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**INTEGRAL SOLUTIONS TO THE INCIDENCE EQUATION FOR
FINITE PROJECTIVE PLANE CASES OF ORDERS $n \equiv 2$
(mod 4)**

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INTEGRAL SOLUTIONS TO THE INCIDENCE EQUATION FOR FINITE PROJECTIVE PLANE CASES OF ORDERS $n \equiv 2 \pmod{4}$

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A finite projective plane of order $n \geq 2$ can be considered as a $\langle v, k, \lambda \rangle$ design where $v = n^2 + n + 1$, $k = n + 1$, and $\lambda = 1$. As such, it can be characterized by its point-line $0, 1$ incidence matrix A of order v satisfying the incidence equation

$$(*) \quad AA^T = nI + J,$$

where J is the matrix of order v consisting entirely of 1's. Thus, if a plane of order n exists then $(*)$ has an integral solution A . Ryser has shown that if A is a normal integral solution to $(*)$ or if A is merely an integral solution to $(*)$ where n is odd, then A can be made into an incidence matrix for a plane of order n by suitably multiplying its columns by -1 . Such an integral solution to $(*)$ we shall call a type I solution. When A is merely an integral solution to $(*)$ where n is even, then A may be a type I solution but may also be not of this type. These latter integral solutions to $(*)$ we shall call type II solutions. Ryser has constructed type II solutions for $n = 2$ and for all $n \equiv 0 \pmod{4}$ for which there exists a Hadamard matrix of order n , and Hall and Ryser have constructed a type II solution for $n = 10$. In this paper we construct type II solutions for some infinite classes of values of $n \equiv 2 \pmod{4}$. Basic to these constructions is a special class of $\langle v, k, \lambda \rangle$ designs called skew-Hadamard designs whose incidence matrices form a part of the substructure of our type II solutions. We exhibit examples for $n = 26$ and 50 and also derive examples for $n = 10$ and 18 .

A $\langle v, k, \lambda \rangle$ design is an arrangement of v elements x_1, x_2, \dots, x_v into v sets S_1, S_2, \dots, S_v such that every set contains exactly k elements, every pair of sets has exactly λ elements in common, and to avoid certain degenerate situations, $0 \leq \lambda < k \leq v - 1$. A $\langle v, k, \lambda \rangle$ design can be characterized by its incidence matrix $A = [a_{ij}]$ by writing the elements x_1, x_2, \dots, x_v in a row and the sets S_1, S_2, \dots, S_v in a column and setting $a_{ij} = 1$ if $x_j \in S_i$ and $a_{ij} = 0$ if $x_j \notin S_i$. This matrix A , of order v , consists entirely of 0's and 1's and, by the conditions given above, is easily seen to satisfy the incidence equation:

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$$(1.1) \quad AA^T = (k - \lambda)I + \lambda J \equiv B,$$

where A^T is the transpose of A , I is the identity matrix of order v , and J is the matrix of order v consisting entirely of 1's. Conversely, if $0 \leq \lambda < k \leq v - 1$, a matrix A of order v consisting entirely of 0's and 1's and satisfying equation (1.1) is an incidence matrix for some $\langle v, k, \lambda \rangle$ design. Ryser [13] showed for a $\langle v, k, \lambda \rangle$ design with incidence matrix A that $\lambda(v - 1) = k(k - 1)$ and that A is *normal*, i.e., $A^T A = AA^T = B$, which means that every element is contained in exactly k of the sets and every pair of elements are together in exactly λ of the sets. When $\lambda = 0$ or $k = v - 1$ we have the $\langle v, 1, 0 \rangle$ or $\langle v, v - 1, v - 2 \rangle$ designs, respectively. These designs exist for every integer $v \geq 2$ and are quite trivial. Two classes of $\langle v, k, \lambda \rangle$ designs will be of particular interest to us here. These are the *finite projective planes* of orders $n \geq 2$ where $v = n^2 + n + 1$, $k = n + 1$, $\lambda = 1$, and the *Hadamard designs* where $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$, $m \geq 1$ on integer.

We now let A be an integral solution to the incidence equation. Although an integral solution to the incidence equation is more general than a 0, 1 solution, Ryser [14] has shown that if A is normal or if $\gcd(k, \lambda)$ is squarefree and $k - \lambda$ is odd, then by suitable multiplication of the columns of A by -1 we can obtain a 0, 1 incidence matrix for a $\langle v, k, \lambda \rangle$ design. Hence, for odd n the existence of a finite projective plane of order n is equivalent to the existence of an integral solution to the corresponding incidence equation. For even n , however, we do not have this equivalence. When n is even, more exotic integral solutions may and do occur. We may, of course, have design type integral solutions like those for odd n , which we shall call type *I* solutions, or we may have integral solutions which are not of that type, which we shall call type *II* solutions. Ryser [14] showed that a type *II* solution exists for $n = 2$ and for $n \equiv 0 \pmod{4}$ whenever n is the order of a Hadamard matrix, and Hall and Ryser [11] exhibit a type *II* solution for $n = 10$. Here we shall construct type *II* solutions for some infinite classes of values of $n \equiv 2 \pmod{4}$ which satisfy the Bruck-Ryser criterion [4]. This criterion is equivalent to saying that $n = a^2 + b^2$ where a and b are odd integers. It rules out the existence of integral solutions for all orders $n \equiv 6 \pmod{8}$ along with some orders $n \equiv 2 \pmod{8}$. Basic to these constructions is a special class of Hadamard designs called skew-Hadamard designs, whose incidence matrices form part of the substructure of our integral solutions.

2. Skew-Hadamard matrices and designs. Let $H = [h_{ij}]$ be a matrix of order n where $h_{ij} = 1, -1$; $j = 1, \dots, n$. We call H a

Hadamard matrix if $HH^T = nI$. By an inequality of Hadamard [10], H is a Hadamard matrix if and only if $|\det(H)| = n^{n/2}$. We immediately see that a Hadamard matrix is normal. It is easy to show that a Hadamard matrix can only exist when $n = 1, 2$ or $n = 4m, m \geq 1$ an integer, and that a direct product of two Hadamard matrices is a Hadamard matrix, which means that from Hadamard matrices of orders m and n we can construct one of order mn . In [19] J. A. Todd showed that from a Hadamard matrix of order $4m$ we can obtain a related Hadamard design incidence matrix of order $4m - 1$, and conversely, $m \geq 1$ an integer. Hadamard matrices and their related Hadamard designs have been studied extensively [1], [2], [3], [5], [7], [8], [9], [10], [12], [16], [17], [18], [19], [20], [21]. Hadamard matrices exist for infinitely many orders $4m, m \geq 1$ an integer, and are conjectured to exist for all such orders. We call a Hadamard matrix H *skew-Hadamard* if $H + H^T = 2I$. These also exist for infinitely many orders, as will be shown later. We also call a Hadamard design and its corresponding incidence matrix A *skew-Hadamard* if $A + A^T = J - I$. This agreement in terminology will be justified by the next theorem. Skew-Hadamard design incidence matrices are a special type of round robin tournament matrix [15]. As such, they occur in the statistical method of paired comparisons [6]. Corresponding to Todd's result for Hadamard matrices and designs, we have the following result for skew-Hadamard matrices and designs.

