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# NONNEGATIVE PROJECTIONS ON $C_0(X)$

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**Let $X$ be a locally compact Hausdorff space, $C_0(X)$ the space of continuous real-valued functions on $X$ which vanish at infinity, and let $C_0(X)$ be equipped with the supremum norm. Let $E: C_0(X) \to C_0(X)$ be a nonnegative projection ($x \geqq 0 \Rightarrow Ex \geqq 0$; $E^2 = E$) of norm 1. The first theorem states that $E(xEy) = E(ExEy)$ for all $x, y \in C_0(X)$. Let $X_0 = \bigcap\{x^{-1}[\{0\}]: x \geqq 0, Ex = 0\}$. The second theorem states (in part) that $M = E[C_0(X)]$ under the norm and order it inherits from $C_0(X)$ is a Banach lattice, that the mapping $x \to x \mid X_0$ (=restriction of $x$ to $X_0$) is an isometric vector lattice homomorphism (=linear map which preserves the lattice operations) of $M$ onto a subalgebra of $C_0(X_0)$, and that for $t \in X_0$, $E(xEy)(t) = (ExEy)(t)$ for all $x, y \in C_0(X)$.**

**The paper concludes with a characterization of the conditional expectation operators $L^1$ of a probability space.**

The characterization is complementary to (and inspired by) one given by Moy [5; p. 61]. As a corollary to our first theorem we obtain the theorem of Kelley [2; p. 219] which states that $E[C_0(X)]$ is a subalgebra of $C_0(X)$ if and only if $E(xEy) = ExEy$ for all $x, y \in C_0(X)$.

Preliminaries. An *M-space* is a Banach lattice whose norm satisfies the condition $x, y \geqq 0 \Rightarrow \| x \vee y \| = \max(\| x \|, \| y \|)$ ($x \vee y$ is the maximum of $x$ and $y$). An element $u$ of a Banach lattice is a *unit* if and only if $\{x: 0 \leqq x \leqq u\} = \{x: x \geqq 0, \| x \| \leqq 1\}$. If a Banach lattice has a unit, it has only one and is an $M$-space.

LEMMA 1. *Let $M$ be an $M$-space with unit $u$. Then*

(i) $X = \{x^* \in M^*: x^*u = 1,$ *$x^*$ is a vector lattice homomorphism*$\}$ *is $\sigma(M^*, M)$-compact;*

(ii) *the natural mapping of $M$ into $C(X)$ ($X$ has the relative $\sigma(M^*, M)$-topology) is an isometric vector lattice homomorphism onto.*

*If, in additions, $M$ is order-complete*[1], *then*

(iii) *$X$ is Stonian*[2];

(iv) *$M$ is the (norm-)closed linear span of the set $U$ of extreme points of $\{x \in M: 0 \leqq x \leqq u\}$, and $x \in M$ belongs to $U$ if and only if $x \wedge (u - x) = 0$.*

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[1] That is, as a lattice $M$ is conditionally complete.

[2] $X$ is Stonian if and only if it is compact and its open subsets have open closures.

*Proof.* (i) and (ii) are proved in [1] (pp. 1000–1006). (iii) is proved in [7] (p. 185). We now prove (iv). By (i)–(iii) we may assume that $M = C(X)$ for some Stonian $X$. $x$ is an extreme point of $\{y \in C(X)\colon 0 \leqq y \leqq 1\}$ if and only if it is the characteristic function of an open closed subset of $X$. This proves the second part of (iv). The linear span $A$ of $U$ is a subalgebra and (since $X$ is totally disconnected) separates the points of $X$. By the Stone-Weierstrass Theorem A is dense in $C(X)$.

The adjoint of a Banach lattice with its natural norm and order ($x^* \geqq y^* \Leftrightarrow x^*x \geqq y^*x$ for all $x \geqq 0$) is an order-complete Banach lattice (that the adjoint is a lattice is proved in [6], p. 36). In particular, if $X$ is a locally compact Hausdorff space, then both $C_0(X)^*$ and $C_0(X)^{**}$ are order-complete Banach lattices.

