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ON TOPOLOGICALLY INDUCED GENERALIZED PROXIMITY RELATIONS. II

MICHAEL LODATO

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In the theory of proximity spaces of Efremovic, (The geometry of proximity, Mat. Sbornic, N.S. 31 (73), (1952), 189-200.) the result:

A set X with a binary relation "A close to B" is a proximity space if and only if there exists a compact Hausdorff space Y in which X can be imbedded so that A is close to B in X if and only if \overline{A} meets \overline{B} in Y (\overline{A} denotes the closure of the set A) (Y. M. Smirnov, on proximity spaces, Mat. Sbornic, N.S. 31 (73), (1952), 543-574.)

Raises the question: Can we display a set of axioms for a binary relation δ on the power set of a set X so that the system (X,δ) satisfies these axioms if and only if there is a topological space Y in which X can be imbedded so that

(1.1) $A \delta B$ in X if and only if $\overline{A} \cap \overline{B} \neq \phi$ in Y.

In (M.W. Lodato, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. vol. 15, no. 3, June 1964, pp. 417-422), it is shown that an affirmative answer can be given if Y is T_1 and if X is regularly dense in Y. The clusters of S. Leader, On clusters in proximity spaces, Fund. Math. 47 (1959), 205-213, were used in (M.W. Lodato, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. vol. 15, no. 3, June 1964, pp. 417-422). The present paper generalized this notion and thus relaxes the condition that X be regularly dense in Y. We actually characterize every system (X, \hat{o}) for which there exists a mapping f (not necessarily one-to-one) of X into a Hausdorff space Y such that

(1.2) $A\delta B$ in X if and only if $A\overline{f} \cap f\overline{B} \neq \phi$ in Y.

- 2. P_s -Spaces. Recall from [3] that a symmetric generalized proximity space or P_s -space is a system (X, δ) where δ is a binary operation on the power set of X satisfying
 - (P. 1) $A\delta(B \cup C)$ implies that either $A\delta B$ or $A\delta C$
 - (P. 2) $A\delta B$ implies that $A \neq \phi$ and $B \neq \phi$
 - (P. 3) $A \cap B \neq \phi$ implies $A \delta B$
 - (P. 4) $A\delta B$ and $b\delta C$ for all points b in B imply that $A\delta C$
 - (P. 5) $A\delta B$ implies $B\delta A$

We read the symbols " $A\delta B$ " as "A is close to B"; and we say that "A is remote from B"-in symbols, " $A\phi B$ "-if A is not close to B.

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(2.1) The following facts are evident: (1) If $A \delta B$, $A \subset C$, and $B \subset D$ then $C \delta D$. (2) Define

$$A^{\delta} = \{x \in X : x \delta A\}$$

then in a P_s -space $(A^{\delta})\delta(B^{\delta})$ if and only if $A\delta B_s$.

- 3. Bunches. Let X be a P_s -space. A bunch over X is a class σ of subsets of X satisfying:
 - (B. 1) $A\delta B$ for all $A, B \in \sigma$
 - (B. 2) $A \cup B \in \sigma$ implies that $A \in \sigma$ or $B \in \sigma$
 - (B, 3) $X \in \sigma$
 - (B. 4) If $A \in \sigma$ and $a\delta B$ for all a in A then $B \in \sigma$.
 - (3.1) The following facts are easily established:
 - (1) Every cluster is a bunch.
- (2) For x, a point in a P_s -space X, the class σ_x of all subsets A of X such that $x \delta A$ is a bunch over X.
- (3) If a point x of X belongs to a bunch σ , then σ is identical to the class σ_x of all subsets A of X such that $x \delta A$.
- (4) Any bunch σ from a P_s -space (X, δ) is closed under the operation of supersets: If σ is a bunch from X, $A \in \sigma$ and $A \subseteq B$, then $B \in \sigma$.

