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**Let  $V$  be a group of operators acting in fixed point free manner on a group  $G$  and suppose  $V$  has order relatively prime to  $|G|$ . Work of several authors has shown that if  $V$  is cyclic of prime order or has order four,  $G'$  is nilpotent. In this paper it is proved that  $G'$  is nilpotent if  $V$  is non-abelian of order six, but that  $G'$  need not be nilpotent for any further groups other than those just mentioned. A side result is that  $G$  has nilpotent length at most 2 when  $V$  is non-abelian of order  $pq$ ,  $p$  and  $q$  primes (non-Fermat, if  $|G|$  is even).**

A fundamental theorem of Thompson [7] states that if  $G$  is a group admitting a fixed free automorphism of prime order, then  $G$  is nilpotent. It appears to be well known that if, in this theorem, the group of prime order is replaced by any group of automorphisms of composite order acting in fixed point free manner on  $G$ , one can no longer conclude that  $G$  is nilpotent. (For the sake of completeness, this fact is proved at the end of § 1.) However, one can frequently draw weaker conclusions concerning  $G$  in these cases. For example, D. Gorenstein and I. N. Herstein [4] proved that a group,  $G$ , which admits a fixed point free automorphism of order four, has nilpotent length at most two. S. Bauman [1] in 1961 obtained a similar result for the case that the fixed point free operator group was the four-group. Other more general results giving bounds for the nilpotent length of a solvable group,  $G$ , admitting various fixed point free operator groups,  $V$ , of order prime to  $|G|$  can be found in Hoffman [5], Thompson [8] and Shult [6]. In summarizing these results we remark only that the bounds are best possible when  $V$  is abelian and subject to a certain restriction on the prime divisors of its order (a restriction which vanishes when  $|V|$  and  $|G|$  are both odd), but that the bounds are very large otherwise.

In the case that  $V$  has order 4, something rather special obtains. Not only does  $G$  have nilpotent length 2, but moreover  $G$  has a nilpotent commutator subgroup. These findings raise the following question: Let  $G$  admit a fixed point free group of operators,  $V$ , of order prime to  $|G|$ . For what groups,  $V$ , does this imply nilpotence of the commutator subgroup? From the above-mentioned results of Thompson, Gorenstein

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and Herstein, and Bauman, this situation obtains whenever  $V$  is cyclic of prime order, or has order four. In this paper, it is shown that  $G'$  is also nilpotent when  $V$  is  $S_3$ , the symmetric group of degree three, but that this implication does *not* hold for any further groups,  $V$ .

In order to address the general question concerning which fixed point free operator groups,  $V$ , yield the nilpotence of  $G'$ , it would seem imperative that one have available general information concerning groups admitting fixed point free groups of operators, particularly information concerning nilpotent length 2. Information of this type can be obtained for the special case that  $V$  is abelian (Shult [6]), but so far only bounds on nilpotent length which exceed 5 are available when  $V$  is non-abelian (Thompson [8]). In the second section of this paper we produce a special result for the case that  $V$  is non-abelian of order  $pq$ ,  $p$  and  $q$  primes. Here, if  $G$  admits  $V$  as a fixed point free operator group,  $G$  has nilpotent length at most 2, provided neither  $p$  nor  $q$  are Fermat primes when  $G$  is even. Although this meagre result barely scratches the surface for the case that  $V$  is non-abelian, it turns out to be sufficient to answer the central question of this paper: when is  $G'$  necessarily nilpotent? In § 2, it is proved that  $G'$  is nilpotent when  $V \simeq S_3$ . Unlike proofs for the case  $V$  has order four, this proof does not merely hinge on the fact that a group fixed point free under an automorphism of order two is abelian. Rather, the proof asserts that a group which admits a fixed point free automorphism of order three in a very special way (a special case of condition (3) of (\*) in Theorem 1) is abelian. The final section merely consists in showing the existence of groups  $G$ , fixed point free under  $V$ , for which  $G'$  is not nilpotent whenever  $V$  is not cyclic of prime order, not of order four and not  $S_3$ .

1. **Technical preliminaries.** The purpose of this section is to standardize notation and to list a few preliminary results which are used repeatedly in the arguments in the main sections of the paper. Throughout all groups considered are finite and  $E$  denotes the identity group. The symbol  $O_\pi(G)$  denotes the maximal normal  $\pi$ -subgroup of  $G$ , where  $\pi$  is a fixed collection of primes.

If  $V$  is a group of operators acting on a group,  $G$ , the following subgroups are distinguished:

$$G_V = \{g: g \in G, v(g) = g \text{ for all } v \in V\}$$

$$(V, G) = \text{the subgroup generated by } \{v(g)g^{-1}: g \in G, v \in V\}$$

$V$  is said to act *fixed point free* on  $G$  if  $G_V = E$ . If  $N$  is a normal  $V$ -invariant subgroup of  $G$ ,  $V$  will also be regarded as a group of operators acting on  $N$  and  $G/N$ . The following two lemmas are obvious.

- LEMMA 1.1. (a)  $(V, G)$  is always normal in  $G$ .  
 (b) If  $W \triangleleft V, G_W$  and  $(W, G)$  are both  $V$ -invariant.

- LEMMA 1.2. If  $N$  is a normal  $V$ -invariant subgroup of  $G$ , then  
 (a)  $(G/N)_V = G/N$  if and only if  $(V, G) \subseteq N$   
 (b) if  $V$  is fixed point free on  $G/N, G_V \subseteq N$ .

From part (b) it can be seen that if  $V$  is fixed point free on both  $G/N$  and  $N$ , then  $V$  is fixed point free on  $G$ .

The following lemma is essentially a special case of a result of Glauberman [3].

LEMMA 1.3. Let  $V$  and  $G$  have relatively prime orders and let  $N$  be normal and  $V$ -invariant. Then

- (a)  $G = G_V(V, G)$   
 (b)  $(G/N)_V = G_V N/N$ .

*Proof.* If the coset  $xN$  is fixed by  $V$ , the theorem of Glauberman asserts that  $xN = yN$  for some  $y \in G_V$ , whence (b). (a) follows from (b) upon setting  $(V, G) = N$ , and using Lemma 1.1 (a) and Lemma 1.2 (a).

Theorem 1 of § 2 requires a technical theorem which is a special case of Theorem 4.1 proved in [6]:

THEOREM (A). Let  $U$  be cyclic of prime order,  $p$ , and suppose  $U$  is a group of automorphisms acting on a solvable group  $G$  of order relatively prime to  $p$ . Let  $H = GU$  be the semidirect product and suppose  $A$  is a faithful indecomposable  $KH$ -module, where  $K$  is any field whose characteristic is not  $p$ . Then if  $U$  acts in fixed point free manner on the module  $A, U$  centralizes  $G$  provided (i)  $O_r(G) = E$  when the characteristic of  $K$  is  $r$ , and (ii)  $p$  is not a Fermat prime when  $|G|$  is even.

We say that a group theoretic property,  $P$ , is *residually complete* if for any group  $G$  and any collection of two or more normal subgroups  $N_1, \dots, N_s$  intersecting at  $E$ , the fact that  $G/N_i$  has property  $P$  for  $i = 1, \dots, s$  implies  $G$  has property  $P$ . In short,  $P$  is residually complete if the collection of finite  $P$ -groups is closed under taking subdirect products. It is easy to show that (a) having nilpotent commutator subgroup and (b) having nilpotent length  $\leq k$ , are residually complete group theoretic properties, and these facts are assumed throughout the remainder of the paper.

We now settle in the negative the question whether there are groups,  $V$ , other than those which are cyclic of prime order, which

imply the nilpotence of groups,  $G$ , admitting  $V$  as a fixed point free group of automorphisms.