LEMMA 2. *Let $X$ be a locally compact Hausdorff space. Then $C_0(X)^{**}$ is an M-space with unit, and when it is equipped with the multiplication it so acquires, the natural embedding of $C_0(X)$ in $C_0(X)^{**}$ is multiplicative.*

*Proof.* The mapping $\mu \to ||\mu||$ is additive and nonnegatively homogeneous on $\{\mu \in C_0(X)^*\colon \mu \geqq 0\}$ and so has a unique linear extension to all of $C_0(X)^*$. This extension, which we denote by 1, is clearly a unit for $C_0(X)^{**}$.

Let $\Omega = \{\xi \in C_0(X)^{***}\colon \xi 1 = 1,\ \xi$ a vector lattice homomorphism$\}$. Let $\kappa\colon C_0(X) \to C_0(X)^{**}$ be the natural embedding. We show the existence of a meagre subset $H$ of $\Omega$ such that for $x$ and $y$ in $C_0(X)$, $\kappa(x)\kappa(y)$ and $\kappa(xy)$, when regarded as functions on $\Omega$, agree on $\Omega \sim H$. $\kappa$ is a vector lattice homomorphism [6; p. 39] so that for $\xi \in \Omega$, $\xi \circ \kappa$ is a vector lattice homomorphism, i.e., $\xi \circ \kappa$ is a nonnegative multiple of evaluation at some point of $X$. Thus if $||\xi \circ \kappa|| = 1$, then $\xi \circ \kappa$ is evaluation at some point of $X$ and so is multiplicative. We now show that $H = \{\xi \in \Omega\colon ||\xi \circ \kappa|| < 1\}$ is meagre. Let $A = \{\kappa(x)\colon x \geqq 0, ||x|| \leqq 1\}$. $A$ is directed by $\leqq$ and is bounded above. Thus $\bigvee A$ (=supremum of $A$ in $C_0(X)^{**}$) exists and for $\mu$ a nonnegative member of $C_0(X)^*$, $(\bigvee A)(\mu) = \sup_{f \in A} f(\mu)$. $\sup_{f \in A} f(\mu) = \sup\{\mu(x)\colon x \geqq 0, ||x|| \leqq 1\} = ||\mu|| = 1(\mu)$ whenever $\mu \geqq 0$. Thus $\bigvee A = 1$. Since the supremum of a subset of $C(\Omega)$ and the pointwise supremum agree off some meagre set, we have $1 = \xi(1) = \sup\{\xi(f)\colon f \in A\} = \sup\{(\xi \circ \kappa)(x)\colon x \geqq 0, ||x|| \leqq 1\} = ||\xi \circ \kappa||$ save for $\xi$ in some meagre set. Thus, $\kappa(xy)$ and $\kappa(x)\kappa(y)$, when regarded as functions on $\Omega$, agree on $\Omega \sim H$, i.e., $\kappa(xy) = \kappa(x)\kappa(y)$

LEMMA 3. *Let $X$ be a compact Hausdorff space, and let $E\colon C(X) \to C(X)$ be a nonnegative projection of norm 1. Then $E[C(X)]$ with the norm and order it inherits from $C(X)$ is an M-space and has $E1$*

