Pacific Journal of Mathematics

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Vol. 17, No. 1 January 1966

PACIFIC JOURNAL OF MATHEMATICS
Vol. 17, No. 1, 1966

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The two sections of this note are unrelated, except that both are concerned with the exposed points of a compact convex subset K of a locally convex space E. In § 1 it is proved that if K is of finite dimension d, then the set of all its exposed points can be expressed as the union of a G_δ set, an F_σ set, and $d-2$ sets each of which is the intersection of a G_δ set with an F_σ set. A sharper assertion is proved for the three-dimensional case, and some related results are obtained for certain infinite-dimensional situations. Section 2 describes a compact convex set in the space $\mathfrak{R}^c$ which has no algebraically exposed points. Both sections contain unsolved problems.

When E is finite-dimensional, a point p of K is said to be *exposed* provided $\{p\}$ is the intersection of K with some supporting hyperplane of K, or, equivalently, provided there is a linear form on E whose K-maximum is attained precisely at p. The set of all exposed points of K will be denoted by exp K. It is well known that exp K is a dense subset of ext K, the set of all extreme points of K. The set ext K is a G_δ set, and in the two-dimensional case exp K is also a G_δ set. However, there are three-dimensional sets K for which exp K is not a G_δ set [1, 4], and in Corson's example [1] the set exp K is not even the union of a G_δ set and an F_σ set. This suggests the problem of determining the Borel type of exp K, and the answer provided in § 1 seems to be fairly complete.

For infinite-dimensional convex sets, the notion of exposed point may be defined in several different ways, all of which are equivalent in the finite-dimensional case. The weakest notion is that of an *algebraically exposed point*, this being a point of K such that $p \in \exp(K \cap P)$ for every two-dimensional flat P through p. The example in § 2 provides a negative answer for questions raised by Phelps [7] and Klee [4].

1. The Borel type of the set of exposed points.

Let us begin with the main finite-dimensional result.

THEOREM 1.1. *Suppose that K is a closed convex set of finite dimension d, and let* exp K *denote the set of all exposed points of K. Then*

(a) *if $d = 2$,* exp K *is a G_δ set;*

Received November 5, 1964. This paper was written at the University of Washington while the first author was a visiting Walker-Ames Professor and the others were supported by a grant from the National Science Foundation (NSF-GP-378).

(b) *if* $d = 3$, exp K *is the union of a* G_δ *set and a set which is the intersection of a* G_δ *set and an* F_σ *set;*

(c) *for arbitrary* d, exp K *is the union of a* G_δ *set, an* F_σ *set, and* $d - 2$ *sets each of which is the intersection of a* G_δ *set with an* F_σ *set.*

Proof. For (a), it suffices to note that if $d = 2$, then the set ext $K \sim$ exp K is countable. For (b), let us define a *face* of a convex set A as a maximal convex subset of the relative boundary of A. A face of dimension j will be called a j-*face.* Let S denote the union of all 0-faces of K. Let T denote the set of all points x of K such that $\{x\}$ is the intersection of a 1-face or a 2-face of K with a supporting hyperplane of K. Finally, let U denote the set of all endpoints of 1-faces of 2-faces of K. Then S is a G_δ set, T is an F_σ set, and U is countable. Further, exp $K = S \cup (T \sim V)$ for some $V \subset U$, and the desired conclusion follows.

For an alternative proof, let R denote the set of all points of K at which K admits a unique supporting hyperplane, let $S = R \cap \exp K$, let T be the union of all 1-faces of K which are contained in $K \sim R$, and let U be the set of all endpoints of these 1-faces. Then the above statements about S, T and U and still correct.

For (c), we assume that K is a closed convex subset of Euclidean d-space $\mathfrak{E}^d$. Let X denote the boundary of K and let Y denote the unit sphere $\{y \in \mathfrak{E}^d : \| y \| = 1\}$. By means of the usual inner product, Y will be regarded as a set of functions on X. For each point x_0 of X, we define

(1) $x_0^s = \{y_0 \in Y : y_0(x_0) = \sup y_0 X\}$,

and for each point y_0 of Y we define

(2) $y_0^e = \{x_0 \in X : y_0(x_0) = \sup y_0 X\}$.