**THEOREM.** *Suppose  $V$  is a group of composite order. Then there exists a non-nilpotent group,  $G$ , admitting  $V$  as a fixed point free group of operators.*

*Proof.* Let  $T$  be a proper subgroup of  $V$  and suppose  $|T|$  is composite. Then by induction there exists a nonnilpotent group,  $G_1$ , which admits  $T$  as a fixed point free group of automorphisms. Set

$$V = x_1T + x_2T + \cdots + x_kT \text{ with } x_1T = T$$

and let  $G$  be a sum of  $k$  isomorphic copies of  $G_1$ , so  $G = G_1 \times \cdots \times G_k$ . The action of  $V$  on  $G$  is defined by letting  $V$  permute the components,  $G_i$ , as wholes, with  $T$  being the subgroup leaving  $G_1$  invariant, and  $G_1^{x_i} = G_i$ .  $T$  acts in fixed point free manner on  $G_1$  and  $T^{x_i}$  acts in fixed point free manner on  $G_i$ . In effect,  $G$  may be regarded as a normal subgroup of the semidirect product,  $VG$ , consisting of all  $k$ -tuples  $(g_1, x_2^{-1}g_2x_2, \cdots, x_k^{-1}g_kx_k)$ ,  $g_i \in G_1$ , on which  $V$  acts by component-wise conjugation. Then if  $u = (u_1, \cdots, u_k)$  was a fixed point,  $t^{-1}u_it = u_i$  for all  $t \in T^{x_i}$ . As  $T^{x_i}$  is fixed point free on  $G_i$ , each  $u_i = 1$  so  $u = 1 \in G$ . Thus  $G$  is a nonnilpotent group admitting  $V$  as a fixed point free group of automorphisms. Thus we may suppose that all proper subgroups of  $V$  are cyclic of order  $p$ . Then  $V$  is metacyclic of order  $pq$  ( $p$  and  $q$  primes), and we may suppose that  $U$ , the  $q$ -subgroup of  $V$ , is normal in  $V$ . Let  $R$  be cyclic of prime order  $r \equiv 1 \pmod{p}$  and let  $V$  act on  $R$  in such manner that  $U$  acts trivially on  $R$ . Then in the semidirect product,  $X = VR$ ,  $U$  is normal and  $X/U$  is the non-abelian group of order  $pr$ . Let  $s$  be a prime such that  $s \equiv 1 \pmod{qr}$ , and let  $M_1$  be the faithful irreducible  $R$ -module of dimension 1 over  $GF(s)$ , and convert  $M_1$  into a  $UR$ -module by letting  $U$  act by scalar multiplication on  $M_1$ . Now let  $M$  be the induced  $GF(s)X$ -module,

$$M = M_1 \otimes_{GF(s)UR} GF(s)X,$$

affording the representation,  $\rho$ , of  $X$ . Then  $G = MR$ , is a subgroup of the semidirect product  $XM$ , and  $V$  can then be regarded as a group of automorphisms acting on  $G$ . Then  $V$  is fixed point free on  $G/M \simeq R$  since  $VG/UM \simeq X/U$  is non-abelian of order  $pr$ . Also  $U$  is fixed point free on  $M$  since  $M$  is a sum of conjugate 1-dimensional faithful  $U$ -modules. Thus  $V$  is fixed point free on  $G$ . Also  $M$  is a sum of conjugate faithful  $R$ -modules and  $r$  is prime to  $s$ ; hence  $[R, M] = M$  and so  $G$  is not nilpotent.

2. **Groups with metacyclic fixed point free operator groups of order  $pq$ .** Throughout this section,  $V$  denotes a non-abelian metacyclic group of order  $pq$  where  $p$  divides  $q - 1$ . We suppose  $v$  and  $w$  are elements in  $V$  such that  $v^p = w^q = 1, v^{-1}wv = w^a$ , where  $a^p \equiv 1 \pmod q$ . We set  $W = \{w\}$ , the subgroup of order  $q$  in  $V$ .

**THEOREM 1.** *Let  $G$  be a group admitting  $V$  as a fixed point free group of operators, where  $|V|$  and  $|G|$  are coprime and, if  $|G|$  is even,  $p$  and  $q$  are not Fermat Primes. Form the semidirect product  $H = GV$  and let  $A$  be a faithful  $KH$ -module where  $K$  is a splitting field for  $H$  chosen so that if  $\text{char } K = r$ ,  $r$  does not divide  $pq$  and  $O_r(G) = E$ .  $G$  is assumed to be solvable. Then if the representation,  $\alpha$ , afforded by  $A$  is such that  $V$  acts in fixed point free manner on the non-trivial elements of  $A$ , then  $G$  has the following properties*

- (\*) (1)  $G = G_1 \times G_2$  where  $G_i$  is  $V$ -invariant ( $i = 1, 2$ ).
- (2)  $G_1$  is fixed elementwise by  $W$ .
- (3)  $G_2$  contains a set of normal subgroups,  $N_1(G), \dots, N_q(G)$ , such that

- (i) the  $N_i(G)$  have trivial meet
- (ii)  $w(N_i(G)) = N_{i+1(\pmod q)}(G)$
- (iii) if  $v = v_1, v_2, \dots, v_q$  are the successive conjugates of  $v$  under  $w^{-1}$  (i.e.  $wv_iw^{-1} = v_{i+1(\pmod q)}$ ), then  $v_i$  leaves  $N_i(G)$  invariant and fixes  $G/N_i(G)$  elementwise.

Before proceeding to the proof of Theorem 1, we first establish a number of lemmas. The first few of these concern several aspects of the property (\*).

**LEMMA 2.1.** *Let  $G$  be a group admitting  $V$  as a fixed point free operator group. Then if  $G$  enjoys property (3) in (\*),  $G$  is fixed point free under  $w$ .*

*Proof.* Since  $W \triangleleft V$ , by Lemma 1.1 (b),  $G_w$  is a  $V$ -invariant subgroup of  $G$ . Since  $V$  is fixed point free on  $G$ ,  $v$  is fixed point free on  $G_w$ , whence, by Lemma 1.3 (a)  $G_w = (v, G_w)$ . But since  $v_i$  fixes  $G/N_i(G)$  elementwise,  $(v_i, G) \subseteq N_i(G)$ , by Lemma 1.2 (a). Now  $v_{i+1} = w^i v w^{-i}$ , and we may write every element in  $G$  which is of the form  $v_{i+1}(x)x^{-1} = w^i v w^{-i}(x)x^{-1}$  as  $w^i(v(y)y^{-1})$  by setting  $y = w^{-i}(x)$ . Thus

$$(v_{i+1}, G) = (v, G)^{w^i}, \quad i = 0, 1, 2, \dots, q - 1.$$

We now have

$$G_w = (v, G_w) \subseteq (v, G)^{w^{i-1}} = (v_i, G) \subseteq N_i(G), \quad i = 1, \dots, q.$$

Since the  $N_i(G)$  have trivial meet,  $G_w = E$ .

LEMMA 2.2. *If a group,  $G$ , satisfies condition (\*), then  $G$  is nilpotent.*

*Proof.* From (\*),  $G = G_1 \times G_2$ . By Lemma 2.1,  $G_2$  is fixed point free under an automorphism,  $w$ , of order  $q$  and so, by the theorem of Thompson, is nilpotent. Also,  $G_1$  is fixed point free under an automorphism,  $v$ , of order  $p$ , whence  $G$  is also nilpotent.

LEMMA 2.3. *The condition (\*) is inherited by  $V$ -invariant subgroups.*

*Proof.* Let  $H$  be a  $V$ -invariant subgroup of  $G$ . Then by Lemma 1.3 (a),  $H = H_w(W, H)$ . From Lemma 1.1 (a), we always have  $(W, H) \triangleleft H$ . Since  $G_w = G_1$  is normal in  $G$ ,  $H_w = H \cap G_w \triangleleft H$ . Finally  $H_w \cap (W, H) \subseteq G_w \cap (W, G) = E$ . Under these circumstances,  $H = H_w \times (W, H)$ . Since  $W \triangleleft V$ , by Lemma 1.1 (b), each of these direct factors are  $V$ -invariant. Setting  $H_1 = H_w$  and  $H_2 = (W, H)$ ,  $H$  satisfies (1) and (2) of (\*).

Now set  $N_i(H) = H \cap N_i(G)$ . Then, because of the  $v_i$ -isomorphism  $H/N_i(H) \simeq HN_i(G)/N_i(G)$ ,  $v_i$  fixes  $H_2/N_i(H)$  elementwise, proving (iii). Now

$$\begin{aligned} w(N_i(H)) &= w(H \cap N_i(G)) \\ &= H^w \cap N_i(G)^w \\ &= H \cap N_{i+1}(G) \\ &= N_{i+1}(H) \end{aligned}$$

where the subscripts are taken mod  $q$ . This proves (ii). Finally the intersection of the  $N_i(H)$  is necessarily trivial, so (i) holds.