*for a unit.*

*Proof.* To show that $M = E[C(X)]$ is a vector lattice it is enough to prove that for $x \in M$, the maximum in $M$ of $x$ and 0 exists. Let $x \in M$. $x^+ \geqq x, 0 \Rightarrow Ex^+ \geqq x, 0 (x^+ = x \vee 0)$. If $y \in M$, and $y \geqq x, 0$, then $y \geqq x^+$ so that $y = Ey \geqq Ex^+$. Thus $Ex^+$ is the maximum in $M$ of $x$ and 0. Let $u = E1$. We show that for $x \in M$, $\|x\| = \inf\{\alpha: -\alpha u \leqq x \leqq \alpha u\}$. This will show that $M$ is a Banach lattice, and that $u$ is a unit for $M$. Let $x \in M$. $-\|x\| \leqq x \leqq \|x\| \Rightarrow -\|x\| u = E(-\|x\|) \leqq Ex = x \leqq E(\|x\|) = \|x\| u$; if $-\alpha u \leqq x \leqq \alpha u$, then $-\alpha \leqq -\alpha u \leqq \alpha u \leqq \alpha$ so that $\alpha \geqq \|x\|$.

## Main Theorems.

THEOREM 1. *Let $X$ be a locally compact Hausdorff space, and let $E: C_0(X) \to C_0(X)$ be a nonnegative projection of norm 1. Then $E(xEy) = E(ExEy)$ for all $x, y \in C_0(X)$.*

*Proof.* We shall show that by passing to $E^{**}$ and $C_0(X)^{**}$ it is enough to prove the theorem under the additional hypotheses

(a) $X$ is Stonian;

(b) if $\{x_i\}_{i\in I}$ is an increasing net in $C(X)$ with $x = \bigvee_{i\in I} x_i$, then $Ex = \bigvee_{i\in I} Ex_i$.

First we prove the theorem under the additional hypotheses. Let $M = E[C(X)]$. If $\{x_i\}_{i\in I}$ is an increasing net in $M$ with $\bigvee_{i\in I} x_i = x \in C(X)$, then $Ex = \bigvee_{i\in I} Ex_i = \bigvee_{i\in I} x_i = x$ so that $M$ is an order-complete $M$-space with unit $u = E1$. By Lemma 1 $M$ is the closed linear span of the set $\mathscr{U}$ of extreme points of $U = \{x \in M: 0 \leqq x \leqq u\}$. By the bilinearity and continuity of $(x, y) \to xy$ it is enough to prove that $E(xy) = E(xEy)$ whenever $x \in \mathscr{U}$ and $0 \leqq y \leqq 1$. Set $z = E(xy) - E(xEy)$. $x + z = E(x + xy - xEy) = E(x(1 + y - Ey))$, and, since $0 \leqq x \leqq 1$ and $1 + y - Ey \geqq 0$ (indeed, $1 - Ey \geqq 0$), we have $0 \leqq E(x(1 + y - Ey)) \leqq E(1 + y - Ey) = E1 = u$. Thus $x + z \in U$. Similarly, $x - z \in U$. Since both $x + z$ and $x - z$ belong to $U$ and $x \in \mathscr{U}$ we must have $z = 0$. This proves the theorem under the additional hypotheses.

Now let $X$ and $E$ be as in the theorem. $E^{**}$ is a nonnegative projection of norm 1, and by Lemmas 1 and 2 there is a Stonian space $\Omega$ such that $C_0(X)^{**} = C(\Omega)$. Let $\{f_i\}_{i\in I}$ be an increasing net in $C_0(X)^{**}$ with $f = \bigvee_{i\in I} f_i$. For $\mu$ a nonnegative member of $C_0(X)^*$, $f(\mu) = \sup_i f_i(\mu) = \lim_i f_i(\mu)$. Since any member of $C_0(X)^*$ is the difference of nonnegative members, we have $f(\mu) = \lim_i f_i(\mu)$ for all $\mu \in C_0(X)^*$. Since $E^{**}$ is $\sigma(C_0(X)^{**}, C_0(X)^*)$-continuous, $\{E^{**}f_i\}_{i\in I}\sigma(C_0(X)^{**}, C_0(X)^*)$-converges to $E^{**}f$, which, together with the monotonicity of $\{E^{**}f_i\}_{i\in I}$, implies that $E^{**}f = \bigvee_{i\in I} E^{**}f_i$. Thus $E^{**}$ and $\Omega$ satisfy the additional hypotheses. Let $\kappa: C_0(X) \to C_0(X)^{**}$ be the natural embedding. For $x, y \in C_0(X)$, $\kappa(E(xEy)) = E^{**}(\kappa(xEy)) = E^{**}(\kappa(x)\kappa(Ey)) = E^{**}(\kappa(x)E^{**}(\kappa(y))) = E^{**}(E^{**}(\kappa(x))E^{**}(\kappa(y))) = E^{**}(\kappa(Ex)\kappa(Ey)) = E^{**}(\kappa(ExEy)) = \kappa(E(ExEy))$.