4. Extensions characterized by bunches.

- (4.1) THEOREM. Given a set X and some binary relation δ on the power set of X, the following are equivalent:
- (I) There exists a T_2 topological space Y and a mapping f of X into Y with $\overline{fx} = Y$ and such that (1.2) holds.
 - (II) δ is a P_s -relation satisfying the additional axiom:
 - (P.7) There exists a family Σ of bunches from X such that
- (i) AdB implies that there exists a $\sigma \in \Sigma$ such that $A, B \in \sigma$, and
- (ii) if σ and σ' are in Σ and either $A \in \sigma$ or $B \in \sigma'$ for all subsets A and B of X such that $A \cup B = X$, then $\sigma = \sigma'$.
- *Proof.* Suppose that (I) holds and define δ by (1.2). (P.1), (P.2), (P.3), and (P.5) are trivial consequences of the properties of closure. For (P.4) suppose that $A\delta B$ and $b\delta C$ for all b in B. Then $\overline{fA}\cap \overline{fB}\neq \phi$ $\overline{fb}\cap \overline{fC}\neq \phi$ for all b in B, which since Y is T_2 , implies that $fb\in \overline{fC}$ for all b in B. Thus $fB\subset \overline{fC}$ or $\overline{fB}\subset \overline{fC}$ so that $\overline{fA}\cap \overline{fC}\neq \phi$ showing that $A\delta C$. For (P.7), define $\sigma_y=\{A\subseteq X:y\in \overline{fA}\}$ for each point $y\in Y$. Clearly, σ_y is a bunch.

Now let $\Sigma = {\sigma_y \colon y \in Y}$ and we will show that Σ satisfies (i) and (ii).

- (i) If $A\delta B$, then $f\overline{A}$ meets $f\overline{B}$ in Y so we can take a point y in $\overline{fA}\cap f\overline{B}$ and σ_y will be a bunch containing both A and B.
- (ii) Suppose $\sigma_x \neq \sigma_y$. Then $x \neq y$ in Y so that, using the T_2 property, there exist disjoint open sets U and V, containing x and y respectfully. Thus, $y \notin Y V = \overline{Y V}$ and $x \notin Y U = \overline{Y U}$ so that $y \notin \overline{fX V}$ and $x \notin \overline{fX U}$. Hence, $A = f^{-1}(fX V) \notin \sigma_y$ and $B = f^{-1}(fX U) \notin \sigma_x$ and

$$f(A \cup B) = (fX - V) \cup (fX - U) = fX - (V \cap U) = fX$$

so that $A \cup B = X$.

For the converse suppose that (II) holds. Given x in X the class $\sigma_x = \{A \subseteq X : x \delta A\}$ is a bunch from X, by (3.1), (2.). Thus for any subset A of X, let $\mathscr M$ be the set of all bunches σ_a determined by the points a in A and let $\mathscr M$ be the set of all bunches in Σ which have A as a number. Define the correspondence, $f(x) = \sigma_x$ between X and $\mathscr M = fX$ by identifying each x in X with the bunch σ_x determined by it. Let $Y = \Sigma$, the family of bunches satisfying (i) and (ii).

We first show that $fX \subseteq \Sigma$. Consider any σ_x in fX. Then since by (P. 3) $x \delta x$, by (i) there exists a σ in Y such that $x \in \sigma$. But by (3.1), (3.), $\sigma_x = \sigma$, hence $\sigma_x \in Y$ and $fX \subseteq Y$.

By (P. 3), $A \in \sigma_a$ for each a in A and so $\mathscr{A} \subset \widetilde{\mathscr{A}}$.

A subset A of X absorbs a subset Φ of Y if and only if A belongs to every bunch in Φ , i.e., if and only if \mathscr{A} contains Φ . For any subset Φ of Y we define the closure, $cl(\Phi)$, of Φ by

(4.2) $\sigma \in cl(\Phi)$ if and only if every subset E of X which absorbs Φ is in σ .

We next show that

$$(4.3) \quad cl(\mathscr{A}) = \bar{\mathscr{A}}.$$

For if $\sigma \in cl(\mathscr{A})$ then since A absorbs \mathscr{A} , $A \in \sigma$ so that $\sigma \in \mathscr{A}$. On the other hand, if $\sigma \in \mathscr{A}$ then $A \in \sigma$. Now let P be in every σ_a in \mathscr{A} , i.e., $P \delta a$ for every a in A and hence $A \subset P^{\delta}$. Thus, by (B.4), $P \in \sigma$ so that $\sigma \in cl(\mathscr{A})$.

We now show that the Kuratowski closure axioms are satisfied by the closure defined by (4.2).

- (K.1) $\mathscr{O} \subset cl(\mathscr{O})$: This is trivial since if E absorbs \mathscr{O} then $E \in \sigma$ for every $\sigma \in \mathscr{O}$.
- (K.2) $cl(\phi) = \phi$: Suppose $\sigma \in cl(\phi)$. Since it is vacuously true that every subset of X absorbs ϕ , we then have that every subset of X is in σ . In particular, ϕ and X are in σ . Thus, $\phi \partial X$, by (B.1), contradicting (P.2).

