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# ON TOPOLOGICALLY INDUCED GENERALIZED PROXIMITY RELATIONS. II

MICHAEL LODATO

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# ON TOPOLOGICALLY INDUCED GENERALIZED PROXIMITY RELATIONS II

## MICHAEL W. LODATO

In the theory of proximity spaces of Efremovic, (The geometry of proximity, Mat. Sbornic, N.S. 31 (73), (1952), 189-200,) the result:

A set X with a binary relation "A close to B" is a proximity space if and only if there exists a compact Hausdorff space Y in which X can be imbedded so that A is close to B in X if and only if  $\overline{A}$  meets  $\overline{B}$  in Y ( $\overline{A}$  denotes the closure of the set A) (Y. M. Smirnov, on proximity spaces, Mat. Sbornic, N.S. 31 (73), (1952), 543-574.)

Raises the question: Can we display a set of axioms for a binary relation  $\delta$  on the power set of a set X so that the system  $(X, \delta)$  satisfies these axioms if and only if there is a topological space Y in which X can be imbedded so that

(1.1)  $A\delta B$  in X if and only if  $\overline{A} \cap \overline{B} \neq \phi$  in Y.

In (M.W. Lodato, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. vol. 15, no. 3, June 1964, pp. 417-422), it is shown that an affirmative answer can be given if Y is  $T_1$  and if X is regularly dense in Y. The clusters of S. Leader, On clusters in proximity spaces, Fund. Math. 47 (1959), 205-213, were used in (M.W. Lodato, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. vol. 15, no. 3, June 1964, pp. 417-422). The present paper generalized this notion and thus relaxes the condition that X be regularly dense in Y. We actually characterize every system  $(X, \partial)$  for which there exists a mapping f (not necessarily one-to-one) of X into a Hausdorff space Y such that

(1.2)  $A\delta B$  in X if and only if  $\overline{Af} \cap \overline{fB} \neq \phi$  in Y.

2.  $P_s$ -Spaces. Recall from [3] that a symmetric generalized proximity space or  $P_s$ -space is a system  $(X, \delta)$  where  $\delta$  is a binary operation on the power set of X satisfying

(P.1)  $A\delta(B\cup C)$  implies that either  $A\delta B$  or  $A\delta C$ 

(P.2)  $A\delta B$  implies that  $A \neq \phi$  and  $B \neq \phi$ 

(P. 3)  $A \cap B \neq \phi$  implies  $A \delta B$ 

(P.4)  $A\delta B$  and  $b\delta C$  for all points b in B imply that  $A\delta C$ 

(P.5)  $A\delta B$  implies  $B\delta A$ 

We read the symbols " $A\partial B$ " as "A is close to B"; and we say that "A is remote from B"-in symbols, " $A\phi B$ "-if A is not close to B.

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(2.1) The following facts are evident: (1) If  $A\partial B$ ,  $A \subset C$ , and  $B \subset D$  then  $C\partial D$ . (2) Define

$$A^{\delta} = \{x \in X : x \delta A\}$$

then in a  $P_s$ -space  $(A^{\delta})\delta(B^{\delta})$  if and only if  $A\delta B$ .

3. Bunches. Let X be a  $P_s$ -space. A bunch over X is a class  $\sigma$  of subsets of X satisfying:

(B.1)  $A\delta B$  for all  $A, B \in \sigma$ 

(B.2)  $A \cup B \in \sigma$  implies that  $A \in \sigma$  or  $B \in \sigma$ 

(B.3)  $X \in \sigma$ 

(B. 4) If  $A \in \sigma$  and  $a\delta B$  for all a in A then  $B \in \sigma$ .

(3.1) The following facts are easily established:

(1) Every cluster is a bunch.

(2) For x, a point in a  $P_s$ -space X, the class  $\sigma_x$  of all subsets A of X such that  $x \delta A$  is a bunch over X.

(3) If a point x of X belongs to a bunch  $\sigma$ , then  $\sigma$  is identical to the class  $\sigma_x$  of all subsets A of X such that  $x\delta A$ .

(4) Any bunch  $\sigma$  from a  $P_s$ -space  $(X, \delta)$  is closed under the operation of supersets: If  $\sigma$  is a bunch from X,  $A \in \sigma$  and  $A \subseteq B$ , then  $B \in \sigma$ .

4. Extensions characterized by bunches.

(4.1) THEOREM. Given a set X and some binary relation  $\delta$  on the power set of X, the following are equivalent:

(I) There exists a  $T_2$  topological space Y and a mapping f of X into Y with  $\overline{fx} = Y$  and such that (1.2) holds.

(II)  $\delta$  is a  $P_s$ -relation satisfying the additional axiom:

(P.7) There exists a family  $\Sigma$  of bunches from X such that

(i)  $A\delta B$  implies that there exists a  $\sigma \in \Sigma$  such that  $A, B \in \sigma$ , and

(ii) if  $\sigma$  and  $\sigma'$  are in  $\Sigma$  and either  $A \in \sigma$  or  $B \in \sigma'$  for all subsets A and B of X such that  $A \cup B = X$ , then  $\sigma = \sigma'$ .