COROLLARY. *(Kelley) $E[C_0(X)]$ is a subalgebra of $C_0(X)$ if and only if $E(xEy) = ExEy$ for all $x, y \in C_0(X)$.*

*Proof.* $E[C_0(X)]$ is a subalgebra of $C_0(X)$ if and only if $ExEy = E(ExEy)$ for all $x, y \in C_0(X)$.

DEFINITION. Let $L$ and $M$ be vector lattices, and let $T: L \to M$ be a nonnegative linear map. $|\mathrm{Ker}|(T) = \{x \in L: T(|x|) = 0\}$ ($|x| = x \vee (-x)$).

Note that $|\mathrm{Ker}|(T)$ is a vector lattice ideal in $L$, that is, $|\mathrm{Ker}|(T)$ is a linear subspace of $L$ and $x \in |\mathrm{Ker}|(T)$, $|y| \leqq |x| \Rightarrow y \in |\mathrm{Ker}|(T)$.

THEOREM 2. *Let $X$ be a locally compact Hausdorff space and $E: C_0(X) \to C_0(X)$ a nonnegative projection of norm 1. Let $X_0 = \bigcap \{x^{-1}[\{0\}]: x \in |\mathrm{Ker}|(E)\}$, $Y$ be the set of level sets (sets of constancy) of $M = E[C_0(X)]$, $X_1 = \bigcup \{A \in Y: A \cap X_0 \neq \emptyset\}$, and let $Z = \bigcap \{x^{-1}[\{0\}]: x \in M\}$. Then*

(i) *$M$ with the norm and order it inherits from $C_0(X)$ is a Banach lattice;*

(ii) *$x \to x \mid X_0$ is an isometric vector lattice homomorphism from $M$ to $C_0(X_0)$;*

(iii) *for $x, y \in M$, $xy \mid X_0 = E(xy) \mid X_0$; in particular, $\{x \mid X_0: x \in M\}$ is a subalgebra of $C_0(X_0)$;*

(iv) *$X_1 \cup Z = \{s \in X: E(xEy)(s) = (ExEy)(s)$ for all $x, y \in C_0(X)\}$.*

*Proof.* We saw in the proof of Lemma 3 that $M$ is a vector lattice under the order it inherits from $C_0(X)$. (ii) will imply that $M$ is a Banach lattice. First we prove that $x \to x \mid X_0$ is a vector lattice homomorphism. Let $x \in M$. We have seen that the maximum of $x$ and 0 in $M$ is $Ex^+$. Thus we must show that $Ex^+ \mid X_0 = x^+ \mid X_0$. $Ex^+ \geqq x$, $0 \Rightarrow Ex^+ \geqq x^+$. $Ex^+ - x^+ \geqq 0$, $E(Ex^+ - x^+) = 0 \Rightarrow Ex^+ - x^+ \in |\mathrm{Ker}|(E) \Rightarrow Ex^+ - x^+$ vanishes on $X_0$. Thus $x \to x \mid X_0$ is a vector lattice homomorphism of $M$ to $C_0(X_0)$. Note that $|\mathrm{Ker}|(E)$ is a closed algebraic ideal in $C_0(X)$ and so is equal $\{x \in C_0(X): x \mid X_0 = 0\}$. Let $y \in C_0(X)$ be an extension of $x \mid X_0$ with norm $\|x \mid X_0\|$. Since $x$ and $y$ agree on $X_0$, $Ey = Ex = x$. We thus have $\|x \mid X_0\| = \|y\| \geqq \|Ey\| = \|x\| \geqq \|x \mid X_0\|$. Thus $x \to x \mid X_0$ is an isometry from $M$ into $C_0(X_0)$.