LEMMA 2.4. *The property (\*) is preserved under taking direct products.*

*Proof.* Let  $G$  and  $H$  be two groups admitting  $V$  as a fixed point free group of automorphisms and suppose (\*) holds for each group. Set  $L = G \times H$ . Then  $L$  admits  $V$  in a natural way and is fixed point free under  $V$ . Set  $L_i = G_i \times H_i$  ( $i = 1, 2$ ) so  $L = L_1 \times L_2$ , each  $L_i$  is  $V$ -invariant, and  $L_w = L_1$ ,  $(W, L) = L_2$ . Thus (1) and (2) hold for  $L$ . Now define  $N_i(L) = N_i(G) \times N_i(H)$ . Then  $N_i(L)$  is  $v_i$ -invariant and normal in  $L_2$ , and the  $N_i(L)$  have trivial meet.

Consider any left coset of  $N_i(L)$  in  $L_2$ , say  $(x, y)N_i(L)$  where  $x \in G_2$  and  $y \in H_2$ . Since  $v_i$  fixes  $G/N_i(G)$  and  $H/N_i(H)$  elementwise,  $v_i(x) = xn, v_i(y) = yn'$  where  $n \in N_i(G)$  and  $n' \in N_i(H)$ . Then

$$v_i(x, y)N_i(L) = (x, y)(n, n')N_i(L) = (x, y)N_i(L)$$

since  $(n, n') \in N_i(L)$ . Thus  $v_i$  fixes  $L_2/N_i(L)$  elementwise. Clearly, the  $N_i(L)$  have trivial meet, and  $w(N_i(L)) = N_{i+1}(L)$ . Thus (i), (ii), and (iii), and hence (3), hold for  $L$ . Thus  $L$  satisfies the condition (\*).

*Proof of Theorem 1.* Suppose  $A$  is decomposable as a  $KH$ -module ( $H = GV$ ): Then  $A = A_1 + A_2 + \dots + A_s$  where each  $A_i$  is indecomposable.

*Case I.* Either  $s > 1$  or at least one  $A_i$  is reducible.

Let  $B_i$  be a proper maximal submodule of  $A_i$  and consider the module  $A_0$  defined by the external direct product

$$(1) \quad A_0 = A_1/B_1 + A_2/B_2 + \dots + A_s/B_s,$$

and let  $\alpha_i$  and  $\mu$  be the representations afforded by  $A_i/B_i$  and  $A_0$  respectively,  $i = 1, 2, \dots, s$ . We now set out to show that  $\mu$  is faithful. If  $\text{char } K = 0$ , each  $A_i$  is irreducible, whence  $B_i = 0$  so  $A_0$  coincides with  $A$ . Then  $\mu$  is faithful, since, by hypothesis,  $A$  is a faithful  $KH$ -module.

On the other hand, if  $\text{char } K = r, O_r(G) = E$ . Since  $G$  is solvable, we must have, in this case,  $C_i = O_r(\ker \alpha_i \cap G) \neq E$ . Then  $C_i$  fixes  $A_i/B_i$  elementwise. Now as an additive group,  $A_i$  is a finite elementary abelian  $r$ -group, acted on by a group of operators,  $C_i$  of order prime to  $r$ . Then, since  $C_i$  centralizes  $A_i/B_i$ , by Lemma 1.3(b), we must have  $A_i = (A_i)_{C_i}B_i$ , so  $(A_i)_{C_i} \neq E$ . Since  $C_i$  is  $V$ -invariant and normal in  $G, C_i \triangleleft H$ . Then by Lemma 1.1 (b)  $(A_i)_{C_i}$  and  $(C_i, A_i)$  are  $KH$ -submodules of  $A_i$ . Moreover, by Maschke's theorem,  $A_i = (A_i)_{C_i} \oplus (C_i, A_i)$ . Then, because of the indecomposability of  $A_i$  and the fact that  $(A_i)_{C_i} \neq E, (C_i, A_i) = E$ , whence  $C_i$  centralizes all of  $A_i$ .

Now suppose  $\ker \mu \cap G \neq E$ . Then, since  $G$  is solvable and  $O_r(G) = E, O_r(\ker \mu \cap G) \neq E$ . Also,  $O_r(\ker \mu \cap G)$  is a normal  $r'$ -subgroup of  $\ker \alpha_i \cap G$  and so  $O_r(\ker \mu \cap G) \subseteq C_i, i = 1, \dots, s$ . Then  $O_r(\ker \mu \cap G)$  centralizes each  $A_i$  and hence all of  $A$ . Since  $A$  is faithful

$$O_r(\ker \mu \cap G) = E,$$

contrary to our assumption that  $\ker \mu \cap G \neq E$ . Thus  $\mu|_{\sigma}$  is a faithful



representation of  $G$ .

Now each  $A_i/B_i$  is an irreducible  $KH$ -module and  $G/(\ker \alpha_i \cap G)$  has no normal  $r$ -groups (since any normal  $r$ -subgroup of  $G$  necessarily acts trivially on  $A_i/B_i$ ). Further, both  $A_i/B_i$  and  $G/(\ker \alpha_i \cap G)$  are fixed point free under the action of  $V$ . Now if  $s > 1$ , or  $s = 1$  and  $B_1 \neq E$  (which will be the case if  $A_1$  is not irreducible),  $\dim_K(A_i/B_i) < \dim_K A$ ,  $i = 1, \dots, s$ . Thus, by induction,  $G/(\ker \alpha_i \cap G)$  satisfies (\*). Then by Lemma 2.3,

$$L = \prod_{i=1}^s (G/(\ker \alpha_i \cap G))$$

satisfies (\*). But because of the decomposition (1),  $\mu(G)$  is isomorphic to a subgroup of  $L$  and hence, by Lemma 2.1, also enjoys (\*). Since  $\mu$  is faithful,  $G$  itself satisfies (\*).

*Case II.*  $s = 1, A_1 = A$  is an irreducible  $KH$ -module.

Here we may apply Clifford's theory [2]. Since  $G \triangleleft H$ ,  $A$  decomposes into homogeneous  $KG$ -components,  $D_1, \dots, D_t$ , which are permuted transitively by  $V$ . Thus  $t$  divides  $pq$ .

Subcase (a).  $t = pq$ .

Here, the permutation representation of  $V$  afforded by the permutations  $V$  effects upon the  $D_i$ , is the regular representation. Consequently,  $D_i = u_i D_1$  for some unique  $u_i \in V$ . Thus, selecting  $d_i \in D_1$ ,  $d_1 \neq 0$ ,  $u_i(d_1) \in D_i$ , and so

$$d = \sum_{u \in V} u(d_1)$$

is a nonzero element of  $A$ , fixed by  $V$ . This contradicts our assumption that  $V$  is fixed point free on  $A$  and so subcase (a) cannot occur.

Subcase (b).  $t = q$ .

Here, the permutation representation is isomorphic to that induced by multiplication in  $V$  on left cosets of a subgroup of index  $q$ . Since all such subgroups are conjugate in  $V$ , without loss of generality we may take this subgroup to be  $\{v\}$ . The upshot of this is that  $w$  permutes the  $D_i$  in a cycle of length  $q$ , while  $v$  fixes one component, say  $D_1$ , and permutes the remaining  $q - 1$  components in cycles of length  $p$ . If  $D_1$  contained a point  $d_0$  fixed by  $v$ , then

$$d = \sum_{i=1}^q w^{i-1}(d_0)$$

would be a nonzero point of  $A$  fixed by all of  $V$ . Thus  $v$  acts in fixed point free manner on  $D_i$ . Let  $\delta_i$  be the representation of  $G$  afforded by  $D_i$ . Now  $\delta_i$  can be extended to a representation of  $K(G/\ker \delta_i)\{v\}$ , and, by another result of Clifford's is indecomposable since  $G\{v\}$  is the stability group in  $H$  for the submodule  $D_i$ . Set  $H_i = O_r(G/\ker \delta_i)$ . Since  $D_i$  is a sum of equivalent irreducible  $KG$ -modules, each  $D_i$  is also a sum of conjugate irreducible  $KH_i$ -modules. But each of these is trivial since  $H_i$  is an  $r$ -group and  $\text{char } K = r$ . Thus  $H_i = E$  and so none of the groups  $G/\ker \delta_i$  have normal  $r$ -subgroups. If  $[G; \ker \delta_i]$  is even so is  $|G|$ , and in that case our hypotheses guarantee that  $p$  is not a Fermat prime. The groups  $\{v\}$ ,  $G/\ker \delta_i$  and module,  $D_i$ , now satisfy the conditions of Theorem (A). Thus  $v$  fixes  $G/\ker \delta_i$  elementwise. Then also,  $v_i = w^{i-1}vw^{-(i-1)}$  leaves  $D_i$  invariant and fixes  $G/\ker \delta_i$  elementwise. Finally, since  $A$  is faithful, the groups  $\ker \delta_i$ ,  $i = 1, \dots, 2$ , have trivial meet. Thus  $G$  satisfies condition (3) of (\*), and so (for the case  $G_1 = E$ ) also satisfies (\*).