THEOREM 2.1. *From a skew-Hadamard matrix of order $4m$ we can obtain a skew-Hadamard design incidence matrix of order $4m - 1$, and conversely, $m \geq 1$ an integer.*

Proof. By multiplying the appropriate rows and the corresponding columns of a skew-Hadamard matrix by -1 , we can bring this matrix to the form

$$H = \left(\begin{array}{c|c} 1 & 1 \dots 1 \\ \hline -1 & \\ \vdots & H_1 \\ -1 & \end{array} \right).$$

Without loss of generality, assume that our original skew-Hadamard matrix is H . Here H_1 consists of 1's and -1 's and satisfies

$$H_1 H_1^T = 4mI - J$$

and

$$H_1 + H_1^T = 2I.$$

Now let $A = (J - H_1)/2$. Then A consists of 0's and 1's and satisfies

$$\begin{aligned} AA^t &= \frac{1}{4}(J^2 - JH_1^t - H_1J + H_1H_1^t) \\ &= \frac{1}{4}((4m - 1)J - J - J + 4mI - J) \\ &= mI + (m - 1)J \end{aligned}$$

and

$$\begin{aligned} A + A^t &= J - \frac{1}{2}(H_1 + H_1^t) \\ &= J - I. \end{aligned}$$

Hence A is a skew-Hadamard design incidence matrix of order $4m - 1$. By reversing the above argument, we have the converse.

We note that the matrices [1] of order 1 and

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

of order 2 are skew-Hadamard. Among the matrices of order $4m$ with entries 1 and -1 , $m \geq 1$ an integer, we can characterize those that are skew-Hadamard by the following theorem.

THEOREM 2.2. *Let $H = [h_{ij}]$, $h_{ij} = 1, -1$ be a matrix of order $n = 4m$, $m \geq 1$ an integer, and let $G = H + H^t - 2I$. Then the following statements are equivalent:*

- (a) H is a skew-Hadamard matrix.
- (b) $H^2 - 2H + nI = 0$.
- (c) The eigenvalues of H are $1 + i\sqrt{n-1}$ and $1 - i\sqrt{n-1}$, each with multiplicity $2m$.
- (d) H is a Hadamard matrix and $\text{tr}(G^2) = 0$.

Proof. We shall show that (a) implies (b) implies (c) implies (d) implies (a). Let H be a skew-Hadamard matrix. Then $HH^t = nI$ and $H + H^t = 2I$ imply (b). Now suppose that (b) holds. Since H cannot satisfy a first degree polynomial, $\lambda^2 - 2\lambda + n$ must be its minimal polynomial, whence only $1 + i\sqrt{n-1}$ and $1 - i\sqrt{n-1}$ are its eigenvalues. Now the trace of H is real; hence these two complex eigenvalues must occur with the same multiplicity, namely, $2m$. Now assume that (c) holds. Then

$$\det(H) = (1 + i\sqrt{n-1})^{2m} (1 - i\sqrt{n-1})^{2m} = n^{n/2}$$

whence H is a Hadamard matrix. Since the eigenvalues of H^2 are $2 - n + 2i\sqrt{n-1}$ and $2 - n - 2i\sqrt{n-1}$, each with multiplicity $2m$,

we have, moreover, that

$$\begin{aligned} \text{tr}(G^2) &= \text{tr}[H^2 + (H^T)^2 + 4I + HH^T + H^T H - 4H - 4H^T] \\ &= 2\text{tr}(H^2) + 4\text{tr}(I) + 2\text{tr}(nI) - 8\text{tr}(H) \\ &= 2[2m(4 - 2n)] + 4n + 2n^2 - 8[2m \cdot 2] \\ &= 16m - 8mn + 4n + 2n^2 - 32m \\ &= 0, \end{aligned}$$

hence (d) is satisfied. Now suppose (d) holds. Since G is symmetric, $\text{tr}(G^2) = 0$ implies that the sum of the squares of the elements of G is 0. Hence $G = 0$ and H is a skew-Hadamard matrix.

We now inquire as to whether there is a direct product type of construction for skew-Hadamard matrices as there is for Hadamard matrices. Such a result can be obtained as a corollary to the following lemma of Williamson [20] in which I_r denotes the identity matrix of order r and $\dot{\times}$ denotes the direct product.

LEMMA 2.3. *Let C be a matrix of order n such that $C^T = \varepsilon C$, $\varepsilon = 1, -1$, and $CC^T = (n - 1)I_n$, and let D and E be two matrices of order m satisfying $DD^T = EE^T = mI_m$ and $DE^T = -\varepsilon ED^T$. Then the matrix $K = D\dot{\times}I_n + E\dot{\times}C$ satisfies $KK^T = mnI_{mn}$.*

The result of interest to us here for skew-Hadamard matrices is the following corollary.

COROLLARY 2.4. *Let $C + I$ be a skew-Hadamard matrix of order n , and let D be a skew-Hadamard and E a symmetric Hadamard matrix of order m such that $DE^T = ED^T$. Then the matrix $K = D\dot{\times}I_n + E\dot{\times}C$ is a skew-Hadamard matrix of order mn .*

Proof. Clearly K consists entirely of 1's and -1 's. Since $C + I$ is a skew-Hadamard matrix, $C^T = -C$ and $CC^T = (n - 1)I_n$, and since D and E are both Hadamard matrices, $DD^T = EE^T = mI_m$. Now $\varepsilon = -1$ and we have $DE^T = ED^T$. Thus, by Lemma 2.3, we have $KK^T = mnI_{mn}$. Now since D is skew-Hadamard and E is symmetric,

$$\begin{aligned} K + K^T &= D\dot{\times}I_n + E\dot{\times}C + (D\dot{\times}I_n + E\dot{\times}C)^T \\ &= D\dot{\times}I_n + E\dot{\times}C + D^T\dot{\times}I_n + E^T\dot{\times}C^T \\ &= (D + D^T)\dot{\times}I_n + E\dot{\times}C - E\dot{\times}C \\ &= 2I_m\dot{\times}I_n \\ &= 2I_{mn}. \end{aligned}$$

Hence K is a skew-Hadamard matrix of order mn .

Williamson [20] obtained special cases of this corollary for $m = 2$ and $m = p^\alpha + 1 \equiv 0 \pmod{4}$, p a prime, $\alpha \geq 1$ an integer, by obtaining the desired pair of matrices of order m . In a different vein, Goldberg [8] constructed a skew-Hadamard design incidence matrix of order $(m - 1)^3$ from one of order $m - 1$, in effect obtaining a skew-Hadamard matrix of order $(m - 1)^3 + 1$ from one of order m . We summarize these results in the following theorem.

THEOREM 2.5. *If there exists a skew-Hadamard matrix of order n then there exists one of order*

- (i) $2n$.
- (ii) $n(p^\alpha + 1)$; $p^\alpha + 1 \equiv 0 \pmod{4}$, p a prime, $\alpha \geq 1$ an integer.
- (iii) $(n - 1)^3 + 1$.

TABLE 1.
The Existence of Skew-Hadamard Matrices for Orders $4 \leq n \leq 200$

n	Form	Exists	n	Form	Exists
4	2^2	SH	104	$103 + 1$	SH
8	2^3	SH	108	$107 + 1$	SH
12	$11 + 1$	SH	112	$2^2(3^3 + 1)$	SH
16	2^4	SH	116		
20	$19 + 1$	SH	120	$2(59 + 1)$	SH
24	$2(11 + 1)$	SH	124		h
28	$3^3 + 1$	SH	128	2^7	SH
32	2^5	SH	132	$131 + 1$	SH
36		h	136	$2(67 + 1)$	SH
40	$2(19 + 1)$	SH	140	$139 + 1$	SH
44	$43 + 1$	SH	144	$2(71 + 1)$	SH
48	$2^2(11 + 1)$	SH	148		h
52		h	152	$151 + 1$	SH
56	$2(3^3 + 1)$	SH	156		h
60	$59 + 1$	SH	160	$2^3(19 + 1)$	SH
64	2^6	SH	164	$163 + 1$	SH
68	$67 + 1$	SH	168	$2(83 + 1)$	SH
72	$71 + 1$	SH	172		h
76		h	176	$2^2(43 + 1)$	SH
80	$2^2(19 + 1)$	SH	180	$179 + 1$	SH
84	$83 + 1$	SH	184		h
88	$2(43 + 1)$	SH	188		
92		h	192	$2^4(11 + 1)$	SH
96	$2^3(11 + 1)$	SH	196		h
100		h	200	$199 + 1$	SH

Since there exist skew-Hadamard matrices of orders 2 and $p^\alpha + 1 \equiv 0 \pmod{4}$, p a prime, $\alpha \geq 1$ an integer [12] [20], we can apply Theorem 2.5 to obtain the following existence theorem.