- (K. 3) $cl(cl(\Phi)) \subseteq cl(\Phi)$: Suppose $\sigma \in cl(cl(\Phi))$ and that E absorbs Φ . By (4.2) E absorbing Φ implies that E absorbs $cl(\Phi)$. Hence $E \in \sigma$ showing that $\sigma \in cl(\Phi)$.
- (K. 4) $cl(\mathcal{O} \cup \mathcal{O}') = cl(\mathcal{O}) \cup cl(\mathcal{O}')$: Suppose that $\sigma \in cl(\mathcal{O} \cup \mathcal{O}')$ and that A absorbs \mathcal{O} and A' absorbs \mathcal{O}' . Then by (3.1), (4.), $A \cup A'$ absorbs $\mathcal{O} \cup \mathcal{O}'$ so that $A \cup A' \in \sigma$. But by (B. 2) this means that either $A \in \sigma$ or $A' \in \sigma$, i.e., $\sigma \in cl(\mathcal{O})$ or $\sigma \in cl(\mathcal{O}')$. On the other hand, $\sigma \in cl(\mathcal{O}) \cup cl(\mathcal{O}')$ implies that either $\sigma \in cl(\mathcal{O})$ or $\sigma \in cl(\mathcal{O}')$. Now if E absorbs $\mathcal{O} \cup \mathcal{O}'$, then E absorbs \mathcal{O} and also absorbs \mathcal{O}' . Hence, $E \in \sigma$ showing that $\sigma \in cl(\mathcal{O} \cup \mathcal{O}')$ and (K. 4) holds.

Thus, (4.2) defines a topology on Y.

To show that fX is dense in Y, we just note that by (4.3), $cl(\mathcal{X}) = \bar{\mathcal{X}} = Y$.

To show that the topology is T_2 we must show that if σ and σ' are in Y such that $\sigma \neq \sigma'$, then there exist subsets $\mathscr O$ and $\mathscr O'$ of Y such that $\sigma \notin cl(\mathscr O)$, $\sigma' \notin cl(\mathscr O')$ and $cl(\mathscr O) \cup cl(\mathscr O') = Y$.

So suppose $\sigma \neq \sigma'$, then by (ii) there exist subsets A and B of X such that $A \notin \sigma$, $B \notin \sigma'$ and $A \cup B = X$. Thus, $\mathscr M$ and $\mathscr B$ are subsets of Y such that $\sigma \notin \overline{\mathscr A}$ and $\sigma' \notin \overline{\mathscr B}$, (since for instance A absorbs $\mathscr M$ but $A \notin \sigma$) and $\overline{\mathscr A} \cup \overline{\mathscr B} = \overline{\mathscr A} \cup \overline{\mathscr B} = \overline{\mathscr X} = Y$.

To finish the proof we need only show that (1.2) holds: $A\delta B$ in X if and only if $\bar{\mathscr{A}}$ meets $\bar{\mathscr{G}}$ in Y. If $A\delta B$ there exists, by (i) a $\sigma \in Y$ to which both A and B belong. Thus, by definition of $\bar{\mathscr{A}}$, we have $\sigma \in \bar{\mathscr{A}} \cap \bar{\mathscr{G}}$. On the other hand, if $\sigma \in \bar{\mathscr{A}} \cap \bar{\mathscr{G}}$ then A and B are in σ so that by (B.1), $A\delta B$.

The proof is now complete.

- 5. Symmetric P_1 -Spaces. A P_s -Spaces (X, δ) in which δ satisfies the additional axiom.
- (5.1) $x \delta y$ implies x = y is called a *symmetric* P_1 -space (see [4]). The following theorem follows directly from (B. 1) and (5.1).
- (5.2) Theorem. Every bunch σ from a symmetric P_1 -space (X, δ) possesses at most one point.
- (5.3) THEOREM. Given a set X and a binary relation, δ , on the power set of X, the following are equivalent:
- (I') There exists a T_z topological space Y in which X can be imbedded so that (1.1) holds.
 - (II') δ is a symmetric P₁-relation satisfying (P. 7).

Proof. The demonstration is very similar to that of theorem (4.1). To see that (I') implies (5.1), note that $\bar{x} \cap \bar{y} \neq \phi$ implies that $x \cap y \neq \phi$, or x = y.

Finally we note that, because of (5.2), the correspondence between X and $\mathscr X$ induced by the identification of x with the bunch σ_x determined by it is one-to-one.

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