A point x of K is exposed if and only if $x \in X$ and $\{x\} = y^e$ for some $y \in Y$.

In the sphere Y, we define an *open* j-*ball of radius* ε as a set of the form $N(y, \varepsilon) \cap L \cap Y$, where $y \in Y, 0 < \varepsilon < \sqrt{2}$, $N(y, \varepsilon)$ is the ε-neighborhood of y in $\mathfrak{E}^d$, and L is a $(j + 1)$-dimensional linear subspace of $\mathfrak{E}^d$. Let $D(j, \varepsilon)$ denote the set of all points $x \in X$ such that x^s contains an open j-ball of radius ε. Let $C_1, C_2, \cdots$ be a sequence of compact sets whose union is X, and let $Q(j, k, \varepsilon)$ denote the set of all points $x \in X$ for which x^s contains an open j-ball B of radius ε such that for some $y \in B$, diam $y^e \geqq \varepsilon$ and y^e intersects C_k. It can be verified that each of the sets $D(j, \varepsilon)$ and $Q(j, k, \varepsilon)$ is closed, and that each of the sets $Q(d - 1, k, \varepsilon)$ is empty. The set exp K is the union of the d sets

$$X_0 = D(0, 1) \sim \bigcup_{k,\varepsilon} Q(0, k, \varepsilon) \, ,$$

$$X_j = \bigcup_\varepsilon D(j, \varepsilon) \sim \bigcup_{k,\varepsilon} Q(j, k, \varepsilon) \qquad (1 \leqq j \leqq d - 2) \, ,$$

and

$$X_{d-1} = \bigcup_\varepsilon D(d-1, \varepsilon) .$$

This completes the proof of 1.1.

PROBLEM 1.2. *If K is a three-dimensional compact convex set, must* exp K *be the intersection of a G_δ set and an F_σ set?*[1] *If the answer is affirmative, what about the general finite-dimensional case?*

In the remainder of this section, X will denote a set and Y will denote a set of real-valued functions on X. The sets x_0^s (for $x_0 \in X$) and y_0^e (for $y_0 \in Y$) are defined as in (1) and (2) above. A point x_0 of X will be called *Y-smooth* provided the set x_0^s consists of a single point of Y; that is, provided precisely one member of Y attains its maximum at x_0. And the point x_0 of X will be called *Y-exposed* provided $\{x_0\} = y_0^e$ for some $y_0 \in Y$; that is, provided some member of Y attains its maximum precisely at x_0. The sets of all Y-smooth and Y-exposed points of X will be denoted respectively by $\mathrm{sm}_Y X$ and $\exp_Y X$.

The following remarks are elementary but useful. Their proofs are left to the reader.

LEMMA 1.3. *If Y is convex (with respect to the pointwise addition and scalar multiplication of real-valued functions on X), then the set x^s is convex for each $x \in X$.*

LEMMA 1.4. *Suppose that x_0 is a point of X, the functions $y_1, y_2, \cdots$ are members of x_0^s, the numbers $\lambda_1, \lambda_2, \cdots$ are strictly positive, and the series $\Sigma_i \lambda_i y_i$ is pointwise convergent to a function $y_0 \in Y$. Then $y_0 \in x_0^s$ and $y_0^e = \bigcap_i y_i^e$.*

LEMMA 1.5. *Suppose that X is a metric space, each member of Y is upper semicontinuous on X, and Y has the topology of uniform convergence on X. Then*

(a) *if Y is compact, the set-valued transformation $x^s \mid x \in X$ is upper semicontinuous;*

(b) *if X is compact, the set-valued transformation $y^e \mid y \in Y$ is upper semicontinuous.*

When X and Y are as in 1.1, it follows from 1.1 that the set $\exp_Y X$ is both a $G_{\delta\sigma}$ set and an $F_{\sigma\delta}$ set. This may be extended in one direction as follows.

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[1] If K has no 2-faces, exp K is the union of a G_δ set and an F_σ set, but the example of [1] shows that this is not true in general.

