'} X$ is an analytic set. If X is weakly compact, then both $\text{sexp}_{E'} X$ and $\text{exp}_{E'} X$ are analytic sets under the weak topology.*

Proof. Apply 1.9 much as it was applied in 1.10 and 1.11, noting that convergence in E' implies uniform convergence on X , and that if X is weakly compact, then (with E' separable) X is metrizable under the weak topology.

When X is a subset of a topological linear space E , a point p of X will be called *topologically exposed* provided there is a linear form E whose restriction to X is continuous and attains its maximum precisely at p . The following is an immediate consequence of 1.10 or 1.11.

COROLLARY 1.13. *If X is a metrizable compact subset of a topological linear space, then the set of all topologically exposed points of X is an analytic set.*

PROBLEM 1.14. *If K is a compact convex subset of a Banach space E , must the set $\text{exp}_{E'} K$ be analytic or even Borelian?*

2. A compact convex set having no algebraically exposed point.
For some results and examples concerning the existence of exposed points of infinite-dimensional compact convex sets, see [4] and its references, and especially [5]. Here we shall construct a compact convex set which has no algebraically exposed point, thus settling problems raised in [4] (p. 97) and [7].

PROPOSITION 2.1. *Suppose that I is an uncountable set of indices. Let $U = [-1, 1]^I$, $V = \{x \in \mathfrak{R}^I: \sum_{i \in I} x_i^2 \leq 1\}$, and $K = U + V$. Then K is a symmetric compact convex subset of the locally convex space \mathfrak{R}^I , but no point of K is algebraically exposed.*

Proof. Let $p = u + v$, with $u \in U$ and $v \in V$, and suppose that p is an algebraically exposed point of K . Then of course, p is an extreme point of K , and it follows that the points u and v are extreme in U and

V respectively. Thus $u \in \{-1, 1\}^I$, and since the sets U and V are both invariant with respect to permutation and change of sign of coordinates, we may assume without loss of generality that all the coordinates of u are equal to 1. Since the index set I is uncountable and $\sum_{i \in I} v_i^2 = 1$, there exists $j \in I$ such that $v_j = 0$. Let the point w of \mathfrak{R}^I be such that $w_j = 1$ but $w_i = 0$ for all $i \neq j$, and let Q denote the plane consisting of all linear combinations of the points v and w . Then the intersection $K \cap (p + Q)$ contains the circular disk $u + (V \cap Q)$ as well as the points p and $p - w$. Since the line determined by $p - w$ and p is tangent to the disk, p cannot be an exposed point of the intersection $K \cap (p + Q)$. This completes the proof of 2.1.

If I has the cardinality of the continuum, the space \mathfrak{R}^I is separable (has a countable dense set) [6], but the set K of 2.1 is not separable. We do not know of a separable compact convex set which has no algebraically exposed points.

A point p of a convex set X will be called an *angular point* of K provided there exists a two-dimensional flat P through p such that the intersection $K \cap P$ is two-dimensional and has more than one line of support through p . As can be seen directly or by using the fact that the space $m(I)$ does not admit a smooth norm consistent with its topology [2], the set K of 2.1 has many angular points. We do not know of a compact convex set which has neither angular points nor algebraically exposed points. A possible approach toward constructing such a set is indicated by the following remark.

PROPOSITION 2.2. Let K be a convex F_σ set in a topological linear space E . Suppose that K has no angular point and that no one-pointed subset of K is a G_δ set in K . Then no point of K is algebraically exposed.

Proof. Supposing the contrary, we may assume without loss of generality that E is the linear hull of K and 0 is an algebraically exposed point of K . Let $M = \bigcup_{\mu > 0} \mu K$, an F_σ set in E , and let \mathcal{P} be the family of all two-dimensional linear subspaces of E whose intersection with K is also two-dimensional. For each $P \in \mathcal{P}$, 0 is an exposed point but not an angular point of the intersection $K \cap P$, and it follows readily that the set $(M \cap P) \sim \{0\}$ is an open halfplane in P . This implies that the set $M \sim \{0\} (= \bigcup_{P \in \mathcal{P}} (M \cap P) \sim \{0\})$ is an algebraically open halfspace in E , whence $M \sim \{0\}$ is the union of countably many translates of the F_σ set M . But $M \sim \{0\}$ is an F_σ set in E , whence $\{0\}$ is a G_δ set in K and the contradiction completes the proof.

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Pacific Journal of Mathematics

Vol. 17, No. 1

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