We first prove (iii) under the additional hypothesis that $X$ is compact. $M_0 = \{x \mid X_0: x \in M\}$ is a closed vector sublattice of $C(X_0)$. By the proof of the Stone-Weierstrass theorem in [4] (p. 8) $M_0$ is a subalgebra if it contains the constants. For this it is enough to prove $1 \mid X_0 = E1 \mid X_0$. $1 - E1 \geqq 0, E(1 - E1) = 0 \Rightarrow 1 - E1 \in |\mathrm{Ker}|(E) \Rightarrow 1 - E1$ vanishes on $X_0$. Now let $x, y \in M$. There exists $z \in M$ such that $z \mid X_0 = xy \mid X_0$. $xy$ and $z$ agree on $X_0$ so that $E(xy) = Ez = z$. Thus $xy \mid X_0 = E(xy) \mid X_0$.

Now let us return to the general case. $C_0(X)^{**} = C(\Omega)$ for some compact $\Omega$, and $E^{**}$ is a nonnegative projection of norm 1. By the above $E^{**}(fg) - fg \in |\mathrm{Ker}|(E^{**})$ whenever $f, g \in E^{**}[C_0(X)^{**}]$. In particular, if $x, y \in M$, then $E^{**}(\kappa(x)\kappa(y)) - \kappa(x)\kappa(y) \in |\mathrm{Ker}|(E^{**})$, where $\kappa: C_0(X) \to C_0(X)^{**}$ is the natural embedding. Thus $0 = E^{**}(|E^{**}(\kappa(x)\kappa(y)) - \kappa(x)\kappa(y)|) = E^{**}(|E^{**}(\kappa(xy)) - \kappa(xy)|) = E^{**}(|\kappa(E(xy) - xy)|) = E^{**}(\kappa(|E(xy) - xy|)) = \kappa(E(|E(xy) - xy|))$ so that $E(|E(xy) - xy|) = 0$, i.e., $E(xy) - xy \in |\mathrm{Ker}|(E)$. Thus $E(xy)$ and $xy$ agree on $X_0$ whenever $x, y \in M$.

Let the set on the right in (iv) be denoted by $W$. Clearly, $Z \subset W$. To prove that $X_1 \subset W$ it is enough to prove that $X_0 \subset W$. Let $x, y \in C_0(X)$. By (iii) $ExEy$ and $E(ExEy)$ agree on $X_0$ and by Theorem 1 $E(ExEy) = E(xEy)$. Thus $ExEy$ and $E(xEy)$ agree on $X_0$. Now let $s \in W \sim Z$. Set $M_0 = \{x \mid X_0 : x \in M\}$. Let $\varphi \in M_0^*$ be defined by $\varphi(x \mid X_0) = x(s)$, $x \in M$. For $x, y \in M$, $\varphi((x \mid X_0)(y \mid X_0)) = \varphi(xy \mid X_0) = \varphi(E(xy) \mid X_0) = E(xy)(s) = E(xEy)(s) = (ExEy(s) = (xy)(s) = \varphi(x)\varphi(y)$. Thus $\varphi$ is a nonzero multiplicative linear functional on $M_0$. Therefore there exists $t \in X_0$ such that $\varphi(x \mid X_0) = x(t)$, $x \in M$, i.e., the level set of $M$ which contains $s$ intersects $X_0$. Thus $s \in X_1$.

DEFINITION. Let $X$ be a locally compact Hausdorff space. For $t \in X$, $\delta_t \in C_0(X)^*$ is evaluation at $t$.

