# Pacific Journal of Mathematics

# ASYMPTOTIC PROPERTIES OF GROUPS GENERATION

OSCAR S. ROTHAUS

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# ASYMPTOTIC PROPERTIES OF GROUP GENERATION

# O. S. ROTHAUS

Let G be a finite group, A and B two elements of G, which generate a subgroup L of order  $\lambda$ . We call an expression of the form  $A^{\alpha_1}B^{\beta_1}A^{\alpha_2}\cdots B^{\beta_2}$  with  $\alpha_i, \beta_i \geq 0$  a word in A and B and  $\sum_i (\alpha_i + \beta_i)$  the weight of the word. For any  $g \in G$  define  $f_m(g)$  as the number of words of weight m which are equal to g. Our purpose in this paper is to investigate the asymptotic dependence of  $f_m(g)$  on m. Subject to some simple side conditions, it turns out that the elements of L all occur with relative equal frequency as m approaches infinity. We also have an estimate of the smallest weight for which all elements of L can be realized.

Now define the matrix  $F_m$ , whose rows and columns are indexed by the elements of G, for which the entry in the gth row and hth column is  $f_m(g^{-1}h)$ . By virtue of the obvious identity:

$$f_{m+n}(g) = \sum_{h \in G} f_m(h) f_n(h^{-1}g)$$

we have  $F_{m+n} = F_m$ ,  $F_n$ , more particularly  $F_m = F_1^m$ . Note that  $F_1$  is the sum of the permutation matrices of A and B in the regular representation in G.

The matrix  $P = (1/2)F_1$  is doubly stochastic, and may be thought of as the matrix of transition probabilities of a Markov chain. In its study then, we take over the language of Markov chains as found in [1]. The irreducible sets of states are now easily described; they are the left cosets of L in G. A state is periodic if and only if the weights of all words equal to the identity have a greatest common divisor other than one. It is possible to have periodicity; if the symmetric group is generated by two odd permutations then all representations of the identity will have even weight.

Let us agree to call two generators A and B periodic of period d if the weights of all words in A and B equal to the identity have greatest common divisor d > 1. If d = 1, we will say A and B are aperiodic. (A simple way to insure aperiodicity is to have the periods of A and B relatively prime.)

THEOREM. Let A and B be periodic of period d. Then the group Received December 17, 1964. generated by A and B has a normal subgroup for which the factor group is cyclic of order d. Moreover, A and B both belong to a coset of the normal subgroup which generates the cyclic factor group.

*Proof.* Imagine the group generated by A and B presented in terms of the generators A and B and relations. Without loss of generality we may suppose that the exponents in all these relations are positive. Since the weight of every relation is a multiple of d, the mapping  $A \rightarrow w, B \rightarrow w$ , where w is a primitive dth root of unity is a homomorphism of the group onto a cyclic group of order d. The theorem follows.

The following converse is also clearly true; i.e., if A and B are both selected from the same coset of a proper normal subgroup for which the factor group is cyclic, then A and B are periodic.

As immediate consequences we have the following facts. A and B generating the symmetric group are periodic if and only if both odd, and then the period is 2. A and B generating a noncyclic simple group are aperiodic. Hence A and B generating an alternating group are aperiodic except for the alternating group on 4 letters. In that case (123) and (134) give a periodic generation of period 3.

We are now in a position to invoke the familiar statements about the limiting behavior of finite irreducible aperiodic doubly stochastic matrices.

Let M be the  $\lambda$  by  $\lambda$  matrix all of whose entries are  $1/\lambda$ . Then we have:

THEOREM. Let aperiodic A and B belonging to G generate a subgroup L of order  $\lambda$ . Construct the matrix P as before, but ordering the indices sequentially within the left cosets of L in G. Then we have:

$$\lim_{m
ightarrow\infty}P^{\,m}=egin{bmatrix} M&0\ &M\ &0&M\end{bmatrix}$$

where the number of M blocks on the diagonal is the index of L in G. In particular if L = G, we have:

$$\lim P^{\,m} = M$$

An alternative statement is that the elements of the group generated

by aperiodic A and B are asymptotically equidistributed over the words of weight m.

COROLLARY. For some weight m (and all larger weights) the elements of the group generated by aperiodic A and B are all realized by words of weight m. (There are corresponding statements for periodic generation.)