THEOREM 1.6. *Suppose that X is a compact metric space and Y is a convex set of upper semicontinuous real-valued functions on X. Suppose that Y is compact in the topology of uniform convergence. Then the set of all Y-exposed points of X is an $F_{\sigma\delta}$ set.*

Proof. Define

$$M = \bigcup_{x \in X} \{x\} \times x^s \subset X \times Y ,$$

and for each $\varepsilon > 0$ define

$$W_\varepsilon = \{y \in Y\colon \mathrm{diam}\ y^e < \varepsilon\} .$$

It follows from 1.5 (a) that M is closed and from 1.5 (b) that W_ε is open. Hence the set $M \cap (X \times W_\varepsilon)$ is an F_σ subset of the compact set $X \times Y$, and its projection A_ε on X must also be an F_σ. To complete the proof it suffices to show that

$$\exp_Y X = \bigcap_{\varepsilon > 0} A_\varepsilon .$$

Inclusion in one direction is obvious. For the other, we consider an arbitrary point $x_0 \in \bigcap_{\varepsilon>0} A_\varepsilon$ and want to show that $x_0 \in \exp_Y X$. From the definition of A_ε it follows that for each $\varepsilon > 0$ there exists $y(\varepsilon) \in x_0^s$ such that diam $y(\varepsilon)^e < \varepsilon$. Since the set x_0^s is convex by 1.3 and compact by 1.5 (a), it must include the function $y_0 = \Sigma_i 2^{-i} y(2^{-i})$. Then 1.4 implies that diam $y_0^e = 0$, and the desired conclusion follows.

The remaining theorems of the present section are proved by refinements of the reasoning of 1.1 and 1.6. Indeed, 1.1 (c) could be derived as a corollary of 1.7 below[2], and 1.6 as a corollary of 1.9. However, the simpler arguments were given first as an aid to clarity.

When j is a nonnegative integer and Z is a subset of a linear space E, the *j-interior* of Z ($\mathrm{int}_j Z$) is defined as the set of all points $z \in Z$ such that for some j-dimensional flat F through z, z is interior to the set $Z \cap F$ with respect to the natural topology of F.

THEOREM 1.7. *Suppose that X and Y are as in 1.6. For $j = 0, 1, \cdots$, let X_j denote the set of all points $x \in X$ such that* $\dim x^s \geqq j$ *and $y^e = \{x\}$ for all $y \in \mathrm{int}^j x^s$. Then X_0 is a G_δ set and each of the sets $X_1, X_2, \cdots$ is the intersection of a G_δ set and an F_σ set.*

Proof. For the sake of simplicity, we assume at first that every

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[2] Of course, the Euclidean sphere Y of 1.1 is not convex. However, it can be replaced in the proof of 1.1 by the boundary of a cube, which is the union of a finite number of convex sets. Another technical complication in deriving 1.1 directly from 1.7 would result from the fact that the set X in 1.1 need not be compact but only σ-compact.

member of Y is bounded below as well as (automatically) above. Then Y is a compact convex subset of the Banach space E of all bounded real-valued functions on X. For $\varepsilon > 0$ and for $j = 0, 1, 2, \cdots$, let $D(j, \varepsilon)$ denote the set of all points $x \in X$ such that x^s contains an open j-ball of radius ε.[3] Let $Q(j, \varepsilon)$ denote the set of all points $x \in X$ such that x^s contains an open j-ball B of radius ε with diam $y^e \geqq \varepsilon$ for some $y \in B$. It is evident that

$$X_j = \bigcup_{\varepsilon>0} D(j, \varepsilon) \sim \bigcup_{\varepsilon>0} Q(j, \varepsilon) .$$

Further, $D(j, \varepsilon)$ and $Q(j, \varepsilon)$ are both antitone functions of ε, and $D(0, \varepsilon) = D(0, 1)$ for all $\varepsilon > 0$. Thus, it suffices to show that each of the sets $D(j, \varepsilon)$ and $Q(j, \varepsilon)$ is closed. This follows from 1.5, but the case in which $j \geqq 1$ may require some explanation.