*Proof.* Suppose that (I) holds and define  $\delta$  by (1.2). (P. 1), (P. 2), (P.3), and (P.5) are trivial consequences of the properties of closure. For (P. 4) suppose that  $A\delta B$  and  $b\delta C$  for all b in B. Then  $\overline{fA} \cap \overline{fB} \neq \phi$  $\overline{fb} \cap \overline{fC} \neq \phi$  for all b in B, which since Y is  $T_2$ , implies that  $fb \in \overline{fC}$ for all b in B. Thus  $fB \subset \overline{fC}$  or  $\overline{fB} \subset \overline{fC}$  so that  $\overline{fA} \cap \overline{fC} \neq \phi$  showing that  $A\delta C$ . For (P.7), define  $\sigma_y = \{A \subseteq X : y \in \overline{fA}\}$  for each point  $y \in Y$ . Clearly,  $\sigma_y$  is a bunch.

Now let  $\Sigma = \{\sigma_y : y \in Y\}$  and we will show that  $\Sigma$  satisfies (i) and (ii).

(i) If  $A\delta B$ , then  $f\overline{A}$  meets  $f\overline{B}$  in Y so we can take a point y in  $\overline{fA} \cap \overline{fB}$  and  $\sigma_y$  will be a bunch containing both A and B.

(ii) Suppose  $\sigma_x \neq \sigma_y$ . Then  $x \neq y$  in Y so that, using the  $T_2$  property, there exist disjoint open sets U and V, containing x and y respectfully. Thus,  $y \notin Y - V = \overline{Y - V}$  and  $x \notin Y - U = \overline{Y - U}$  so that  $y \notin \overline{fX - V}$  and  $x \notin \overline{fX - U}$ . Hence,  $A = f^{-1}(fX - V) \notin \sigma_y$  and  $B = f^{-1}(fX - U) \notin \sigma_x$  and

$$f(A\cup B)=(fX-V)\cup(fX-U)=fX-(V\cap U)=fX$$

so that  $A \cup B = X$ .

For the converse suppose that (II) holds. Given x in X the class  $\sigma_x = \{A \subseteq X: x \delta A\}$  is a bunch from X, by (3.1), (2.). Thus for any subset A of X, let  $\mathscr{N}$  be the set of all bunches  $\sigma_a$  determined by the points a in A and let  $\mathscr{N}$  be the set of all bunches in  $\Sigma$  which have A as a number. Define the correspondence,  $f(x) = \sigma_x$  between X and  $\mathscr{H} = fX$  by identifying each x in X with the bunch  $\sigma_x$  determined by it. Let  $Y = \Sigma$ , the family of bunches satisfying (i) and (ii).

We first show that  $fX \subseteq \Sigma$ . Consider any  $\sigma_x$  in fX. Then since by (P.3)  $x \delta x$ , by (i) there exists a  $\sigma$  in Y such that  $x \in \sigma$ . But by (3.1), (3.),  $\sigma_x = \sigma$ , hence  $\sigma_x \in Y$  and  $fX \subseteq Y$ .

By (P.3),  $A \in \sigma_a$  for each a in A and so  $\mathscr{A} \subset \mathscr{A}$ .

A subset A of X absorbs a subset  $\varphi$  of Y if and only if A belongs to every bunch in  $\varphi$ , i.e., if and only if  $\mathscr{A}$  contains  $\varphi$ . For any subset  $\varphi$  of Y we define the closure,  $cl(\varphi)$ , of  $\varphi$  by

(4.2)  $\sigma \in cl(\Phi)$  if and only if every subset E of X which absorbs  $\Phi$  is in  $\sigma$ .

We next show that

 $(4.3) \quad cl(\mathscr{A}) = \mathscr{A}.$ 

For if  $\sigma \in cl(\mathscr{A})$  then since A absorbs  $\mathscr{A}$ ,  $A \in \sigma$  so that  $\sigma \in \mathscr{A}$ . On the other hand, if  $\sigma \in \mathscr{A}$  then  $A \in \sigma$ . Now let P be in every  $\sigma_a$  in  $\mathscr{A}$ , i.e.,  $P\delta a$  for every a in A and hence  $A \subset P^{\delta}$ . Thus, by (B.4),  $P \in \sigma$  so that  $\sigma \in cl(\mathscr{A})$ .

We now show that the Kuratowski closure axioms are satisfied by the closure defined by (4.2).

(K.1)  $\emptyset \subset cl(\emptyset)$ : This is trivial since if E absorbs  $\emptyset$  then  $E \in \sigma$  for every  $\sigma \in \emptyset$ .

(K.2)  $cl(\phi) = \phi$ : Suppose  $\sigma \in cl(\phi)$ . Since it is vacuously true that every subset of X absorbs  $\phi$ , we then have that every subset of X is in  $\sigma$ . In particular,  $\phi$  and X are in  $\sigma$ . Thus,  $\phi \partial X$ , by (B.1), contradicting (P.2).