THEOREM 2.6. *There exists a skew-Hadamard matrix of order n where n is of the form*

- (i) $2^c \prod_{i=1}^r (p_i^{\alpha_i} + 1)$; $c \geq 0$, $r \geq 0$ are integers,
 $p_i^{\alpha_i} + 1 \equiv 0 \pmod{4}$, p_i a prime, $\alpha_i \geq 1$ an integer,
 $i = 1, \dots, r$, where $\prod_{i=1}^r (p_i^{\alpha_i} + 1) = 1$ for $r = 0$.
- (ii) N , where N is derivable from (i) by Theorem 2.5.

Table 1 gives the existence of skew-Hadamard matrices for orders $4 \leq n \leq 200$ according to Theorem 2.6. For comparison, this table also gives the currently known existence of Hadamard matrices for the same range of n , based on constructions in the references mentioned earlier. The symbols SH indicate that a skew-Hadamard matrix exists, while the symbol h indicates that only non-skew-Hadamard matrices are known to exist.

3. Constructions. By § 4 of [11] we know that we can put any type II solution $A = [a_{ij}]$ of order $v = n^2 + n + 1$ for the finite projective plane case of order n into a form where $a_{11} = 0$, $a_{i1} = 1$ for $2 \leq i \leq v$, $a_{1j} = 1$ for $j \equiv 2 \pmod{n}$ and $a_{ij} = 0$ for $j \not\equiv 2 \pmod{n}$ where $2 \leq j \leq v$, and where the remaining entries form a submatrix C of order $v - 1 = n(n + 1)$ which has n 1's and n^2 0's in each of the $n + 1$ columns under a 1 in row 1 of A and which satisfies the matrix equation $CC^T = C^T C = nI$. The constructions given in [11] and [14] have C in the form $C = A_n \dot{+} A_n \dot{+} \dots \dot{+} A_n$, where this direct sum contains A_n , of order n , $n + 1$ times and where A_n has all entries in column 1 equal to 1 and satisfies the matrix equation $A_n A_n^T = nI$. These conditions on A_n are sufficient for the construction of a type II solution for order n . We shall confine ourselves here to this form of type II solution. This restriction reduces the construction of a type II solution A of order $n^2 + n + 1$ to that of an integral matrix A_n of order n satisfying the above conditions. Type II solutions need not, however, be of this direct sum form to within permutations of rows and columns of A . This can be seen from the following example for $n = 4$. Here the entries in the blank parts of A are 0's.

$$A = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc}
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & -1 & \\
 1 & 1 & 1 & -1 & 1 & \\
 1 & 1 & -1 & & & & & & 1 & -1 & \\
 1 & 1 & -1 & & & & & & -1 & 1 & \\
 \hline
 1 & & & & 1 & 1 & 1 & 1 & \\
 1 & & & & -1 & -1 & 1 & 1 & \\
 1 & & & & & & 1 & -1 & & & & & & 1 & -1 & & & & & & & & & & & & & & & & & & \\
 1 & & & & & & 1 & -1 & & & & & & -1 & 1 & & & & & & & & & & & & & & & & & & \\
 \hline
 1 & & & & & & & & 1 & 1 & 1 & 1 & \\
 1 & & & & & & & & -1 & -1 & 1 & 1 & \\
 1 & & & & & & & & & & 1 & -1 & & & & & & & & & & & 1 & -1 & & & & & & & & & & \\
 1 & & & & & & & & & & 1 & -1 & & & & & & & & & & & -1 & 1 & & & & & & & & & & \\
 \hline
 1 & & & & & & & & & & & & 1 & 1 & 1 & 1 & & & & & & & & & & & & & & & & & & \\
 1 & & & & & & & & & & & & -1 & -1 & 1 & 1 & & & & & & & & & & & & & & & & & & \\
 1 & & & & & & & & & & & & & & 1 & -1 & & & & & & & & & 1 & -1 & & & & & & & & \\
 1 & & & & & & & & & & & & & & 1 & -1 & & & & & & & & & -1 & 1 & & & & & & & & \\
 \hline
 1 & & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & & & & & & & & & & & & & & & \\
 1 & & & & & & & & & & & & & & & & -1 & -1 & 1 & 1 & & & & & & & & & & & & & & & \\
 1 & 1 & -1 & 1 & 1 & & & & & & & \\
 1 & 1 & -1 & -1 & -1 & & & & & & & \\
 \hline
 \end{array} \right)$$

Let K be a skew-Hadamard design incidence matrix of order $q \equiv 3 \pmod{4}$. Here $v = q = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$, where $m \geq 1$ is an integer,

$$(3.1) \quad KK^t = K^tK = mI + (m - 1)J,$$

and

$$(3.2) \quad K + K^t = J - I.$$

We obtain from K a matrix $K(t, u, x)$ by substituting t for each of the main diagonal 0's, u for each of the remaining 0's and x for each of the 1's. From (3.1) and (3.2), any two rows of $K(t, u, x)$ can be schematically represented as

$$\begin{array}{cccccccccccccccccccc}
 t, & u, & u, & \dots, & u, & u, & \dots, & u, & x, & \dots, & x, & x, & \dots, & x \\
 x, & t, & x, & \dots, & x, & u, & \dots, & u, & x, & \dots, & x, & u, & \dots, & u \\
 & & \underbrace{\hspace{2cm}}_{m-1} & & \underbrace{\hspace{2cm}}_{m-1} & & \underbrace{\hspace{2cm}}_{m-1} & & \underbrace{\hspace{2cm}}_m
 \end{array}$$

where there are $4m - 1$ entries in each row, $2m - 1$ each of u 's and x 's. The inner product of a row of $K(t, u, x)$ with itself is thus

$$(3.3) \quad t^2 + (2m - 1)(x^2 + u^2) = t^2 + \frac{1}{2}(q - 1)(x^2 + u^2) .$$

Also, the inner product of two distinct rows of $K(t, u, x)$ is

$$(3.4) \quad \begin{aligned} t(x + u) + (m - 1)(x^2 + u^2) + (2m - 1)xu \\ = t(x + u) + \frac{1}{4}(q - 1)(x + u)^2 - \frac{1}{2}(x^2 + u^2) . \end{aligned}$$

We now form $Y = [y_{ij}] = K(t_1, u_1, x_1)$ and $Z = [z_{ij}] = K(t_2, u_2, x_2)$ of order q and then form

$$(3.5) \quad N = \begin{bmatrix} Y & Z \\ -Z^x & Y^x \end{bmatrix} .$$

We then set

$$(3.6) \quad w \equiv t_1^2 + t_2^2 + \frac{1}{2}(q - 1)(x_1^2 + u_1^2 + x_2^2 + u_2^2) .$$

LEMMA 3.1. *The matrix equation*

$$(3.7) \quad NN^x = wI$$

is satisfied if and only if

$$(3.8) \quad w = \left[t_1 + \frac{1}{2}(q - 1)(x_1 + u_1) \right]^2 + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) \right]^2 .$$

Proof. By (3.5) we have

$$(3.9) \quad NN^x = \begin{bmatrix} YY^x + ZZ^x, & ZY - YZ \\ (ZY - YZ)^x, & Y^xY + Z^xZ \end{bmatrix} .$$