COROLLARY. *Let $u(s) = \| E^*\delta_s \|$, $s \in X$. Then $E[C_0(X)]$ is a vector sublattice of $C_0(X)$ if and only if $ExEy = uE(xEy)$ for all $x, y \in C_0(X)$.*

*Proof.* Suppose $E[C_0(X)]$ is a vector sublattice of $C_0(X)$. Let $s \in X$. $x \mid X_0 \to x(s)$ is a vector lattice homomorphism of $M_0$ to $\boldsymbol{R}$ so that there exist $t \in X_0$ and $\alpha \in \boldsymbol{R}$ such that $x(s) = \alpha x(t)$ for all $x \in M$. $x \mid X_0 \to x(t)$ is a linear functional of norm 1 on $M_0$ so that $\| E^*\delta_s \| = \sup\{x(s): x \in M, \|x\| \leqq 1, x \geqq 0\} = \alpha \sup\{x(t): x \in M, \|x\| \leqq 1, x \geqq 0\} = \alpha$. Thus $\alpha = u(s)$. Let $x, y \in C_0(X)$. $u(s)E(xEy)(s) = u(s)^2E(xEy)(t) = u(s)^2(Ex)(t)(Ey)(t) = (Ex)(s)(Ey)(s) = (ExEy)(s)$.

Now suppose that $ExEy = uE(xEy)$ for all $x, y \in C_0(X)$. First we show that $x, y \in M$, $x \wedge_M y = 0 \Rightarrow x \wedge y = 0$. $x \wedge_M y = 0 \Rightarrow (x \mid X_0) \wedge (y \mid X_0) = 0 \Rightarrow xy \mid X_0 = 0, x, y \geqq 0 \Rightarrow E(xy) = 0, x, y \geqq 0 \Rightarrow 0 = uE(xy) = ExEy = xy, x, y \Rightarrow x \wedge y = 0$. Now let $x$ be any element of $M$. $Ex^+ = x \vee_M 0$, $Ex^- = (-x) \vee_M 0 \Rightarrow Ex^+ \wedge_M Ex^- = 0 \Rightarrow Ex^+ \wedge Ex^- = 0$. $x = Ex^+ - Ex^-$

and $Ex^+ \wedge Ex^- = 0 \Rightarrow x^+ = Ex^+$ and $x^- = Ex^-$.[3] Thus $x \in M \Rightarrow x^+ \in M$, i.e., $M$ is a vector sublattice of $C_0(X)$.

EXAMPLES. Let $X$ be the discrete space $\{0, 1, 2\}$, and let $E_i: C(X) \rightarrow C(X)$, $i = 1, 2, 3$, be defined by

$$(E_1x)(s) = \begin{cases} x(s) & s = 0, 1 \\ \frac{1}{2}(x(0) + x(1)) & s = 2 \end{cases} \qquad (E_2x)(s) = \begin{cases} \frac{1}{2}x(1) & s = 0 \\ x(1) & s = 1, 2 \end{cases}$$

$$(E_3x)(s) = \begin{cases} 0 & s = 0 \\ x(0) + x(1) & s = 1 \\ x(2) & s = 2 \end{cases}$$

$E_1$, $E_2$, and $E_3$ are nonnegative projections on $C(X)$, $\| E_1 \| = \| E_2 \| = 1$, and $\| E_3 \| = 2$; $E_1[C(X)]$ is not a vector sublattice of $C(X)$; $E_2[C(X)]$ is a vector sublattice of $C(X)$ but not a subalgebra; $E_3[C_3(X)]$ is a subalgebra of $C(X)$, but $E_3$ does not satisfy the conclusion of Theorem 1.

(i) and (ii) were proved (essentially) by Lloyd [3; p. 172] for $X$ compact. Specifically, let $X$ be compact, and let $E$, $M$ and $Y$ be as in Theorem 2; let $Y_0$ be the set of elements of $Y$ at which evaluation is a nonzero extreme point of the nonnegative part of the unit ball of $M^*$; then $Y_0$ is compact (when $Y$ is equipped with the quotient topology), and the natural map of $M$ to $C(Y_0)$ is an order-preserving isometry onto. It can be shown that $Y_0 = \{A \in Y: A \cap X_0 \neq 0\}$ so that (ii) follows from Lloyd's result.