Consider a sequence x_α of points of X, converging to a point $x_0 \in X$. Suppose that $\{x_1, x_2, \cdots\} \subset D(j, \varepsilon)$. Then for each i there are a point $z_i \in E$ and a j-dimensional linear subspace L_i of E such that

$$z_i + N(0, \varepsilon) \cap L_i \subset x_i^s .$$

For each i, L_i admits a basis $\{b_i^1, \cdots, b_i^j\}$ consisting of points of norm ε such that for $1 < h \leqq j$, the point b_i^h is at distance ε from the linear hull of the set $\{b_i^1, \cdots, b_i^{h-1}\}$.[4] The set x_0^s is compact and (by 1.5 (a)) every neighborhood of x_0^s contains all but finitely many of the sets x_i^s. Thus, we may assume without loss of generality that each of the sequences z_α and b_α^h is convergent; say $z_\alpha \rightarrow z_0$ and $b_\alpha^h \rightarrow b_0^h$. We claim that the set $\{b_0^1, \cdots, b_0^j\}$ is linearly independent. Indeed, suppose the contrary, whence for some $h > 1$ we have $b_0^h = \Sigma_{1 \leqq r < h} \lambda_r b_0^r$. But then for each i,

$$\begin{aligned} || b_i^h - \Sigma_{1\leqq r<h}\lambda_r b_i^r || &\leqq || b_i^h - b_0^h || + || \Sigma_{1\leqq r<h}\lambda_r(b_0^r - b_i^r) || \\ &\leqq (1 + \Sigma_{1\leqq r<h} | \lambda_r |) \max_{1\leqq r\leqq h} || b_i^h - b_0^h || , \end{aligned}$$

and for all sufficiently large i this is inconsistent with the way in which the basis $\{b_i^1, \cdots, b_i^j\}$ was chosen. Thus the set $\{b_0^1, \cdots, b_0^j\}$ is linearly independent and its linear hull L_0 is a j-dimensional linear subspace of E. It is easily verified that

$$z_0 + N(0, \varepsilon) \cap L_0 \subset x_0^s ,$$

whence $x_0 \in D(j, \varepsilon)$ and it follows that the set $D(j, \varepsilon)$ is closed.

Now suppose in addition that $\{x_1, x_2, \cdots\} \subset Q(j, \varepsilon)$. Then the points z_i and linear subspaces L_i can be chosen so that diam $y^e \geqq \varepsilon$ for some

[3] Here a j-ball of radius ε is a set of the form $N(y, \varepsilon) \cap F$, where F is a j-dimensional flat through y.

[4] This is an easy application of the Hahn-Banach theorem.

$y \in z_i + N(0, \varepsilon) \cap L_i$. But then diam $z_i^e \geqq \varepsilon$, for it follows from 1.4 that y^e is constant on every open segment contained in a set x_i^s and hence on every open j-ball contained in x_i^s. With diam $z_i^e \geqq \varepsilon$ and $z_\alpha \to z_0$, it follows from 1.5 (b) that diam $z_0^e \geqq \varepsilon$ and consequently the set $Q(j, \varepsilon)$ is closed.

Now to complete the proof of 1.7, we abandon the assumption that all of the members of Y are bounded. Let S denote the linear space of all real-valued functions on X, and for $s_1, s_2 \in S$ let

$$\rho(s_1, s_2) = \sup_{x \in X} | s_1(x) - s_2(x) | .$$

The function ρ satisfies all of the requirements for a metric except that it may have the value $+\infty$. The ρ-topology for S is the topology of uniform convergence. Let

$$R = \{(s_1, s_2) : \rho(s_1, s_2) < \infty\} \subset S \times S$$

Then R is an equivalence relation on S, and each equivalence class is both open and closed. Note that if $(s_1, s_2) \notin R$, then no two points of the segment $[s_1, s_2]$ are in the same equivalence class. Choose $y_0 \in Y$ and let $\zeta(s) = s - y_0$ for all $s \in S$. Then ζ is an affine isometry of S onto S, and ζ carries Y into the Banach space E of all bounded real-valued functions on X. From this point on, the proof is merely a paraphrase of the one already given when $Y \subset E$.