Since, by (3.1), K is a normal matrix, the statements about inner product values of $K(t, u, x)$ are true when the word row(s) is replaced by column(s); hence $K(t, u, x)$ is normal whence Y and Z are normal or

$$(3.10) \quad Y^xY = YY^x \quad \text{and} \quad Z^xZ = ZZ^x .$$

Now

$$\begin{aligned} Y &= t_1I + x_1K + u_1(J - K) - u_1I \\ &= (t_1 - u_1)I + (x_1 - u_1)K + u_1J , \end{aligned}$$

and similarly

$$Z = (t_2 - u_2)I + (x_2 - u_2)K + u_2J .$$

Since I commutes with both K and J and

$$KJ = JK = (2m - 1)J ,$$

i.e., K commutes with J , Y commutes with Z so that

$$(3.11) \quad ZY - YZ = 0 .$$

Then by (3.10) and (3.11), (3.9) becomes

$$(3.12) \quad NN^T = (YY^T + ZZ^T) + (YY^T + ZZ^T) .$$

The diagonal entries of NN^T are, by (3.3) and (3.12),

$$(3.13) \quad t_1^2 + t_2^2 + \frac{1}{2}(q - 1)(x_1^2 + u_1^2 + x_2^2 + u_2^2) = w ,$$

and the nondiagonal entries of the direct summands in (3.12) are, by (3.4),

$$(3.14) \quad t_1(x_1 + u_1) + t_2(x_2 + u_2) + \frac{1}{4}(q - 1)[(x_1 + u_1)^2 + (x_2 + u_2)^2] \\ - \frac{1}{2}(x_1^2 + u_1^2 + x_2^2 + u_2^2) = y .$$

We note that (3.7) is satisfied if and only if $y = 0$. Now solving (3.14) for $(x_1^2 + u_1^2 + x_2^2 + u_2^2)/2$ and substituting the result into (3.13) we obtain

$$(3.15) \quad \left[t_1 + \frac{1}{2}(q - 1)(x_1 + u_1) \right]^2 \\ + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) \right]^2 - (q - 1)y = w .$$

Hence by (3.13), (3.14), and (3.15), we see that (3.7) is true if and only if (3.8) is.

We now define the matrices $E_r = (r + 2)I/2 - J$ of even order r , F_r of size $r \times 2$ consisting entirely of 1's, and G_r of size $r \times 2$ whose first column consists entirely of 1's and whose second column consists entirely of -1 's. In the constructions which follow we shall be taking $t_1 = (r + 2)/2$ and $x_1 + u_1 = 2$. We then note that

$$(3.16) \quad F_r F_r^T + E_r E_r^T = G_r G_r^T + E_r E_r^T = \left[\frac{1}{2}(r + 2) \right]^2 I = t_1^2 I ,$$

$$(3.17) \quad F_r F_r^T + 2E_r = G_r G_r^T + 2E_r = (r + 2)I = (x_1 + u_1)t_1 I ,$$

and

$$(3.18) \quad F_r G_r^T = G_r F_r^T = 0 .$$

We substitute for the entries y_{ii} in Y and Y^T the matrix E_r and for all other entries y_{ij} , $i \neq j$, the matrix $y_{ij}I$ of order r to obtain the matrices Y_* and Y_*^T , respectively, of order rq , and substitute for the entries z_{ij} in Z and Z^T the matrix $z_{ij}I$ of order r to obtain the matrices Z_* and Z_*^T , respectively, also of order rq . These matrices will appear in the constructions which follow, bordered by the matrices F_{rq} and G_{rq} .

We can now obtain two existence theorems for type II solutions to the incidence equation for finite projective plane cases of orders $n \equiv 2 \pmod{4}$. After each one are theorems which cover the various cases of the theorem.

THEOREM 3.2. *Let (3.8) be satisfied in integers t_1, t_2, u_1, u_2, x_1 , and x_2 where $q \equiv 3 \pmod{4}$ is the order of a skew-Hadamard design incidence matrix and w is defined in (3.6), and where $x_1 + u_1 = 2$ and $t_1 = (r + 2)/2$ and $w = 2rq + 2$ for the positive even integer r . Then we can construct a type II solution to the incidence equation for the finite projective plane case of order $n = 2rq + 2$.*

Proof. We have

$$N = \begin{bmatrix} Y & Z \\ -Z^T & Y^T \end{bmatrix}; \quad Y = [y_{ij}], \quad Z = [z_{ij}],$$

where

$$(3.19) \quad y_{ii} = t_1 = \frac{1}{2}(r + 2),$$

$$y_{ij} + y_{ji} = x_1 + u_1 = 2; \quad 1 \leq i \leq q, \quad 1 \leq j \leq q, \quad i \neq j,$$

and

$$(3.20) \quad NN^T = (2rq + 2)I .$$

Since (3.8) is satisfied we have

$$(3.21) \quad \left[\frac{1}{2}(r + 2) + (q - 1) \right]^2 + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) \right]^2 = 2rq + 2,$$

or

$$\left[q - \frac{1}{2}r \right]^2 + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) \right]^2 = 2 .$$

Since q , $r/2$, t_2 , $(q - 1)/2$, x_2 , and u_2 are integers this means that

$$(3.22) \quad q - \frac{1}{2}r = \varepsilon_1, \quad t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) = \varepsilon_2; \quad \varepsilon_1, \varepsilon_2 = 1, -1.$$

We form two matrices U and V of size $2 \times rq$ according to the values of ε_1 and ε_2 as follows:

$$(3.23) \quad \begin{array}{cc} & \begin{array}{c} U \\ \left[\begin{array}{ccc} -1 & \dots & -1 \\ 1 & \dots & 1 \end{array} \right] \end{array} & \begin{array}{c} V \\ \left[\begin{array}{ccc} -1 & \dots & -1 \\ -1 & \dots & -1 \end{array} \right] \end{array} & \text{if } \varepsilon_1 = \varepsilon_2 = 1. \\ & \begin{array}{c} \left[\begin{array}{ccc} 1 & \dots & 1 \\ -1 & \dots & -1 \end{array} \right] \end{array} & \begin{array}{c} \left[\begin{array}{ccc} 1 & \dots & 1 \\ 1 & \dots & 1 \end{array} \right] \end{array} & \text{if } \varepsilon_1 = \varepsilon_2 = -1. \\ & \begin{array}{c} \left[\begin{array}{ccc} -1 & \dots & -1 \\ -1 & \dots & -1 \end{array} \right] \end{array} & \begin{array}{c} \left[\begin{array}{ccc} 1 & \dots & 1 \\ -1 & \dots & -1 \end{array} \right] \end{array} & \text{if } \varepsilon_1 = -\varepsilon_2 = 1. \\ & \begin{array}{c} \left[\begin{array}{ccc} 1 & \dots & 1 \\ 1 & \dots & 1 \end{array} \right] \end{array} & \begin{array}{c} \left[\begin{array}{ccc} -1 & \dots & -1 \\ 1 & \dots & 1 \end{array} \right] \end{array} & \text{if } \varepsilon_1 = -\varepsilon_2 = -1. \end{array}$$

Finally, we construct A_n of order $n = 2rq + 2$:

$$(3.24) \quad A_n = \left(\begin{array}{cc|cc} 1 & 1 & U & V \\ 1 & -1 & & \\ \hline F_{rq} & & Y_* & Z_* \\ G_{rq} & & -Z_*^T & Y_*^T \end{array} \right).$$

By (3.23) the first two rows of A_n are orthogonal and have self inner products equal to $2rq + 2 = n$. Since the row and column sums of Y_* are $q - r/2$ and those of Z_* are $t_2 + (q - 1)(x_2 + u_2)/2$, we have by (3.22) and (3.23) that rows one and two are orthogonal to all the other rows of A_n . We now look upon the submatrix of A_n below row 2 and to the right of F_{rq} and G_{rq} as a matrix with the matrix entries $E_r, u_1I, x_1I, t_2I, u_2I$, and x_2I , all of order r . These matrices naturally divide the entire submatrix of A_n below 2 into r -row blocks. Since these matrices commute with one another they behave multiplicatively among themselves as scalars. Thus (3.16), (3.19) and (3.20) imply that the inner product of an r -row block with itself is $(2rq + 2)I = nI$ of order r , (3.17), (3.19) and (3.20) imply that any two r -row blocks intersecting either F_{rq} or G_{rq} are orthogonal, and (3.18) and (3.20) imply that any r -row block intersecting F_{rq} is orthogonal to any r -row block intersecting G_{rq} . Hence $A_n A_n^T = nI$, and since the first column of A_n consists entirely of 1's we see that we have a type II solution to the incidence equation for the finite projective plane case of order $n = 2rq + 2$.