(K.3)  $cl(cl(\Phi)) \subseteq cl(\Phi)$ : Suppose  $\sigma \in cl(cl(\Phi))$  and that E absorbs  $\Phi$ . By (4.2) E absorbing  $\Phi$  implies that E absorbs  $cl(\Phi)$ . Hence  $E \in \sigma$  showing that  $\sigma \in cl(\Phi)$ .

(K. 4)  $cl(\mathcal{Q} \cup \mathcal{Q}') = cl(\mathcal{Q}) \cup cl(\mathcal{Q}')$ : Suppose that  $\sigma \in cl(\mathcal{Q} \cup \mathcal{Q}')$  and that A absorbs  $\mathcal{Q}$  and A' absorbs  $\mathcal{Q}'$ . Then by (3.1), (4.),  $A \cup A'$  absorbs  $\mathcal{Q} \cup \mathcal{Q}'$  so that  $A \cup A' \in \sigma$ . But by (B. 2) this means that either  $A \in \sigma$  or  $A' \in \sigma$ , i.e.,  $\sigma \in cl(\mathcal{Q})$  or  $\sigma \in cl(\mathcal{Q}')$ . On the other hand,  $\sigma \in cl(\mathcal{Q}) \cup cl(\mathcal{Q}')$ implies that either  $\sigma \in cl(\mathcal{Q})$  or  $\sigma \in cl(\mathcal{Q}')$ . Now if E absorbs  $\mathcal{Q} \cup \mathcal{Q}'$ , then E absorbs  $\mathcal{Q}$  and also absorbs  $\mathcal{Q}'$ . Hence,  $E \in \sigma$  showing that  $\sigma \in cl(\mathcal{Q} \cup \mathcal{Q}')$  and (K. 4) holds.

Thus, (4.2) defines a topology on Y.

To show that fX is dense in Y, we just note that by (4.3),  $cl(\mathscr{X}) = \mathscr{\bar{X}} = Y$ .

To show that the topology is  $T_2$  we must show that if  $\sigma$  and  $\sigma'$  are in Y such that  $\sigma \neq \sigma'$ , then there exist subsets  $\varphi$  and  $\varphi'$  of Y such that  $\sigma \notin cl(\varphi)$ ,  $\sigma' \notin cl(\varphi')$  and  $cl(\varphi) \cup cl(\varphi') = Y$ .

So suppose  $\sigma \neq \sigma'$ , then by (ii) there exist subsets A and B of X such that  $A \notin \sigma$ ,  $B \notin \sigma'$  and  $A \cup B = X$ . Thus,  $\mathscr{A}$  and  $\mathscr{B}$  are subsets of Y such that  $\sigma \notin \overline{\mathscr{A}}$  and  $\sigma' \notin \overline{\mathscr{B}}$ , (since for instance A absorbs  $\mathscr{A}$ but  $A \notin \sigma$ ) and  $\overline{\mathscr{A}} \cup \overline{\mathscr{B}} = \overline{\mathscr{A} \cup \mathscr{B}} = \overline{\mathscr{X}} = Y$ .

To finish the proof we need only show that (1.2) holds:  $A\delta B$  in X if and only if  $\overline{\mathscr{N}}$  meets  $\overline{\mathscr{B}}$  in Y. If  $A\delta B$  there exists, by (i) a  $\sigma \in Y$  to which both A and B belong. Thus, by definition of  $\overline{\mathscr{N}}$ , we have  $\sigma \in \overline{\mathscr{N}} \cap \overline{\mathscr{D}}$ . On the other hand, if  $\sigma \in \overline{\mathscr{N}} \cap \overline{\mathscr{D}}$  then A and B are in  $\sigma$  so that by (B.1),  $A\delta B$ .

The proof is now complete.

5. Symmetric  $P_1$ -Spaces. A  $P_s$ -Spaces  $(X, \delta)$  in which  $\delta$  satisfies the additional axiom.

(5.1)  $x \delta y$  implies x = yis called a symmetric  $P_1$ -space (see [4]). The following theorem follows directly from (B. 1) and (5.1).

(5.2) THEOREM. Every bunch  $\sigma$  from a symmetric  $P_1$ -space  $(X, \delta)$  possesses at most one point.

(5.3) THEOREM. Given a set X and a binary relation,  $\delta$ , on the power set of X, the following are equivalent:

(I') There exists a  $T_2$  topological space Y in which X can be imbedded so that (1.1) holds.

(II')  $\delta$  is a symmetric P<sub>1</sub>-relation satisfying (P.7).

*Proof.* The demonstration is very similar to that of theorem (4.1). To see that (I') implies (5.1), note that  $\overline{x} \cap \overline{y} \neq \phi$  implies that  $x \cap y \neq \phi$ , or x = y.

Finally we note that, because of (5.2), the correspondence between X and  $\mathscr{X}$  induced by the identification of x with the bunch  $\sigma_x$  determined by it is one-to-one.

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