Subcase (c).  $t = p$ .

Here, the  $D_i$  are permuted in a cycle of length  $p$ , by  $v$  (or any  $v_i$ ), and  $w$  leaves each  $D_i$  invariant. Under these circumstances,  $w$  must be fixed point free on each  $D_i$  since otherwise it would be a simple matter to construct a nonzero point in  $A$  fixed by  $V$ . Then since  $q$  is not a Fermat prime, by Theorem (A),  $w$  fixes  $G/\ker \delta_i$  elementwise,  $i = 1, \dots, p$ . Thus  $(w, G) \cong \bigcap_i \ker \delta_i = E$ , whence  $G$  is fixed elementwise by  $w$ . Thus  $G$  satisfies condition (\*) for the special case that  $G_2 = E$ .

Subcase (d).  $t = 1$ .

Here  $G$  is homogeneous as a  $KG$ -module. At this point we can apply Clifford's theorem relative to any normal subgroup of  $H$  lying in  $G$ , i.e. any  $V$ -invariant normal subgroup of  $G$ . Let  $M$  be a maximal  $V$ -invariant normal subgroup of  $G$ . Since  $G$  is solvable,  $G/M$  is an elementary abelian  $r_1$ -group which, as a vector space over the field of  $r_1$  elements, is an irreducible  $GV$ -module. Since  $A$  is an irreducible  $KG$ -module, we may decompose  $A$  into its homogeneous  $KM$ -components,  $E_1, \dots, E_m$ , and, these are permuted transitively by the elements of  $G$  alone. Let  $N$  denote the subgroup of  $G$  which leaves each component invariant. Then if  $x \in N$ ,  $x(E_i) = E_i$  and  $v(x)E_i = v(x)v(E_i) = v(xE_i) = v(E_i) = E_i$ . Thus  $v(x) \in N$  whence  $N$  is  $V$ -invariant. Clearly,  $N \cong M$ . If  $M \subset N$ ,  $N = G$  because of the maximality of  $M$  and the fact that  $N \triangleleft H$ . Since  $G/N$  is abelian, the permutation of the  $E_i$  under the

action of  $G$  is permutation isomorphic to the regular representation of  $G/N$ . If  $G \neq N$ ,  $[G : N] = [G : M] = r_1^k = m$ , the number of homogeneous  $KM$ -components. On the other hand, if  $N = G$ , there is only one component, so  $A$  is a homogeneous  $KM$ -module. Let us consider the two cases separately.

Subsubcase (i).  $N = G$ ;  $A$  is a homogeneous  $KM$ -module.

Since  $M$  is a proper subgroup of  $G$  admitting  $V$  and  $A$  is a  $KM$ -module fixed point free (along with  $M$ ) under the action of  $V$ , by induction,  $M$  is a subgroup with property (\*). By Lemma 2.2,  $M$  is nilpotent and so has a nontrivial center,  $Z(M)$ . Since the hypotheses of the case under investigation demand that  $A$  be a homogeneous  $KM$ -module, all the irreducible  $KZ(M)$ -submodules of  $A$  are conjugate by an element of  $M$ . Since  $Z(M)$  is the center, these submodules are even equivalent. Since  $Z(M)$  is abelian,  $A$  is a homogeneous  $KZ(M)$ -module and  $K$ , being a splitting field for all subgroups of  $H$  is certainly a splitting field for  $Z(M)$ ,  $Z(M)$  must be represented on  $A$  by scalar multiplication by elements of  $K$ . Under these circumstances, aside from the fact that  $Z(M)$  is cyclic, the matrices representing  $V$  commute with those representing  $Z(M)$ . Since  $A$  is a faithful  $KH$ -module, this means that the elements of  $V$  centralize those of  $Z(M)$ , contrary to our hypothesis that  $V$  acts in fixed point free manner on  $G$ .

Subsubcase (ii).  $N = M$ .

Here there are  $[G : M]$  distinct homogeneous  $KM$ -components. By applying induction on  $M$  and using Lemma 2.2, we have already seen that  $M$  is nilpotent, whence  $Z(M) \neq E$ . The components  $E_i$  are permuted by  $H = GV$ , the resulting permutation representation having kernel,  $N$ . Thus the transformation of the  $E_i$  can be associated with a faithful transitive permutation representation,  $\pi$ , of the semidirect product,  $V(G/N) = VG/N = H/N$ , of degree  $r_1^k$ . But this is permutation isomorphic to the permutation representation induced by multiplication of the left cosets of some subgroup of index  $r_1^k$ , in  $V(G/N)$ . Such a subgroup necessarily has index  $r_1^k$ , and so, since  $V(G/N)$  is solvable, is an  $r_1$ -complement and is conjugate to  $V$ . Thus the representation,  $\pi$ , is permutation isomorphic to that induced by multiplication of left cosets of  $V$  in  $V(G/N)$  by elements of  $G/N$ . In such a representation,  $V$  is the subgroup fixing some letter elementwise. Thus, because of the permutation isomorphism, we learn that  $V$  leaves some  $KM$ -component, say  $E_1$ , invariant. Then, by a theorem of Clifford's, since  $VM$  is the

stability group of  $E_1$  (a consequence of our case division),  $E_1$  is a  $VM$ -module. Moreover,  $V$  is fixed point free on  $E_1$ . Since  $E_1$  is a homogeneous  $KM$ -module,  $Z(M)$  is represented by scalar multiplication on  $E_1$ . Then the matrices representing elements of  $V$  commute with the scalar matrices representing  $Z(M)$  on  $E_1$ . Let  $\beta_1$  be the representation of  $VM$  afforded by  $E_1$ . Then if  $Z(M) \not\subseteq \ker \beta_1$ ,  $V$  would centralize  $Z(M) \ker \beta_1 / \ker \beta_1 \neq E$ . Since  $V$  has order prime to  $M$ , by Lemma 1.3 (b), this would imply  $C_M(V) \neq E$  contrary to our hypothesis. Thus  $Z(M) \subseteq \ker \beta_1$ . But the  $E_i$  are conjugate  $EM$ -modules, i.e.  $E_i = \alpha(x_i)E_1$  for some  $x_i M \in G/M$ . Under these circumstances, if  $\ker \beta_i$  is the kernel in  $M$  of the  $KM$ -representation afforded by  $E_i$ ,

$$\ker \beta_i = (\ker \beta)^{x_i^{-1}},$$

whence, since  $Z(M)$  is normal in  $G$ ,

$$Z(M) = Z(M)^{x_i^{-1}} \subseteq (\ker \beta_i)^{x_i^{-1}} = \ker \beta_i, \quad i = 1, \dots, r_1^k.$$

Since  $A$  is faithful (even when restricted to  $M$ ) the  $\ker \beta_i$  have trivial meet. Thus

$$Z(M) \subseteq \bigcap_{i=1}^{r_1^k} \ker \beta_i = E.$$

But this is impossible since  $M$  is nilpotent. The subsubcase (ii) doesn't arise. This completes the proof.

**COROLLARY 1.1.** *Theorem 1 still holds when the condition that  $K$  be a splitting field for all subgroups of  $H = GV$  is dropped.*

*Proof.* Let  $G$  and  $V$  satisfy the conditions of Theorem 1. Let  $K$  be a field chosen so that if  $\text{char } K = r$ ,  $G$  has no normal  $r$ -groups and  $r$  is prime to  $pq$ . Let  $A$  be a faithful  $KH$ -module whose non-zero elements are fixed point free under the action of  $V$ . Let  $L$  be a splitting field for all subgroups of  $H$ , where  $[L : K]$  is finite, and form the module  $A \otimes_K L = A'$ . Then  $\text{char } L$  is  $\text{char } K$ . The remainder of the proof simply consists of the observation that  $A'$  is faithful and fixed point free under  $V$ . An application of Theorem 1 then shows that  $G$  satisfies (\*).