Letting $c = x_2 + u_2$ and combining (3.22) with (3.6), noting that

$t_1 = (r + 2)/2 = q - \varepsilon_1 + 1$, we have

$$(3.25) \quad [q - \varepsilon_1 + 1]^2 + \left[\varepsilon_2 - \frac{1}{2}(q - 1)c \right]^2 \\ + \frac{1}{2}(q - 1)[x_1^2 + (2 - x_1)^2 + x_2^2 + (c - x_2)^2] \\ = 2q \cdot 2(q - \varepsilon_1) + 2,$$

or

$$- \varepsilon_2 c(q - 1) + \frac{1}{4}c^2(q - 1)^2 \\ + \frac{1}{2}(q - 1) \left[2(x_1 - 1)^2 + 2 \left(x_2 - \frac{1}{2}c \right)^2 + \frac{1}{2}c^2 + 2 \right] \\ = 3q^2 - 2\varepsilon_1 q + 2\varepsilon_1 - 2q - 1 \\ = [3q - (2\varepsilon_1 - 1)](q - 1),$$

or

$$- \varepsilon_2 c + \frac{1}{2}c^2(q - 1) + (x_1 - 1)^2 + \left(x_2 - \frac{1}{2}c \right)^2 + \frac{1}{4}c^2 + 1 \\ = 3q - 2\varepsilon_1 + 1,$$

whence

$$(3.26) \quad (12 - c^2)q + 4\varepsilon_2 c - 8\varepsilon_1 = (2x_1 - 2)^2 + (2x_2 - c)^2.$$

By (3.26)

$$(12 - c^2)q + 4\varepsilon_2 c - 8\varepsilon_1 \geq 0,$$

and since $q \geq 3$,

$$(3.27) \quad c^2 - \frac{4\varepsilon_2}{q}c + \frac{4}{q^2} \leq 12 - \frac{8\varepsilon_1}{q} + \frac{4}{q^2} \leq \frac{136}{9}.$$

Since c is an integer we can readily conclude that

$$(3.28) \quad |c| \leq 4.$$

We let $a = 2x_1 - 2$ and $b = 2x_2 - c$. Since $q = 4m - 1$, where $m > 0$ is an integer, we have from (3.26) that

$$(3.29) \quad (12 - c^2)(4m - 1) + 4\varepsilon_2 c - 8\varepsilon_1 = a^2 + b^2.$$

Now suppose for given values of $\varepsilon_1 = 1, -1, \varepsilon_2 = 1, -1$, and c that (3.29) has a solution in integers a and b . If c is even the left side of (3.29) is divisible by 4 whence a and b must both be even, while if c is odd the left side of (3.29) is odd whence one of these integers,

say a , is even while the other, b , is odd. So in either case we can solve the equations $a = 2x_1 - 2$ and $b = 2x_2 - c$ for integral values of x_1 and x_2 . Thus we have a solution to (3.26) in integers x_1, x_2 , and c . These values then determine the values $u_1 = 2 - x_1$ and $u_2 = c - x_2$. Then taking $t_1 = q - \varepsilon_1 + 1$, $t_2 = \varepsilon_2 - (q - 1)c/2$, and $r = 2(q - \varepsilon_1)$ and noting that (3.25) is equivalent to (3.26) we have by (3.25) that

$$t_1^2 + t_2^2 + (q - 1)[x_1^2 + u_1^2 + x_2^2 + u_2^2]/2 = 2rq + 2 = w.$$

Then since (3.21) is equivalent to (3.22) and (3.22) holds we have by (3.21) that

$$[t_1 + (q - 1)(x_1 + u_1)/2]^2 + [t_2 + (q - 1)(x_2 + u_2)/2]^2 = 2rq + 2 = w$$

where $t_1 = (r + 2)/2$. So if $q = 4m - 1$ is the order of a skew-Hadamard design incidence matrix, the conditions of Theorem 3.2 are satisfied and we can construct a type II solution according to this theorem. Now in deciding whether or not (3.29) has a solution in integers a and b we have, by (3.28), nine values of $\varepsilon_2 c$ to consider for each of the values $\varepsilon_1 = 1, -1$. We take the nine cases for $\varepsilon_1 = 1$.

- Case 1.* $\varepsilon_2 c = 4$: $-16m + 12 = a^2 + b^2$, impossible since $-16m + 12 < 0$ for $m > 0$.
- Case 2.* $\varepsilon_2 c = 3$: $12m + 1 = a^2 + b^2$, possible since, e.g., $12(1) + 1 = 13 = 3^2 + 2^2$. Here $3q + 4 = a^2 + b^2$.
- Case 3.* $\varepsilon_2 c = 2$: $8(4m - 1) = a^2 + b^2$ or $4m - 1 = a_1^2 + b_1^2$, a_1, b_1 integers, impossible since $4m - 1 \equiv 3 \pmod{4}$.
- Case 4.* $\varepsilon_2 c = 1$: $44m - 15 = a^2 + b^2$, possible since, e.g., $44(1) - 15 = 29 = 5^2 + 2^2$. Here $11q - 4 = a^2 + b^2$.
- Case 5.* $\varepsilon_2 c = 0$: $48m - 20 = a^2 + b^2$ or $12m - 5 = a_1^2 + b_1^2$, a_1, b_1 integers, impossible since $12m - 5 \equiv 3 \pmod{4}$.
- Case 6.* $\varepsilon_2 c = -1$: $44m - 23 = a^2 + b^2$, possible since, e.g., $44(2) - 23 = 65 = 8^2 + 1^2$. Here $11q - 12 = a^2 + b^2$.
- Case 7.* $\varepsilon_2 c = -2$: $32m - 24 = a^2 + b^2$ or $4m - 3 = a_1^2 + b_1^2$, a_1, b_1 integers, possible since, e.g., $4(2) - 3 = 5 = 2^2 + 1^2$. Here $8q - 16 = a^2 + b^2$ or $q - 2 = a_1^2 + b_1^2$.
- Case 8.* $\varepsilon_2 c = -3$: $12m - 23 = a^2 + b^2$, possible since, e.g., $12(3) - 23 = 13 = 3^2 + 2^2$. Here $3q - 20 = a^2 + b^2$.
- Case 9.* $\varepsilon_2 c = -4$: $-16m - 20 = a^2 + b^2$, impossible since $-16m - 20 < 0$ for $m > 0$.

Now when $\varepsilon_1 = 1$ we have $r = 2(q - 1)$, hence $n = 4q^2 - 4q + 2 = (2q - 1)^2 + 1$. So by Theorem 3.2 we have the following result.

THEOREM 3.3. *There exists a type II solution to the incidence equation for the finite projective plane case of order $n = (2q - 1)^2 + 1$*

whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares: $3q + 4$, $11q - 4$, $11q - 12$, $q - 2$, $3q - 20$.