**An application.** In this section $(S, \Sigma, \mu)$ is a probability space (i.e., $(S, \Sigma, \mu)$ is a totally finite measure space with $\mu(S) = 1$). For $\Sigma_0$ a $\sigma$-subalgebra of $\Sigma$, $E(\cdot, \Sigma_0): L^1(\mu) \rightarrow L^1(\mu)$ is defined by

$$\left.\begin{array}{l} E(x, \Sigma_0) \text{ is } \Sigma_0\text{-measurable} \\ \int_A E(x, \Sigma_0) d\mu = \int_A x d\mu \text{ for all } A \in \Sigma_0 \end{array}\right\} x \in L^1(\mu) ,$$

that is, $E(x, \Sigma_0)$ is the Radon-Nikodým derivative of $(x \cdot \mu) \mid \Sigma_0$ with respect to $\mu \mid \Sigma_0$ ($x \cdot \mu$ is defined by $(x \cdot \mu)(A) = \int_A x d\mu$, $A \in \Sigma$). $E(\cdot, \Sigma_0)$ is the *conditional expectation operator of* $\Sigma_0$. The object of this section is to characterize all such operators.

LEMMA 4. *Let $M$ be an order complete vector sublattice of $L^\infty(\mu)$ which contains* 1. *Then there is a $\sigma$-subalgebra $\Sigma_0$ of $\Sigma$ such that $M = \{x \in L^\infty(\mu): x$ is $\Sigma_0$-measurable$\}$.*

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[3] If $L$ is any vector lattice, $x \in L$, $u, v \in L$, $u \wedge v = 0$, and if $x = u - v$, then $u = x^+$ and $v = x^-$.

*Proof.* $M$ is an order-complete $M$-space with unit and so by Lemma 1 is the closed linear space of the set $U$ of extreme points of the nonnegative part of its unit ball. $U = \{x \in M: x \wedge (1 - x) = 0\}$. Thus $U = \{\chi_A: A \in \Sigma\} \cap M$[4]. Set $\Sigma_0 = \{A \in \Sigma: \chi_A \in M\}$. That $\Sigma_0$ is a $\sigma$-subalgebra of $\Sigma$ follows easily from the fact that $M$ is an order-complete vector sublattice of $L^\infty(\mu)$. The closed linear span of $U$ is thus the set of $\Sigma_0$-measurable members of $L^\infty(\mu)$.

LEMMA 5. *Let $T: L^1(\mu) \to L^1(\mu)$ be a linear map of norm* 1 *such that $T1 = 1$. Then $T$ is nonnegative, and $\int Txd\mu = \int xd\mu$ for all $x \in L^1(\mu)$.*

*Proof.* Let $x \in L^1(\mu)$, $1 \geqq x \geqq 0$. $1 - \int xd\mu = \| 1 - x \|_1 \geqq \| T(1 - x) \|_1 = \int | 1 - Tx | d\mu \geqq 1 - \int Txd\mu$ so that $\int xd\mu \leqq \int Txd\mu \leqq \int | Tx | d\mu = \| Tx \|_1 \leqq \| x \|_1 = \int xd\mu$. Thus, $0 \leqq x \leqq 1 \Rightarrow \int xd\mu = \int |Tx| d\mu = \int Txd\mu$. The second equality shows that $Tx \geqq 0$ whenever $1 \geqq x \geqq 0$, and it follows immediately that $T$ is nonnegative. The equality of $\int xd\mu$ and $\int Txd\mu$ for $0 \leqq x \leqq 1$ implies equality for all $x \in L^1(\mu)$.