**COROLLARY 1.2.** *Let  $G$  be a solvable group admitting  $V$  as a fixed point free group of operators, where  $|V| = pq$  is prime to  $|G|$ . Then for every prime  $r$  dividing  $|G|$ ,  $G/O_{r,r}(G)$  is nilpotent.*

*Proof.* Let  $F_r$  be the Frattini factor group of the  $r$ -group,

$O_{r'r}(G)/O_r(G)$ . Then  $F_r$  is a  $V(G/O_{r'r}(G))$ -module, faithful when restricted to  $G/O_{r'r}(G)$ . Moreover, from Lemma 1.3 (b),  $F_r$  and  $G/O_{r'r}(G)$  are both fixed point free under the action of  $V$ . By Corollary 1.1,  $G/O_{r'r}(G)$  satisfies (\*) and so, by Lemma 2.2, is nilpotent.

**COROLLARY 1.3.** *Let  $G$  be a solvable group admitting  $V$  as a fixed point free operator group and suppose  $|G|$  is prime to  $pq = |V|$ . Then  $G$  has nilpotent length at most two.*

*Proof.* Let  $G \supseteq M(G) \supseteq M^2(G) \supseteq \dots, F(G)$  and  $n(G)$  denote the lower nilpotent series, Fitting subgroup, and nilpotent length of  $G$ , respectively. By Corollary 1.2,  $G/O_{r'r}(G)$  is nilpotent for every  $r$  dividing  $G$ . Thus  $M(G) \subseteq O_{r'r}(G)$  and in general

$$M(G) \subseteq \bigcap_{r|G} O_{r'r}(G) = F(G),$$

(where the intersection is taken over all primes,  $r$ , dividing  $|G|$ ) whence  $M(G)$  is nilpotent. Thus  $M^2(G) = E$  and so  $n(G) \leq 2$ .

**COROLLARY 1.4.** *Let  $G$  be a solvable group admitting  $V$  as a fixed point free group of operators, where  $|V| = pq$  is prime to  $|G|$ . Then  $G$  has  $\pi$ -length at most one, where  $\pi$  is any collection of primes dividing  $|G|$ .*

*Proof.* Since  $F(G) = O_\pi(F(G)) \times O_{\pi'}(F(G))$ ,  $F(G)O_\pi(G)/O_\pi(G)$  is a normal  $\pi$ -subgroup of  $G/O_\pi(G)$  whence  $F(G) \subseteq O_{\pi'}(G)$ . Since, by Corollary 1.3,  $n(G) \leq 2$ ,  $G/F(G)$  is nilpotent and thus its factor group  $G/O_{\pi'}(G)$  is also nilpotent. But in this case,  $G/O_{\pi'}(G)$ , being a nilpotent group with no normal  $\pi$ -groups, is itself a  $\pi'$ -group. Thus  $O_{\pi'\pi'}(G) = G$  and so  $G$  has  $\pi$ -length at most one.

**3. Nilpotence of the commutator subgroup in groups admitting  $S_3$  as a fixed point free group of operators.** Let  $G$  be a group of operators,  $V$ , isomorphic to  $S_3$ , the symmetric group of degree three. Then  $V$  is a metacyclic group of the type discussed in the previous section, with  $p = 2$  and  $q = 3$ . Our object is to show that if  $V$  acts in fixed point free manner on  $G$  and  $G$  is solvable of order prime to 6, then  $G$  is nilpotent. This property is almost entirely the consequence of

**THEOREM 2.** *Let  $G$  be a group of order prime to six admitting  $V = S_3$  as a fixed point free group of operators. Let  $V$  be generated by elements  $w$  and  $v$  such that  $v^2 = w^3 = 1, vw^2 = wv$ . Set  $v_1 = v, v_2 = vw$  and  $v_3 = vw^2$  (all conjugates in  $V$ ). Suppose  $G$  contains three normal subgroups,  $N_1, N_2$ , and  $N_3$  such that*

- (i)  $N_1 \cap N_2 \cap N_3 = E$
- (ii)  $N_i$  is  $v_i$ -invariant,  $i = 1, 2, 3$ ,
- (iii)  $w^2(N_i) = N_{i+1(\text{mod } 3)}$   $i = 1, 2, 3$ ,
- (iv)  $G/N_i$  is fixed elementwise by  $v_i$ ,  $i = 1, 2, 3$ .

Then  $G$  is abelian.

*Proof.* The reader will recognize that (i)–(iv) is the condition (3) in (\*) imposed on the subgroup  $G_2$  in Theorem 1, for the case that  $p = 2$  and  $q = 3$ . Then by Lemma 2.2,  $G$  is nilpotent, since it is fixed point free under the automorphism,  $w$ , of order three. Then, by a theorem of B. H. Neumann  $G$  has nilpotent class 2, i.e.  $G' \subseteq Z(G)$ .

Now let  $H$  be an arbitrary  $V$ -invariant subgroup of  $G$ . Then  $V$  is fixed point free on  $H$ . We now show that the hypotheses (i)–(iv) inherit to  $H$ . Set  $N'_i = H \cap N_i$ ,  $i = 1, 2, 3$ . Then  $N'_1 \cap N'_2 \cap N'_3 = H \cap (N_1 \cap N_2 \cap N_3) = E$ , proving (i). Clearly  $N'_1$  is  $v_1$ -invariant, being the intersection of two  $v_1$ -invariant subgroups of  $G$ . Also,  $w^2(N'_1) = w^2(H \cap N_1) = w^2(H) \cap w^2(N_1) = H \cap N_{1+1} = N'_{2+1}$  (indices taken mod 3). Thus (ii) and (iii) hold. Finally,  $H/N'_i$  is  $v_i$ -isomorphic to  $HN_i/N_i$ , a subgroup of  $G/N_i$ . Since the latter is fixed elementwise by  $v_i$ , so is the former, proving (iv).

Since  $G$  is nilpotent, each of its Sylow subgroups admit  $V$  and satisfy (i)–(iv). If  $G$  is not a  $p$ -group, each of these is proper and, by induction, is abelian. Then their direct product,  $G$ , is also abelian. Thus, without loss of generality, we may assume that  $G$  is a  $p$ -group.

Let  $F$  denote the Frattini factor group,  $G/\mathcal{O}(G)$ . Then  $F$  is a  $V$ -module, and since  $p \nmid |V|$ , by Maschke's theorem,  $F$  is a direct sum of irreducible  $V$ -modules:  $F = F_1 \oplus F_2 \oplus \dots \oplus F_t$ . Let  $G_i$  be chosen so that  $G_i/\mathcal{O}(G) = F_i$ . If  $t > 1$ , each  $G_i$  is a proper  $V$ -invariant subgroup of  $G$  and hence is abelian. In that case  $G \cong C_{\alpha}(\mathcal{O}(G)) \cong \{G_1, \dots, G_t\} = G$  whence  $\mathcal{O}(G) \subseteq Z(G)$ , the center of  $G$ . If, moreover,  $t > 2$ , each of the subgroups  $G_i G_j$  is a proper  $V$ -invariant subgroup of  $G$  and hence is also abelian. In that case,

$$G \cong C_{\alpha}(G_i) \cong \{G_1, \dots, G_i, \dots, G_t\} = G$$

so each  $G_i$  lies in the center of  $G$ , whence  $G$ , which is generated by the  $G_i$ , is itself abelian, and we are done. Thus without loss of generality we may assume  $t \leq 2$ .

Let us take a closer look at the irreducible  $V$ -modules,  $F_i$ . These are modules over the field of  $p$  elements. The kernel of the repre-

sentation of  $V$  which each affords, is a normal subgroup of  $V$  and so is either the identity (in which case each module is faithful) or contains  $W$ , the normal subgroup of order 3. In the latter case,  $F$  contains points fixed by  $W$ . Then by Lemma 1.3 (b),  $G_w \neq E$ , a contradiction. Thus each  $F_i$  is faithful. In this case, we can show that each  $F_i$  is, indeed, 2-dimensional.

First, we observe that if  $p \equiv 2 \pmod 3$ ,

$$(A) \quad v \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

is a faithful irreducible representation of  $V$ . Second, if  $p \equiv 1 \pmod 3$ , there exists an integer  $a \not\equiv 1 \pmod p$  such that  $a^3 \equiv 1 \pmod p$ . Then

$$(B) \quad v \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w \rightarrow \begin{pmatrix} a & 0 \\ 0 & a^2 \end{pmatrix}$$

(where  $a < 3$  is taken as an element of  $GF(p) = Z/(p)$ ) is also a faithful irreducible representation of  $V$  of dimension two. Now each  $F_i$  is isomorphic, as a  $GF(p)V$ -module, to a minimal left ideal of the semi-simple group algebra  $GF(p)V$  of dimension 6. But the modules corresponding to the trivial representation, and to the representation having  $W$  as its kernel are both 1-dimensional and thus account for two one-dimensional minimal left ideals in the direct decomposition of  $GF(p)V$ . This leaves a four-dimensional complement which must contain a two-dimensional minimal left ideal affording one or the other of the representations (A) and (B) given above. Since there are only three conjugate classes in  $V$ , these exhaust the nonisomorphic  $GF(p)V$ -modules. Thus each of the  $F_i$  afford representations equivalent to one of the two matrix representations (A) and (B) given above.