When $\epsilon_1 = -1$ we have $r = 2(q + 1)$ hence $n = 4q^2 + 4q + 2 = (2q + 1)^2 + 1$. Analyzing this case as was done above for $\epsilon_1 = 1$, we have by Theorem 3.2 the corresponding result:

THEOREM 3.4. *There exists a type II solution to the incidence equation for the finite projective plane case of order $n = (2q + 1)^2 + 1$ whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares: $3q - 4$, $11q + 4$, $11q + 12$, $q + 2$, $3q + 20$.*

Both of these theorems yield infinitely many type II solutions. There exist skew-Hadamard design incidence matrices of orders

$$q_1 = 2^{2d-2}(11 + 1) - 1 = 3 \cdot 2^{2d} - 1$$

and

$$q_2 = 2^{2d-2}(43 + 1) - 1 = 11 \cdot 2^{2d} - 1$$

for each integer $d \geq 1$. Then $3q_1 + 4 = (3 \cdot 2^d)^2 + 1^2$, and $11q_2 + 12 = (11 \cdot 2^d)^2 + 1^2$. The first five orders for which each of these theorems yields a type II solution correspond to $q = 3, 7, 11, 15$, and 19 and are $n = 26, 170, 442, 842$, and 1370 , respectively, by Theorem 3.3, and $n = 50, 226, 530, 962$, and 1522 , respectively, by Theorem 3.4. As an example we construct A_{26} . For $n = 26$ we have $q = 3$ and $\epsilon_1 = 1$ hence $r = 4$ whence $t_1 = 3$. Now by case 2 above, $\epsilon_2 c = 3$ and

$$3q + 4 = 13 = 2^2 + 3^2 = (2x_1 - 2)^2 + (2x_2 - c)^2.$$

We take $2x_1 - 2 = 2$ or $x_1 = 2$ and $2x_2 - c = 3$. Letting $\epsilon_2 = 1$, we have $c = 3$ whence $x_2 = 3$ and $t_2 = -2$. Then $u_1 = u_2 = 0$. Now $E_4 = 3I - J$ of order 4 and since $\epsilon_1 = \epsilon_2 = 1$,

$$U = \begin{bmatrix} -1 & \cdots & -1 \\ & & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 & \cdots & -1 \\ -1 & \cdots & -1 \end{bmatrix},$$

of size 2×12 . The matrices F_4 and G_4 are of size 4×2 and a skew-Hadamard design incidence matrix of order 3 is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence we have

$$A_{26} = \left(\begin{array}{cc|ccc|ccc} 1 & 1 & -1 & \dots & -1 & -1 & \dots & -1 \\ 1 & -1 & 1 & \dots & 1 & -1 & \dots & -1 \\ \hline 1 & 1 & 3I - J & 2I & 0 & -2I & 3I & 0 \\ & \vdots & 0 & 3I - J & 2I & 0 & -2I & 3I \\ 1 & 1 & 2I & 0 & 3I - J & 3I & 0 & -2I \\ \hline 1 & -1 & 2I & 0 & -3I & 3I - J & 0 & 2I \\ & \vdots & -3I & 2I & 0 & 2I & 3I - J & 0 \\ 1 & -1 & 0 & -3I & 2I & 0 & 2I & 3I - J \end{array} \right) .$$

The second existence theorem for type II solutions is the following one.

Theorem 3.5. *Let (3.8) be satisfied in integers $t_1, t_2, u_1, u_2, x_1,$ and x_2 where $q \equiv 3 \pmod{4}$ is the order of a skew-Hadamard design incidence matrix and w is defined in (3.6), and where $x_1 + u_1 = 2$ and $t_1 = (r + 2)/2$ and $w = 2rq + 1$ for the positive even integer r . Then we can construct a type II solution to the incidence equation for the finite projective plane case of order $n = 4rq + 2$.*

Proof. We have

$$N = \left[\begin{array}{cc} Y & Z \\ -Z^x & Y^x \end{array} \right]; \quad Y = [y_{ij}], \quad Z = [z_{ij}],$$

where

$$(3.30) \quad y_{ii} = t_1 = \frac{1}{2}(r + 2),$$

$$y_{ij} + y_{ji} = x_1 + u_1 = 2; \quad 1 \leq i \leq q, \quad 1 \leq j \leq q, \quad i \neq j,$$

and

$$(3.31) \quad NN^x = (2rq + 1)I.$$

Since (3.8) is satisfied we have

$$(3.32) \quad \left[\frac{1}{2}(r + 2) + (q - 1) \right]^2 + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) \right]^2 = 2rq + 1,$$

or

$$\left[q - \frac{1}{2}r \right]^2 + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) \right]^2 = 1.$$

Since $q, r/2, t_2, (q - 1)/2, x_2,$ and u_2 are integers this means that

$$(3.33) \quad q - \frac{1}{2}r = \varepsilon_1, \quad t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) = \varepsilon_2;$$

$$\varepsilon_1^2 + \varepsilon_2^2 = 1; \quad \varepsilon_1, \varepsilon_2 = 1, 0, -1.$$

We form two matrices U and V of size $2 \times rq$ according to the values of ε_1 and ε_2 as follows:

$$(3.34) \quad \begin{array}{cc} U & V \\ \left[\begin{array}{ccc} -2 & \dots & -2 \\ 0 & \dots & 0 \end{array} \right] & \left[\begin{array}{ccc} 0 & \dots & 0 \\ -2 & \dots & -2 \end{array} \right] & \text{if } \varepsilon_1 = 1, \varepsilon_2 = 0. \\ \left[\begin{array}{ccc} 2 & \dots & 2 \\ 0 & \dots & 0 \end{array} \right] & \left[\begin{array}{ccc} 0 & \dots & 0 \\ 2 & \dots & 2 \end{array} \right] & \text{if } \varepsilon_1 = -1, \varepsilon_2 = 0. \\ \left[\begin{array}{ccc} 0 & \dots & 0 \\ 2 & \dots & 2 \end{array} \right] & \left[\begin{array}{ccc} -2 & \dots & -2 \\ 0 & \dots & 0 \end{array} \right] & \text{if } \varepsilon_1 = 0, \varepsilon_2 = 1. \\ \left[\begin{array}{ccc} 0 & \dots & 0 \\ -2 & \dots & -2 \end{array} \right] & \left[\begin{array}{ccc} 2 & \dots & 2 \\ 0 & \dots & 0 \end{array} \right] & \text{if } \varepsilon_1 = 0, \varepsilon_2 = -1. \end{array}$$

We set

$$f = t_1 + \frac{1}{2}(q - 1)(x_1 + u_1) = \frac{1}{2}r + q = \varepsilon_1 + r$$

and

$$g = t_2 + \frac{1}{2}(q - 1)(x_2 + u_2) = \varepsilon_2.$$

Then f and g are integers and by (3.8)

$$(3.35) \quad f^2 + g^2 = w = 2rq + 1.$$

Finally, we construct A_n of order $n = 4rq + 2$:

$$(3.36) \quad A_n = \left(\begin{array}{cc|cc|cc} 1 & 1 & & & & & & \\ 1 & -1 & & & & & & \\ & & U & V & 0 & 0 & & \\ \hline F_{rq} & & Y_* & Z_* & fI_{rq} & gI_{rq} & & \\ F_{rq} & & Y_* & Z_* & -fI_{rq} & -gI_{rq} & & \\ \hline G_{rq} & & -Z_*^T & Y_*^T & gI_{rq} & -fI_{rq} & & \\ G_{rq} & & -Z_*^T & Y_*^T & -gI_{rq} & fI_{rq} & & \end{array} \right).$$