1.8. COROLLARY. *Suppose that X and Y are as in* 1.6. *Then the set* $\exp_Y X \cap \mathrm{sm}_Y X$ *is a G_δ set, and the set* $\{x \in \exp_Y X\colon \dim x^s < \infty\}$ *is a $G_{\delta\sigma}$ set.*

Proof. Note that the first set is equal to $X_0 \sim \bigcup_{\varepsilon<0} D(1, \varepsilon)$ (in the notation of 1.7), while the second is equal to

$$\bigcup_{j=0}^{\infty} \left(X_j \sim \bigcup_{\varepsilon>0} D(j+1, \varepsilon)\right) .$$

For a shorter proof that $\exp_Y X \cap \mathrm{sm}_Y X$ is a G_δ set, let M be as in 1.6 and let $N = \{(x, y) \in M\colon \mathrm{diam}\, x^s = 0 = \mathrm{diam}\, y^e\}$. Then N is a G_δ set and the projection of N onto X is a homeomorphism.

Our final aim in this section is to extend Theorem 1.6 and to apply the result thus obtained. For these purposes, we require some additional terminology. Suppose that X is a set, $\boldsymbol{J}_\alpha$ is a sequence of classes of subsets of X, and Y is a set of real-valued functions on X. A point x_0 of X will be called $(Y, \boldsymbol{J}_\alpha)$*-exposed* provided there exists $y_0 \in x_0^s$ such that for each i it is true that $\sup y_0 J_i < y_0(x_0)$ for some $J_i \in \boldsymbol{J}_i$. The set of all such points will be denoted by $\exp_{(Y, \boldsymbol{J}_\alpha)} X$. If X is a metric space, ε_α is a sequence of positive numbers converging to zero, and $\boldsymbol{J}_i$ is the set of all complements of open ε_i-neighborhoods of points of X, then a point x_0 is $(Y, \boldsymbol{J}_\alpha)$-exposed if and only if there exists $y_0 \in Y$ such that the sets $\{x \in X: y_0(x) > y_0(x_0) - \varepsilon\}$ $(\varepsilon > 0)$ form a base of neighborhoods of x_0 in X. Such a point x_0 will be called *strongly Y-exposed* and the set of all such points will be denoted by $\operatorname{sexp}_Y X$. It is evident that $\operatorname{sexp}_Y X \subset \exp_Y X$, with equality when X is compact and the members of Y are all upper semicontinuous. Lindenstrauss [5] has an example in which $\operatorname{sexp}_Y X \neq \exp_Y X$ even though X is a weakly compact convex subset of a Banach space E and Y is the conjugate space of E.

THEOREM 1.9. *Suppose that X is a metric space, $\boldsymbol{J}_\alpha$ is a sequence of classes of subsets of X, F is a complete metric linear space*[5] *of real-valued functions on X, and Y is a closed convex subset of F. Suppose that every member of Y is upper semicontinuous on X, and that convergence in Y implies uniform convergence on every member of $\bigcup_{i=1}^\infty \boldsymbol{J}_i$ as well as on every compact subset of X. Then*

(a) *if Y is σ-compact, the set of all $(Y, \boldsymbol{J}_\alpha)$-exposed points is an $F_{\sigma\delta}$ set in X;*

(b) *if X is an analytic set and Y is separable, the set of all its $(Y, \boldsymbol{J}_\alpha)$-exposed is an analytic set.*[6]

Proof. For each i, let A_i denote the union of all sets y^e such that for some set $J \in \boldsymbol{J}_i$, $\sup yJ_i < \sup yX$. We claim that $\exp_{(Y, \boldsymbol{J}_\alpha)} X = \bigcap_{i=1}^\infty A_i$, where inclusion is obvious in one direction. For the reverse direction, let us consider an arbitrary point $x_0 \in \bigcap_{i=1}^\infty A_i$. For each i, there exist $y_i \in Y$ and $J_i \in Y$ and $J_i \in \boldsymbol{J}_i$ such that

$$\sup y_i J_i < \sup y_i X = y_i(x_0) .$$

We assume without loss of generality that the space E is topologized by means of a metric ρ which is not only complete but also translation-invariant [3]. Let the sequence of numbers $\lambda_1, \lambda_2, \cdots$ be such that always $0 < \lambda_i < 2^{-i} > \rho(0, \lambda_i y_i)$, whence the two series $\Sigma\lambda_n$ and $\Sigma\lambda_n y_n$ converge respectively to a number $\lambda \in [0, 1]$ and a function $z \in F$. With $y_0 = \lambda^{-1}z$, we have $y_0 \in Y$, and since ρ-convergence implies pointwise convergence on X it is evident that for each i,

$$\sup y_0 J_i < \sup y_0 X = y_0(x_0) .$$

Hence $x_0 \in \exp_{(Y, \boldsymbol{J}_\alpha)} X$.