Since  $G' \subseteq Z(G)$ , commutators in  $G$  obey the following laws:

$$(1) \quad \begin{aligned} (x, yz) &= (x, y)(x, z) \\ (xy, z) &= (x, z)(y, z) \\ (x^i, y^j) &= (x, y)^{i+j} \\ (x, y^{-1}) &= (x, y)^{-1} = (x, y) = (x^{-1}, y) . \end{aligned}$$

Now suppose  $t = 1$ . We can no longer assert that  $\mathcal{O}(G)$  lies in the center of  $G$ , although  $\mathcal{O}(G)$  is certainly abelian. Here  $G/\mathcal{O}(G) = F$  is two-dimensional, and so  $G$  is generated by two elements, say  $x_1$  and  $x_2$ . Thus if  $g$  and  $h$  are arbitrary elements in  $G$ , each can be expressed as "words" in  $x_1$  and  $x_2$ , i.e.,

$$g = x_1^{a_1} x_2^{b_1} x_1^{a_2} x_2^{b_2} \cdots x_1^{a_n} x_2^{b_n}$$

$$h = x_1^{c_1} x_2^{d_1} \dots x_1^{c_m} x_2^{d_m} .$$

Then from (1)

$$\begin{aligned} (g, h) &= (x_1, h)^{S a_i} (x_2, h)^{S b_i} \\ &= (x_1, x_2)^{(S d_i)(S a_i) - (S c_i)(S b_i)} \end{aligned}$$

whence  $G'$  is cyclic. But in that case  $G'/\mathcal{O}(G')$  is a one-dimensional  $GF(p)V$ -module, and hence is fixed by  $w$ . Since  $w$  is fixed point free on  $G$ , it is also on  $G'/\mathcal{O}(G')$  whence  $G'/\mathcal{O}(G') = E$ , i.e.  $G' = \mathcal{O}(G') = E$ . Thus  $G$  is abelian.

We are left with the case that  $t = 2$ . Here  $F = F_1 \oplus F_2$ , and  $\mathcal{O}(G)$  lies in the center of  $G$ . Then the commutators  $(x, y)$  all have order  $p$ , for  $(x, y)^p = (x, y^p) = 1$  since  $y^p \in \mathcal{O}(G) \subseteq Z(G)$ . Thus  $G'$  is elementary abelian and can also be regarded as a  $V$ -module. Commutation now defines a  $V$ -homomorphism:  $F \times F \rightarrow G'$ , which, being bilinear in each component, can be factored through  $F \otimes_V F$ . Thus if  $x$  and  $y$  belong to the same left coset of  $\mathcal{O}(G)$  in  $G$ ,  $x = yz$  for some  $z$  in the center. Then  $(x, g) = (yz, g) = (y, g)$  and similarly,  $(g, x) = (g, yz) = (g, y)$ , so the map is well defined in the sense that  $F \times F$  can be regarded as its domain. Since, for any  $u \in V$ ,  $u(x, y) = (u(x), u(y))$ , the map is a  $V$ -homomorphism. For convenience we write the elements of the modules,  $F$  and  $G'$ , additively so that  $(x, y + z) = (x, y) + (x, z)$  and  $(x + y, z) = (x, z) + (y, z)$ .

Now suppose  $p \equiv 2 \pmod 3$ . Then both of the modules  $F_1$  and  $F_2$  afford representations equivalent to (A). Thus we may select a basis  $\{x_1, x_2, x_3, x_4\}$  for  $F$  such that

$$\begin{aligned} v(x_1) &= x_2 & v(x_3) &= x_4 \\ w(x_1) &= x_2 & w(x_3) &= x_4 \\ w(x_2) &= -x_1 - x_2 & w(x_4) &= -x_3 - x_4 . \end{aligned}$$

Let  $\hat{x}_1$  and  $\hat{x}_3$  be elements of  $G$  such that under the homomorphism  $f: G \rightarrow G/\mathcal{O}(G)$ ,  $f(\hat{x}_1) = x_1, f(\hat{x}_3) = \hat{x}_3$ . Then  $f(w(\hat{x}_1)) = x_2$  and  $f(w(\hat{x}_3)) = x_4$ . Then  $G$  is itself generated by  $\hat{x}_1, w(\hat{x}_1), \hat{x}_3$  and  $w(\hat{x}_3)$ . The groups  $G_i$ , chosen so that  $f(G_i) = F_i, i = 1, 2$ , are abelian, whence  $(\hat{x}_1 w(\hat{x}_1)) = 1$  and  $(\hat{x}_3, w(\hat{x}_3)) = 1$ . Thus in module notation

$$(x_1, x_2) = (x_3, x_4) = 0$$

and  $G'$  is generated by the four elements

$$(x_i, x_j) , \quad i = 1, 2; j = 3, 4 .$$

Now



$$(2) \quad w^2(x_1, x_3) = w(x_2, x_4) = (-x_1 - x_2, -x_3 - x_4) = \Sigma(x_i, x_j), \\ i = 1, 2; j = 3, 4.$$

Since  $w$  is fixed point free on  $G'$ , for any element  $c \in G'$ ,

$$(1 + w + w^2)c = c + w(c) + w^2(c) = 0.$$

Setting  $c = (x_1, x_3)$ , we have from (2)

$$(3) \quad 2(x_1, x_3) + 2(x_2, x_4) + (x_1, x_4) = (x_2, x_3) = 0.$$

Similarly, setting  $c = (x_1, x_4)$ , we obtain

$$(4) \quad (1 + w + w^2)(x_1, x_4) = 0 \\ = -(x_1, x_3) - (x_2, x_4) + (x_1, x_4) - 2(x_2, x_3).$$

Solving for  $(x_1, x_4)$  in (4) and substituting for  $(x_1, x_4)$  in (3), we obtain

$$3(x_1, x_3) + 3(x_2, x_4) + 3(x_2, x_3) = 0$$

or

$$(5) \quad (x_1, x_3) + (x_2, x_4) + (x_2, x_3) = 0.$$

Adding (4) to (5) yields

$$(6) \quad (x_1, x_4) = (x_2, x_3).$$

Thus  $G'$  is at most two-dimensional, and from (5) and (6) is generated by  $(x_1, x_3)$  and  $(x_2, x_4)$ .

Now  $N_1$  is a normal subgroup of  $G$  and contains  $(v, G)$ . Then  $v(x_i)x_i^{-1} \in N_1$  for  $i = 1, 2, 3, 4$ . Since  $N_1$  is normal,  $(h, g) \in N_1$  for any  $h \in N_1$  and  $g \in G$ . Thus the commutators  $(v(x_1)x_1^{-1}, x_3)$  and  $(v(x_1)x_1^{-1}, w(x_3))$  lie in  $N_1 \cap G'$ . Thus

$$(x_2 - x_1, x_3) = (x_2, x_3) - (x_1, x_3)$$

and

$$(x_2 - x_1, x_4) = (x_2, x_4) - (x_1, x_4) \\ = (x_2, x_4) - (x_2, x_3),$$

by (6), all belong to  $N_1 \cap G'$ . Thus

$$(x_2, x_3) \equiv (x_2, x_4) \equiv (x_1, x_4) \equiv (x_1, x_3) \pmod{N_1 \cap G'}.$$

Then from (5)  $3(x_1, x_3) \equiv 3(x_2, x_4) \equiv 3(x_2, x_3) \equiv 0 \pmod{N_1 \cap G'}$ . Since  $p \nmid 3$ ,  $(x_1, x_3)$  and  $(x_2, x_4)$ , the generators of  $G'$ , both lie in  $N_1$ . Thus  $N_1 \cong G'$ . Then  $N_2 = w^2(N_1) \cong G'$  and  $N_3 = w(N_1) \cong G'$ . Since the  $N_i$  have trivial meet,  $G' = E$  and  $G$  is abelian.