By (3.34) the first two rows of A_n are orthogonal and have self inner products equal to $4rq + 2 = n$. Since the row and column sums of Y_* are $q - r/2$ and those of Z_* are $t_2 + (q - 1)(x_2 + u_2)/2$, we have by (3.33) and (3.34) that rows one and two are orthogonal to all

the other rows of A_n . We now look upon the submatrix of A_n below row 2 and to the right of the F_{rq} 's and G_{rq} 's as a matrix with the matrix entries $E_r, u_1I, x_1I, t_2I, u_2I,$ and $x_2I,$ all of order r . These matrices naturally divide the entire submatrix of A_n below row 2 into r -row blocks. Since these matrices commute with one another they behave multiplicatively among themselves as scalars. Thus (3.16), (3.17), (3.30), (3.31), and (3.35) imply that the inner product of an r -row block with itself is $(4rq + 2)I = nI$ of order r and that any two r -row blocks both intersecting F_{rq} 's or both intersecting G_{rq} 's are orthogonal, and (3.18) and (3.31) imply that any r -row block intersecting an F_{rq} is orthogonal to any r -row block intersecting a G_{rq} . Hence $A_n A_n^T = nI,$ and since the first column of A_n consists entirely of 1's we see that we have a type *II* solution to the incidence equation for the finite projective plane case of order $n = 4rq + 2.$

Letting $c = x_2 + u_2$ and combining (3.33) with (3.6), noting that $t_1 = (r + 2)/2 = q - \epsilon_1 + 1,$ we have

$$\begin{aligned}
 (3.37) \quad & [q - \epsilon_1 + 1]^2 + \left[\epsilon_2 - \frac{1}{2}(q - 1)c \right]^2 \\
 & + \frac{1}{2}(q - 1)[x_1^2 + (2 - x_1)^2 + x_2^2 + (c - x_2)^2] \\
 & = 2q \cdot 2(q - \epsilon_1) + 1,
 \end{aligned}$$

which, because of (3.33), again yields (3.26). Since the argument from (3.26) to (3.28) depends only on $|\epsilon_1|, |\epsilon_2| \leq 1$ and $q \geq 3,$ and since this is true here too, we obtain (3.28). Again, letting $a = 2x_1 - 2, b = 2x_2 - c,$ and $q = 4m - 1, m > 0$ an integer, we obtain as before

$$(3.38) \quad (12 - c^2)(4m - 1) + 4\epsilon_2 c - 8\epsilon_1 = a^2 + b^2,$$

where

$$(3.39) \quad |c| \leq 4.$$

Now suppose for given values of $\epsilon_1 = 1, -1, \epsilon_2 = 0$ or $\epsilon_1 = 0, \epsilon_2 = 1, -1$ and c that (3.38) has a solution in integers a and $b.$ We can then show, as we did before, that if $q = 4m - 1$ is the order of a skew-Hadamard design incidence matrix, then the conditions of Theorem 3.5 are satisfied and we can construct a type *II* solution according to that theorem.

Now in deciding whether or not (3.38) has a solution in integers a and b we have, by (3.39), five values of $|c|$ to consider for each of the two sets of values $\epsilon_1 = 1, \epsilon_2 = 0$ and $\epsilon_1 = -1, \epsilon_2 = 0$ and nine values of $\epsilon_2 c$ to consider for the value $\epsilon_1 = 0.$ We take the five cases for $\epsilon_1 = 1, \epsilon_2 = 0.$

- Case 1.* $|c| = 4$: $-16m - 4 = a^2 + b^2$, impossible since $-16m - 4 < 0$ for $m > 0$.
- Case 2.* $|c| = 3$: $12m - 11 = a^2 + b^2$, possible since, e.g., $12(2) - 11 = 13 = 3^2 + 2^2$. Here $3q - 8 = a^2 + b^2$.
- Case 3.* $|c| = 2$: $32m - 16 = a^2 + b^2$ or $2m - 1 = a_1^2 + b_1^2$, a_1, b_1 integers, possible since, e.g., $2(3) - 1 = 5 = 2^2 + 1^2$. Here $8q - 8 = a^2 + b^2$ or $q - 1 = a_2^2 + b_2^2$, a_2, b_2 integers.
- Case 4.* $|c| = 1$: $44m - 19 = a^2 + b^2$, possible since, e.g., $44(1) - 19 = 25 = 5^2 + 0^2$. Here $11q - 8 = a^2 + b^2$.
- Case 5.* $|c| = 0$: $48m - 20 = a^2 + b^2$ or $12m - 5 = a_1^2 + b_1^2$, a_1, b_1 integers, impossible since $12m - 5 \equiv 3 \pmod{4}$.

Now when $\varepsilon_1 = 1$ we have $r = 2(q - 1)$, hence $n = 8q^2 - 8q + 2 = 2(2q - 1)^2$. So by Theorem 3.5 we have the following result.

THEOREM 3.6. *There exists a type II solution to the incidence equation for the finite projective plane case of order $n = 2(2q - 1)^2$ whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares: $3q - 8, q - 1, 11q - 8$.*

When $\varepsilon_1 = -1$ we have $r = 2(q + 1)$, hence $n = 8q^2 + 8q + 2 = 2(2q + 1)^2$. Analyzing this case as was done above for $\varepsilon_1 = 1$, we have by Theorem 3.5 the corresponding result:

THEOREM 3.7. *There exists a type II solution to the incidence equation for the finite projective plane case of order $n = 2(2q + 1)^2$ whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares: $3q + 8, q + 1, 11q + 8$.*

When $\varepsilon_1 = 0$ we have $r = 2q$, hence $n = 8q^2 + 2 = (2q - 1)^2 + (2q + 1)^2$. Analyzing this case as was done for Theorem 3.3 we have by Theorem 3.5 the following result.

THEOREM 3.8. *There exists a type II solution to the incidence equation for the finite projective plane case of order $n = (2q - 1)^2 + (2q + 1)^2$ whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares: $3q + 12, q + 1, 11q + 4, 3q, 11q - 4, q - 1, 3q - 12$.*

All three theorems yield infinitely many type II solutions. There exist skew-Hadamard design incidence matrices of orders

$q_1 = 4(3^{2d-1} + 1) - 1 = 4 \cdot 3^{2d-1} + 3$ and $q_2 = 2^{2d} - 1$ for each integer $d \geq 1$. Then $3q_1 - 8 = (2 \cdot 3^d)^2 + 1^2$, and $q_2 + 1 = 2^{2d} + 0^2$. The first four orders for which each of these theorems yields a type *II* solution correspond to $q = 3, 7, 11, \text{ and } 15$ and are $n = 50, 338, 882, \text{ and } 1682$, respectively, by Theorem 3.6, $n = 98, 450, 1058, \text{ and } 1922$, respectively, by Theorem 3.7, and $n = 74, 394, 970, \text{ and } 1802$, respectively, by Theorem 3.8. As an example we construct A_{50} . For $n = 50$ we have $q = 3, \varepsilon_1 = 1, \text{ and } \varepsilon_2 = 0$ hence $r = 4$ whence $t_1 = 3$. Now by case 4 above, $|c| = 1$ and

$$11q - 8 = 25 = 0^2 + 5^2 = (2x_1 - 2)^2 + (2x_2 - c)^2.$$

We take $2x_1 - 2 = 0$ or $x_1 = 1$ and $2x_2 - c = 5$. Letting $c = 1$ we have $x_2 = 3$ and $t_2 = -1$. Then $u_1 = 1$ and $u_2 = -2, f = 5$ and $g = 0$. Now $E_4 = 3I - J$ of order 4 and since $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$,

$$U = \begin{bmatrix} -2 & \dots & -2 \\ 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & \dots & 0 \\ -2 & \dots & -2 \end{bmatrix},$$