Now let

[5] Addition and scalar multiplication in F are assumed to be jointly continuous.

[6] An *analytic set* is a continuous image of a Borelian subset of the Hilbert cube.

$$M = \bigcup_{x \in X} \{x\} \times x^s \subset X \times Y ,$$

and for each i let W_i denote the set of all points $y \in Y$ such that

$$\sup yJ_i < \sup yX$$

for some $J_i \in \boldsymbol{J}_i$. Since convergence in Y implies pointwise convergence on X as well as uniform convergence on every member of $\bigcup_{i=1}^{\infty} \boldsymbol{J}_i$, each set W_i is open. The set M is closed, for if $(x_1, y_1), (x_2, y_2), \cdots$ is a sequence in M with $x_\alpha \rightarrow x_0$ and $y_\alpha \rightarrow y_0$, then

$$\begin{aligned} y_0(x_0) &\underset{(1)}{\geqq} \limsup_{i \to \infty} y_0(x_i) \underset{(2)}{=} \limsup_{i \to \infty} y_i(x_i) \\ &\underset{(3)}{=} \limsup_{i \to \infty} (\sup y_i X) \underset{(4)}{\geqq} \sup y_0 X . \end{aligned}$$

Here (1) is a consequence of the upper semicontinuity of y_0, (2) of the uniform convergence of y_α to y_0 on the compact set $\{x_0, x_1, x_2, \cdots\}$, (3) of the fact that $(x_i, y_i) \in M$, and (4) of the pointwise convergence of y_α to y_0 on X.

Since M is closed and W_i is open, the set $M \cap (X \times W_i)$ is an F_σ subset of $X \times Y$. The projection of $M \cap (X \times W_i)$ on X is exactly A_i. If X is analytic and Y is separable, then $X \times Y$ is analytic, whence each A_i is an analytic set and thus the same is true of the set $\exp_{(Y, \boldsymbol{J}_\alpha)} X$. If Y is σ-compact then each set A_i is an F_σ set (and hence $\exp_{(Y, \boldsymbol{J}_\alpha)} X$ is an $F_{\sigma\delta}$ set in X), for when Z is a compact subset of Y the projection on X of a closed subset of $X \times Z$ must be closed in X. This completes the proof of 1.9.

COROLLARY 1.10. *Suppose that X is a metric space, $C_B(X)$ is the Banach space of all bounded continuous real-valued functions on X (in the topology of uniform convergence), and Y is a closed convex subset of $C_B(X)$. Then the set* $\operatorname{sexp}_Y X$ *of all strongly Y-exposed points of X is*

an analytic set if X is analytic and Y is separable;
an $F_{\sigma\delta}$ subset of X if Y is σ-compact.

Proof. Apply 1.9, taking as $\boldsymbol{J}_i$ the set of all complements of open 2^{-i}-neighborhoods of points of X.

COROLLARY 1.11. *Suppose that X is a locally compact separable metric space, $C(X)$ is the space of all continuous real-valued functions on X (in the topology of uniform convergence on compact sets), and Y is a closed convex subset of $C(X)$. Then the set* $\exp_Y X$ *of all Y-exposed points of X is*

an analytic set if Y is separable;
an $F_{\sigma\delta}$ subset of X if Y is σ-compact.

Proof. Let X be metrized by means of a metric η such that all η-bounded sets have compact closure. Then apply 1.9, taking as J_i the family of all sets of the form $\{x: 2^{-i} \leqq \eta(x, x_0) \leqq 2^i\}$ for $x_0 \in X$.

COROLLARY 1.12. *Suppose that E is a Banach space whose conjugate space E' is separable. If X is a bounded analytic set in E, then* $\mathrm{sexp}_{E'}X$ *is an analytic set. If X is weakly compact, then both* $\mathrm{sexp}_{E'}X$ *and* $\exp_{E'} X$ *are analytic sets under the weak topology.*

Proof. Apply 1.9 much as it was applied in 1.10 and 1.11, noting that convergence in E' implies uniform convergence on X, and that if X is weakly compact, then (with E' separable) X is metrizable under the weak topology.