THEOREM 3. *Let $E: L^1(\mu) \to L^1(\mu)$ be a projection of norm* 1 *such that $E1 = 1$. Then there is a $\sigma$-subalgebra $\Sigma_0$ of $\Sigma$ such that $E = E(\cdot, \Sigma_0)$.*

*Proof.* By Lemma 5 $E$ is nonnegative. Since $E1 = 1$ and $E > 0$, $E$ maps $L^\infty(\mu)$ into $L^\infty(\mu)$. The restriction $E_0$ of $E$ to $L^\infty(\mu)$ is thus a nonnegative projection of norm 1. We first show that $|\mathrm{Ker}|(E_0) = \{0\}$. Let $x \geqq 0$, and suppose $E_0x = 0$. Since $1 \wedge x = 0 \Rightarrow x = 0$, and since $E_0(1 \wedge x) = 0$, we may assume $0 \leqq x \leqq 1$. $1 - \int xd\mu = \| 1 - x \|_1 \geqq \| E_0(1 - x) \|_1 = \| E1 \|_1 = 1$. Thus $x = 0$. $L^\infty(\mu) = C(\Omega)$ for some compact $\Omega$ so that we may apply Theorem 2. Thus $E_0(xE_0y) = E_0xE_0y$ for all $x, y \in L^\infty(\mu)$, and $E_0[L^\infty(\mu)] = M$ is a vector sublattice of $L^\infty(\mu)$. We assert that $M$ is an order-complete vector sublattice. Let $\{x_i\}_{i \in I}$ be an increasing net in $M$ with $x = \bigvee_{i \in I} x_i$. $\{x_i\}_{i \in I}$ $L^1$-converges to $x$ so that $E_0x = L^1\text{-}\lim_i E_0x_i = L^1\text{-}\lim_i x_i = x$, i.e., $x \in M$. By Lemma 4 there is a $\sigma$-subalgebra $\Sigma_0$ of $\Sigma$ such that $M = \{x \in L^\infty(\mu): x \text{ is } \Sigma_0\text{-measurable}\}$. We conclude the proof by showing that $E$ and $E(\cdot, \Sigma_0)$ agree on $L^\infty(\mu)$. Let $x \in L^\infty(\mu)$. $Ex$ and $E(x, \Sigma_0)$ are $\Sigma_0$-measurable and so are equal if and only if $\int_A E(x, \Sigma_0)d\mu = \int_A Exd\mu$ for all $A \in \Sigma_0$. Let $A \in \Sigma_0$. $\int_A Exd\mu = \int \chi_A Exd\mu = \int E(\chi_A)Exd\mu = \int E(xE\chi_A)d\mu = \int E(x\chi_A)d\mu =$

[4] We identify bounded $\Sigma$-measurable functions and the corresponding elements of $L^\infty(\mu)$.

$$\int x\chi_A d\mu = \int_A x d\mu = \int_A E(x, \Sigma_0) d\mu.$$

COROLLARY. *(Moy) Let* $E: L^1(\mu) \rightarrow L^1(\mu)$ *be a linear map of norm* 1 *such that*

(a) $E1 = 1$;

(b) $E(xEy) = ExEy$ for all $x, y \in L^\infty(\mu)$.

*Then there is a* $\sigma$*-subalgebra* $\Sigma_0$ *or* $\Sigma$ *such that* $E = E(\cdot, \Sigma_0)$.

*Proof.* For $x \in L^\infty(\sigma)$, $E^2 x = E(1Ex) = E1Ex = Ex$. Thus $E^2$ and $E$ agree on $L^\infty(\mu)$, i.e. $E$ is a projection.

REMARK. As was mentioned in the introduction, Theorem 3 was inspired by Moy's theorem. In particular, had Moy's theorem required that $E$ be nonnegative, it would never have occurred to me that the condition of nonnegativeness could be dropped. The proof of Theorem 3 can, of course, be much shortened by using Moy's theorem. However, our proof is substantially different from hers and for this reason is given.

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