Now suppose  $p \equiv 1 \pmod 3$ . Then the two irreducible  $V$ -modules,  $F_1$  and  $F_2$ , afford representations of  $V$  equivalent to that given in (B). Thus we may select a basis,  $\{x_1, x_2, x_3, x_4\}$  for which

$$(7) \quad \begin{aligned} v(x_1) &= x_2 & v(x_3) &= x_4 \\ w(x_1) &= ax_1 & w(x_3) &= bx_3 \\ w(x_2) &= a^2x_1 & w(x_4) &= b^2x_4 \end{aligned}$$

where  $a$  and  $b$  are scalars in  $GF(p)$ , different from 1, and satisfying  $a^3 = b^3 = 1$ . Now the multiplicative group of nonzero elements in  $GF(p)$  is cyclic and so has a unique subgroup of order 3. Thus, since  $a$  and  $b$  both belong to this subgroup, either  $a = b$  or  $a = b^2$ .

Let  $\hat{x}_1$  and  $\hat{x}_3$  be chosen so that  $f(\hat{x}_i) = x_i, i = 1, 3$ . Then, setting  $\hat{x}_2 = v(\hat{x}_1)$  and  $\hat{x}_4 = v(\hat{x}_3), f(\hat{x}_j) = x_j$  for  $j = 2, 4$ . If  $G_i$  is chosen so that  $G_i/\mathcal{O}(G) = F_i, i = 1, 2$ , then, by induction, the  $G_i$  are abelian. From this and (7) we have

$$(8) \quad \begin{aligned} (x_1, x_2) &= (x_3, x_4) = 0 \\ w(x_1, x_3) &= ab(x_1, x_3) \\ w(x_1, x_4) &= ab^2(x_1, x_4) \\ w(x_2, x_3) &= a^2b(x_2, x_3) \\ w(x_2, x_4) &= a^2b^2(x_2, x_4) . \end{aligned}$$

If  $a = b, (x_1, x_4)$  and  $(x_3, x_3)$ , being fixed by  $w$ , must be zero. If  $a = b^2, (x_1, x_3)$  and  $(x_2, x_4)$  are zero. In either case,  $G'$  is generated by two elements. By interchanging the symbols representing  $x_3$  and  $x_4$  if necessary, we can, without loss of generality assume that  $a = b$  so that  $(x_1, x_4) = (x_2, x_3) = 0$ .

Since  $v(x_1)x_1^{-1} \in N_1, (x_2x_1^{-1}, x_j) \in N_1 \cap G'$  for  $j = 3, 4$ . Thus

$$\begin{aligned} (x_2 - x_1, x_3) &= (x_2, x_3) - (x_1, x_3) = (x_2, x_3) \equiv 0 \\ (x_2 - x_1, x_4) &= (x_2, x_4) = (x_1, x_4) = -(x_1, x_4) \equiv 0 \pmod{G' \cap N_1} . \end{aligned}$$

Since  $G' = \{(x_1, x_3), (x_1, x_4)\}, N_1 \cong G'$  whence  $G' \subseteq N_1 \cap w(N_1) \cap w^2(N_1) = E$ . Thus  $G$  is abelian.

**COROLLARY 2.1.** *Let  $G$  be a group admitting  $V = S_3$  as a fixed point free group of operators and suppose  $G$  has order prime to  $|V| = 6$ . Then the commutator subgroup of  $G$  is nilpotent.*

*Proof.* The property that  $G'$  is nilpotent is residually complete, and so, since  $V$  is fixed point free on each factor group, we obtain immediate reduction to the case that  $G$  has a unique minimal normal  $V$ -invariant subgroup,  $M$ . Since  $G$  is solvable,  $M \subseteq O_p(G)$  for some

$p$  dividing  $G$ , and  $O_{p'}(G) = E$ . Thus  $O_p(G) = O_{p,p}(G)$ . Now  $V$  is metacyclic of order 6. Since  $G$  is odd, the restriction on Fermat primes does not apply, and hence we may use Corollaries 1.1 and 1.2 to obtain that  $\bar{G} = G/O_{p,p}(G)$  satisfies (\*). Then  $\bar{G} = G_1 \times G_2$  where  $G_1$  is fixed point free under  $v$ , an automorphism of order 2, and  $G_2$  is a group satisfying conditions (i)—(iv) of Theorem 2. Thus both  $G_1$  and  $G_2$  are abelian, whence  $\bar{G}$  is abelian. Thus  $G' \subseteq O_{p,p}(G) = O_p(G)$ , which shows that  $G'$  is a  $p$ -group and hence is nilpotent.

**4. Nilpotence of the commutator subgroup in groups admitting fixed point free operator groups.** In this section we prove the impossibility of extending the results of Corollary 2.1 to solvable groups,  $V$ , other than those already considered. We begin with

**THEOREM 3.** *Let  $V$  be a solvable group satisfying one of the following properties*

(a)  *$V$  contains a normal subgroup  $W \neq E$  such that  $[V:W]$  is an odd prime,  $p$ .*

(b)  *$V$  has a factor group of order 4,  $|V| \neq 4$ .*

(c)  *$V$  has a dihedral factor group of order  $2p$ ,  $p \geq 5$ .*

*Then there exists a group,  $G$ , having order prime to  $|V|$  which admits  $V$  as a fixed point free group of operators and for which the commutator subgroup is not nilpotent.*

*Proof. Case I. ( $V$  satisfies (a)).*

Since  $V$  is solvable,  $W'$  is a proper subgroup of  $W$ , normal in  $G$ . Select  $U$  maximal with respect to the properties:  $W' \subseteq U \subset W$ , and  $U \triangleleft V$ . Then  $V/U$  is either abelian of order  $p^2$  or  $pq$  or it is metabelian of order  $pq^e$  where  $e$  is the exponent of  $p \bmod q$  defined by letting an element of order  $p$  act irreducibly on the elementary abelian group  $(W/U)$  of order  $q^e$ .

Let  $G$  be a group of order  $r^3s^{r^2}$  having a normal elementary abelian subgroup,  $A$ , of order  $s^{r^2}$  and factor group  $G/A$  isomorphic to the extraspecial group of order  $r^3$ . The primes,  $r$  and  $s$  are chosen so that  $r \equiv 1 \pmod p$  and  $s \equiv 1 \pmod{rq}$  (or  $rp$  if  $[V:U] = p^2$ ). Since  $r \neq s$ ,  $G$  splits over  $A$  and we may write  $G = AR$  where  $R$  is generated by two elements  $x$  and  $y$  such that  $x^r = y^r = 1 = z^r$ , where  $z = (x, y)$  generates the center of  $R$ .

$V$  acts on  $G$  as follows: First  $U$  acts trivially on  $G$ , and  $W$  acts trivially on  $R$ . If  $v$  generates  $V \bmod W$ , set  $v(x) = x^a$ ,  $v(y) = y^a$ ,  $v(z) =$

$z^{a^2}$  where  $a$  is a primitive  $p$ th root mod  $r$ . (Such a root exists since  $r \equiv 1 \pmod p$ .) The action of  $V$  and  $R$  on  $A$  is defined by writing the elements of  $A$  additively, selecting a basis  $a_{11}, a_{12}, \dots, a_{ij}, \dots, a_{r^p, r^p}$  for  $A$ , and letting  $\{a_{11}, a_{12}, \dots, a_{1r}\}$  afford the representation  $\rho_1$  of  $(W/U)R$ , defined by

$$\rho_1(z) = \beta I_r, \quad \rho_1(w_1) = \gamma I_r, \quad \rho_1(w_i) = I_r, \quad i > 1$$

where  $w_1, \dots, w_e$  are a basis for  $W/U$  ( $e = 1$ , and the last matrix is not involved if  $[W:U] = q$  or  $p$ .)  $I_r$  is the  $r$  by  $r$  identity matrix, and  $\beta$  and  $\gamma$  are respectively primitive  $r$ th roots and  $q$ th roots (or  $p$ th if  $[W:U] = p$ ) modulo  $s$ . Also,

$$\rho_1(x) = \text{diag}(1, \beta^{-1}, \dots, \beta^{-r+1})$$

and

$$\rho_1(y) = \begin{pmatrix} 0 & 1 & & 0 \\ & & 1 & \cdot \\ & & & \cdot \\ & & & & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix}$$

If  $H$  denotes the semidirect product  $(V/U)R$ , set

$$\rho(h) = \text{diag}(\rho_1(h), \rho_1(vhv^{-1}), \dots, \rho_1(v^{p-1}hv^{1-p}))$$

for  $h \in (W/U)R$ , and

$$\rho(v) = \begin{pmatrix} 0 & I_r & 0 & \dots & 0 \\ & & I_r & \cdot & \\ & & & \cdot & \\ & & & & I_r \\ uI_r & 0 & 0 & \dots & 0 \end{pmatrix}$$

where  $u = \gamma$  if  $V/U$  is cyclic of order  $p^2$  and 1 otherwise. This completely defines the action of  $VR$  on  $A$ .