When X is a subset of a topological linear space E, a point p of X will be called *topologically exposed* provided there is a linear form E whose restriction to X is continuous and attains its maximum precisely at p. The following is an immediate consequence of 1.10 or 1.11.

COROLLARY 1.13. *If X is a metrizable compact subset of a topological linear space, then the set of all topologically exposed points of X is an analytic set.*

PROBLEM 1.14. *If K is a compact convex subset of a Banach space E, must the set* $\exp_{E'} K$ *be analytic or even Borelian?*

2. A compact convex set having no algebraically exposed point. For some results and examples concerning the existence of exposed points of infinite-dimensional compact convex sets, see [4] and its references, and especially [5]. Here we shall construct a compact convex set which has no algebraically exposed point, thus settling problems raised in [4] (p. 97) and [7].

PROPOSITION 2.1. Suppose that I is an uncountable set of indices. Let $U = [-1, 1]^I$, $V = \{x \in \Re^I: \Sigma_{i \in I} x_i^2 \leqq 1\}$, and $K = U + V$. Then K is a symmetric compact convex subset of the locally convex space $\Re^I$, but no point of K is algebraically exposed.

Proof. Let $p = u + v$, with $u \in U$ and $v \in V$, and suppose that p is an algebraically exposed point of K. Then of course, p is an extreme point of K, and it follows that the points u and v are extreme in U and

V respectively. Thus $u \in \{-1, 1\}^I$, and since the sets U and V are both invariant with respect to permutation and change of sign of coordinates, we may assume without loss of generality that all the coordinates of u are equal to 1. Since the index set I is uncountable and $\Sigma_{i \in I} v_i^2 = 1$, there exists $j \in I$ such that $v_j = 0$. Let the point w of $\mathfrak{R}^I$ be such that $w_j = 1$ but $w_i = 0$ for all $i \neq j$, and let Q denote the plane consisting of all linear combinations of the points v and w. Then the intersection $K \cap (p + Q)$ contains the circular disk $u + (V \cap Q)$ as well as the points p and $p - w$. Since the line determined by $p - w$ and p is tangent to the disk, p cannot be an exposed point of the intersection $K \cap (p + Q)$. This completes the proof of 2.1.

If I has the cardinality of the continuum, the space $\mathfrak{R}^I$ is separable (has a countable dense set) [6], but the set K of 2.1 is not separable. We do not know of a separable compact convex set which has no algebraically exposed points.

A point p of a convex set X will be called an *angular point* of K provided there exists a two-dimensional flat P through p such that the intersection $K \cap P$ is two-dimensional and has more than one line of support through p. As can be seen directly or by using the fact that the space $m(I)$ does not admit a smooth norm consistent with its topology [2], the set K of 2.1 has many angular points. We do not know of a compact convex set which has neither angular points nor algebraically exposed points. A possible approach toward constructing such a set is indicated by the following remark.

PROPOSITION 2.2. Let K be a convex F_σ set in a topological linear space E. Suppose that K has no angular point and that no onepointed subset of K is a G_δ set in K. Then no point of K is algebraically exposed.

Proof. Supposing the contrary, we may assume without loss of generality that E is the linear hull of K and 0 is an algebraically exposed point of K. Let $M = \bigcup_{\mu > 0} \mu K$, an F_σ set in E, and let $\boldsymbol{P}$ be the family of all two-dimensional linear subspaces of E whose intersection with K is also two-dimensional. For each $P \in \boldsymbol{P}$, 0 is an exposed point but not an angular point of the intersection $K \cap P$, and it follows readily that the set $(M \cap P) \sim \{0\}$ is an open halfplane in P. This implies that the set $M \sim \{0\}$ $(= \bigcup_{P \in \boldsymbol{P}} (M \cap P) \sim \{0\})$ is an algebraically open halfspace in E, whence $M \sim \{0\}$ is the union of countably many translates of the F_σ set M. But $M \sim \{0\}$ is an F_σ set in E, whence $\{0\}$ is a G_δ set in K and the contradiction completes the proof.

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INSTITUT HENRI POINCARÉ, PARIS
UNIVERSITY OF WASHINGTON, SEATTLE

Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics
Vol. 17, No. 1 January, 1966