It is some interest to know the first m for which the above conclusion is true. Subsequently, we give a direct proof of the above corollary, which supplies us with an upper bound for the first such m.

It is known [2] that an irreducible doubly stochastic matrix has but a single real eigenvalue of absolute magnitude one, this clearly belonging to the eigenvector all of whose entries are one. So we have:

THEOREM. A necessary and sufficient condition that A and B belonging to a group G shall generate all of G is that the associated matrix P shall have but a single eigenvalue one, and this with eigenvector  $[1, 1, \dots, 1]$ .

This last results admits a simple restatement in the group algebra of G over the complex numbers. For if  $[v_g]$  is an eigenvector of eigenvalue one of the matrix P, we simply read in the group algebra:

$$\left(\sum\limits_{g} v_{g}g\right)(A+B-2I)=0$$
 .

Our conclusion above then says that essentially the only element R of the group algebra for which R(A + B - 2I) = 0 is  $R \equiv \sum_{g} g$ . For a semi-simple ring, if the right ideal  $J_1$  is properly contained in the right ideal  $J_2$  then the left annihilator of  $J_1$  properly contains the left annihilator of  $J_2$ . We conclude:

THEOREM. A necessary and sufficient condition that A and B belonging to a group G shall generate all of G is that the right ideal generated by A + B - 2I in the group algebra of G over the complex numbers consists of all elements of the group algebra whose coefficient sum is zero.

Let now aperiodic A and B generate a group G of order  $\lambda$ . Let the minimum of the periods of A and B be p. We now prove directly that every element of g is realized by a word of weight  $(\lambda - 2)p + 1$ . To this end, note first that the number of distinct group elements realized by words of weight m is a nondecreasing function of m. Let  $g_1, g_2, \dots, g_k$  be the distinct group elements of weight m. To say that the number of distinct group elements of weight m + 1 is still k means that the sets  $\{g_iA\}$  and  $\{g_jB\}$  are the same, or put another way that the sets  $\{g_i\}$  and  $\{g_jBA^{-1}\}$  are the same. To say that the number of distinct group elements of weight m + v is still kmeans more generally that the sets  $\{g_i\}, \{g_iBA^{-1}\}, \{g_iB^2A^{-2}\}, \dots, \{g_iB^vA^{-v}\}$ are all the same, or put another way, that the set  $\{g_i\}$  is invariant under multiplication on the right by any element of the group Hgenerated by  $\alpha_1 = BA^{-1}, \alpha_2 = B^2A^{-2}, \dots$ , and  $\alpha_v = B^vA^{-v}$ . Put v =period of A. Then  $\alpha_v = B^v$  and  $\alpha_{v+b} = \alpha_v\alpha_b$  so that the group Hgenerated by  $\alpha_1, \alpha_2, \dots, \alpha_v$  includes all elements of the form  $B^uA^{-u}$ . Furthermore:

$$egin{aligned} Alpha_{u}A^{\scriptscriptstyle -1} &= lpha_{\scriptscriptstyle 1}^{\scriptscriptstyle -1}lpha_{u+1}\ Blpha_{u}B^{\scriptscriptstyle -1} &= lpha_{u+1}lpha_{\scriptscriptstyle 1}^{\scriptscriptstyle -1} \end{aligned}$$

so that the group H is normal in G.

Again, since  $\alpha_1 = BA^{-1} \in H$ , we have that A and B belong to the same coset of H in G. And finally any element of G, written in terms of A and B, may be reduced modulo H to a power of A. Thus the factor group of G by H is cyclic. Since A and B are aperiodic we are forced to conclude that H = G. All of which implies of course that either  $k = \lambda$  or there are more distinct group elements of weight m + v than of weight m. Since the situation is symmetric in A and B we may assume that v = period of A = P =minimum of the periods of A and B. Starting then with the two distinct group elements of weight P + 1, 4 of weight 2P + 1, and finally at least  $\lambda$  of weight  $(\lambda - 2)P + 1$ . We have proved:

THEOREM. Every element in the group G of order  $\lambda$  generated by aperiodic A and B is realized by a word of weight  $(\lambda - 2)P + 1$ , where P is the minimum of the periods of A and B.

## References

1. W. Feller, Probability Theory and its Applications, Wiley, 1957.

2. F. R. Gantmacher, Applications of the Theory of Matrices, Interscience, 1959.

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