of size 2×12 . The matrices F_4 and G_4 are of size 4×2 , and a skew-Hadamard design incidence matrix of order 3 is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence we have

$$A_{50} = \left(\begin{array}{c|cc|cc|ccc|ccc} 1 & 1 & -2 & \dots & -2 & 0 & \dots & 0 & & & 0 & & \\ 1 & -1 & & & & \times 2 & \dots & -2 & & & & & \\ \hline 1 & 1 & 3I-J & I & I & -I & 3I & -2I & 5I & 0 & 0 & & \\ \vdots & & I & 3I-J & I & -2I & -I & 3I & 0 & 5I & 0 & & 0 \\ 1 & 1 & I & I & 3I-J & 3I & -2I & -I & 0 & 0 & 5I & & \\ \hline 1 & 1 & 3I-J & I & I & -I & 3I & -2I & -5I & 0 & 0 & & \\ \vdots & & I & 3I-J & I & -2I & -I & 3I & 0 & -5I & 0 & & 0 \\ 1 & 1 & I & I & 3I-J & 3I & -2I & -I & 0 & 0 & -5I & & \\ \hline 1 & -1 & I & 2I & -3I & 3I-J & I & I & & & -5I & 0 & 0 \\ \vdots & & -3I & I & 2I & I & 3I-J & I & & & 0 & -5I & 0 \\ 1 & -1 & 2I & -3I & I & I & I & 3I-J & & & 0 & 0 & -5I \\ \hline 1 & -1 & I & 2I & -3I & 3I-J & I & I & & & 5I & 0 & 0 \\ \vdots & & -3I & I & 2I & I & 3I-J & I & & & 0 & 0 & 5I & 0 \\ 1 & -1 & 2I & -3I & I & I & I & 3I-J & & & 0 & 0 & 5I \end{array} \right)$$

The above constructions are all based on the existence of a skew-Hadamard design incidence matrix of a certain order $q \equiv 3 \pmod{4}$.

However, let us examine these constructions to see whether other constructions like these are possible. As a very simple possibility, let us consider replacing the skew-Hadamard design incidence matrix by the matrix [0] of order 1. Here corresponding to (3.5) we have

$$N = \begin{bmatrix} t_1 & t_2 \\ -t_2 & t_1 \end{bmatrix},$$

and setting

$$(3.40) \quad w \equiv t_1^2 + t_2^2$$

we automatically have

$$(3.41) \quad NN^r = wI.$$

Let us consider the form of construction in Theorem 3.2. We let (3.40) be satisfied in integers $t_1 = (r + 2)/2$, t_2 , and $w = 2r + 2$, for the positive even integer r . Then

$$\frac{1}{4}(r + 2)^2 + t_2^2 = 2r + 2,$$

or

$$\frac{1}{4}(r - 2)^2 + t_2^2 = 2,$$

hence

$$1 - \frac{1}{2}r = \varepsilon_1, \quad t_2 = \varepsilon_2; \quad \varepsilon_1, \varepsilon_2 = 1, -1.$$

For $\varepsilon_1 = 1$ we have $r = 0$, hence we get no nontrivial construction. For $\varepsilon_1 = -1$ we obtain $r = 4$ whence $n = w = 10$. We have $E_4 = 3I - J$ of order 4 and F_4 and G_4 , as defined previously, of size 4×2 . Then corresponding to $\varepsilon_2 = 1, -1$ we obtain by the form of construction in Theorem 3.2.

$$A_{10} = \left(\begin{array}{cc|cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 1 & 1 & 3I-J & & & & I & & & \\ 1 & 1 & & & & & & & & \\ \hline 1 & -1 & & & & & & & & \\ 1 & -1 & -I & & & & 3I-J & & & \\ 1 & -1 & & & & & & & & \\ 1 & -1 & & & & & & & & \end{array} \right), \left(\begin{array}{cc|cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 1 & 1 & 3I-J & & & & -I & & & \\ 1 & 1 & & & & & & & & \\ \hline 1 & -1 & & & & & & & & \\ 1 & -1 & & & & & I & & & \\ 1 & -1 & & & & & & & & \\ 1 & -1 & & & & & & & & 3I-J \end{array} \right)$$

respectively, each of which satisfy $A_{10}A_{10}^T = 10I$. These are essentially the same as the A_{10} constructed by Hall and Ryser [11]. Now let us consider the form of construction in Theorem 3.5. We let (3.40) be satisfied in integers $t_1 = (r + 2)/2$, t_2 , and $w = 2r + 1$, for the positive even integer r . Then

$$\frac{1}{4}(r + 2)^2 + t_2^2 = 2r + 1 ,$$

or

$$\frac{1}{4}(r - 2)^2 + t_2^2 = 1 ,$$

hence

$$1 - \frac{1}{2}r = \epsilon_1 , \quad t_2 = \epsilon_2 ; \quad \epsilon_1^2 + \epsilon_2^2 = 1 ; \quad \epsilon_1, \epsilon_2 = 1, 0, -1 .$$

For $\epsilon_1 = 1$ we again get no nontrivial construction. For $\epsilon_1 = 0$ we obtain $r = 2$ whence $n = 2w = 10$. We have $E_2 = 2I - J$ of order 2 and F_2 and G_2 , as defined previously, of size 2×2 . Then corresponding to $\epsilon_2 = 1, -1$ we have $f = 2$ and $g = 1, -1$, respectively, and we obtain by the form of construction in Theorem 3.5

$$A_{10} = \left(\begin{array}{cc|cc|cc|cc} \hline 1 & 1 & 0 & 0 & -2 & -2 & & \\ 1 & -1 & 2 & 2 & 0 & 0 & & \\ \hline 1 & 1 & 1 & -1 & 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & -2 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 0 & -2 & 0 & -1 \\ \hline 1 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -2 & 0 \\ 1 & -1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & -2 \\ 1 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 2 \\ \hline 1 & 1 & 0 & 0 & 2 & 2 & & & & \\ 1 & -1 & -2 & -2 & 0 & 0 & & & & \\ \hline 1 & 1 & 1 & -1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 & 0 & -1 & 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 & 0 & -2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & -1 & 0 & -2 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & -2 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -2 \\ 1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 2 \end{array} \right) ,$$

respectively, each of which satisfy $A_{10}A_{10}^T = 10I$. These, however, are essentially different from the A_{10} 's previously exhibited. This shows that type II solutions of the direct sum type are not necessarily unique to within permutations of the rows and columns of A_n and the multiplication of the columns of A_n by -1 . Finally, for $\epsilon_1 = -1, \epsilon_2 = 0$, we obtain $r = 4$ whence $n = 2w = 18$. We have $E_4 = 3I - J$ of order 4 and F_4 and G_4 , as previously defined, of size 4×2 . Here $f = 3$ and $g = 0$. We obtain by the form of construction in Theorem 3.5

$$A_{18} = \left(\begin{array}{cc|cccc} 1 & 1 & 2 \dots 2 & 0 \dots 0 & & & \\ 1 & -1 & 0 \dots 0 & 2 \dots 2 & 0 & & 0 \\ \hline 1 & 1 & 3I - J & 0 & 3I & & 0 \\ & \vdots & & & & & \\ 1 & 1 & 3I - J & 0 & -3I & & 0 \\ \hline 1 & -1 & 0 & 3I - J & 0 & & -3I \\ & \vdots & & & & & \\ 1 & -1 & 0 & 3I - J & 0 & & 3I \end{array} \right),$$

which satisfies $A_{18}A_{18} = 18I$. Hence, summarizing, we have the following result.

THEOREM 3.9. *There exists a type II solution to the incidence equation for the finite projective plane case orders $n = 10, 18$.*

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