$W$  acts in fixed point free manner on  $A$  since, on each component  $\{a_{i1}, a_{i2}, \dots, a_{ir}\}$  it is represented as scalar multiplication by  $\gamma$ , (the kernel of the representation of  $W/U$  may differ on each component, of course.) Since  $p$  is odd,  $a^2 \equiv 1 \pmod r$ , and so  $v$  acts in fixed point free manner on  $R$ . Summing up, then,  $V$  is fixed point free on  $G/A \simeq R$  whence  $G_v \subseteq A$ . Again,  $G_v \subseteq A_v = E$ , whence  $V$  is fixed point free on  $G$ .

Note that  $G' = A\{z\}$  is not nilpotent.

Case II.  $V$  satisfies (b).

Let  $W$  be the nontrivial subgroup of order 4, and select  $U$  so that  $W' \subseteq U \subset W, U \triangleleft V$  and  $U$  is maximal in this respect. We shall define a group,  $G$ , admitting  $V$  as a group of operators in such manner that  $U$  acts trivially on  $G$ .  $G$  will have the form  $QM$ , where  $M$  is an elementary abelian normal subgroup of  $G$  and  $Q$  is a Hall complement of  $M$  in  $G$ , such that  $M$  becomes a sum of faithful irreducible  $Q$ -modules, so that  $[Q, M] = M$ . Moreover,  $M$  will be fixed point free under the action of  $W$  alone, while at the same time  $Q$  will be  $V$ -invariant and centralized by  $W$ . In this way,  $V$  induces the group  $V/W$  of order 4 on the group  $Q$ . Now  $V/U$  has the normal series,  $E = U/U \triangleleft W/U \triangleleft V_0/U \triangleleft V/U$ , and we select elements  $v_1$  and  $v_2$  in  $V$  so that  $v_1$  generates  $V_0 \text{ mod } W$  and  $v_2$  generates  $V \text{ mod } V_0$  and  $v_2^2 \equiv 1$  or  $v_1 \pmod{W}$  according as  $V/W$  is the 4-group or is cyclic. Now we define  $Q$  as follows. Let  $r$  and  $s$  be odd primes, such that  $(rs, [V:U]) = 1$  and  $s \equiv 1 \pmod{r}$ . Then  $Q$  is a group of order  $rs^2$  defined by

$$\begin{aligned} x^r &= 1 = y_1^s = y_2^s \\ x^{-1}y_1x &= y_1^a, x^{-1}y_2x = y_2^{a^{-1}} \end{aligned}$$

where  $a$  and  $a^{-1}$  are primitive  $r$ th roots mod  $s$ , such that  $aa^{-1} \equiv 1 \pmod{s}$ . Then the action of  $V/W$  on  $Q$  is defined by

$$\begin{aligned} v_2(x) &= x^{-1}, v(x) = x \text{ for all } v \in V_0. \\ v_1(y_i) &= y_i^{-1}, i = 1, 2 \\ v_2(y_1) &= y_2 \text{ and } v_2(y_2) = \begin{cases} y_1 & \text{if } v_2^2 \equiv 1 \pmod{W} \\ y_1^{-1} & \text{if } v_2^2 \equiv v_1 \pmod{W} \end{cases} \end{aligned}$$

$W$  acts trivially on  $Q$ . Then it is easily seen that  $Q$  is fixed point free under  $V/W$  and has nilpotent length 2.

Now  $W/U$  has order 2,  $p$  or  $p^2$ , where  $p$  is an odd prime, and acts as an irreducible  $V/W$ -module. We can then find a factor system,  $m_{ij} \in W/U$ , such that  $v_i v_j \equiv u m_{ij} \pmod{U}$  where  $u$  is the appropriate coset representative,  $1, v_1, v_2, v_1 v_2$ , of  $W/U$  in  $V/U$ . Now let  $t$  be a prime different from 2,  $r$  and  $s$  such that  $t \equiv 1 \pmod{p}$  if  $W/U$  has order  $p$  or  $p^2$ . Let  $M_1$  be a faithful irreducible  $Q$ -module over  $GF(t)$ . We make  $M_1$  into an irreducible  $(U/W)Q$ -module by letting  $W/U$  act nontrivially by scalar multiplication by  $-1$  or by a  $p$ th root according as  $[W:U] = 2$  or is odd. Then, as  $W$  acts trivially on  $Q, WQ \triangleleft VQ$ . Set

$$M = M_1^{r^2} \simeq M_1 \oplus v_1 M_1 \oplus v_2 M_1 \oplus v_1 v_2 M_1 \simeq M_1 \otimes_{WQ} VQ$$

the induced module. Then  $W$  has a conjugate representation which is also scalar multiplication on each component  $xM_1, x = 1, v_1, v_2,$  or  $v_1 v_2$ . Now set  $G$  to be the semidirect product  $QM$ , where  $M$  is a

normal abelian subgroup on which  $VQ$  acts in a manner prescribed by the module construction of  $M$ . Then  $M$  has order prime to  $|VQ|$ , and is a sum of conjugate faithful irreducible  $Q$ -modules, whence  $[Q, M] = M$ . Thus  $M$  is the Fitting subgroup of  $G$  and  $G$  has nilpotent length three.  $V/U$  has order prime to  $G$ ,  $V$  is fixed point free on both  $Q \simeq G/M$  and  $M$  and hence is fixed point free on  $G$ , by the remark following Lemma 1.2. Evidently  $G'$  is not nilpotent.

*Case III.*  $V$  satisfies (c).

$V$  contains a normal subgroup  $U$  such that  $V/U$  is dihedral of order  $2p$ , where  $p \geq 5$ .

In this example we let  $G = RQ$  where  $R$  is a normal elementary abelian  $r$ -group and  $Q$  is a special  $q$ -group of order  $q^8$ . We select the primes  $q$  and  $r$  so that  $r \equiv 1 \pmod q$  and  $q \equiv 1 \pmod p$ , both  $q$  and  $r$  odd.  $Q$  is generated by four elements  $x_1, x_2, x_3, x_4$  subject to the rules:

$$\begin{aligned} x_i^q &= 1, i = 1, 2, 3, 4, & (x_1, x_2) &= (x_3, x_4) = 1 \\ z_1 &= (x_1, x_3), z_2 = (x_1, x_4), z_3 = (x_2, x_3), z_4 = (x_2, x_4) \\ z_i^q &= 1, i = 1, 2, 3, 4, & \{z_1, z_2, z_3, z_4\} &= Z(Q). \end{aligned}$$

$R$  will consist of  $p$  irreducible  $Q$ -modules, each of which has dimension  $q$ . Thus  $R$  has order  $r^{pq}$ .

The action of  $V$  on  $G$  is defined as follows: First,  $U$  is assumed to act trivially on  $G$  so that  $G$  admits  $\bar{V} = V/U$ , the dihedral group of order  $2p$ , as a group of operators. Let  $V$  be generated by elements  $v$  and  $w$  such that  $v^2 = 1, w^p = 1$ , and  $vw = w^{-1}v$ . Since  $p \geq 5$ , we can find four primitive  $p$ th roots modulo  $q, a, a^{-1}, b, b^{-1}$ , such that  $aa^{-1} \equiv bb^{-1} \equiv 1 \pmod q$  and  $b$  is incongruent to both  $a$  and  $a^{-1} \pmod q$ . Then we set

$$w(x_1) = x_1^a, w(x_2) = x_2^{a^{-1}}, w(x_3) = x_3^b, w(x_4) = x_4^{b^{-1}}.$$

Then  $w$  acts in fixed point free manner on  $Q/Z(Q)$ . We must also have

$$w(z_1) = z_1^{ab}, w(z_2) = z_2^{ab^{-1}}, w(z_3) = z_3^{a^{-1}b}, w(z_4) = z_4^{a^{-1}b^{-1}}.$$

The action of  $v$  is given by

$$\begin{aligned} v(x_1) &= x_2, v(x_2) = x_1, v(x_3) = x_4, v(x_4) = x_3 \\ v(z_1) &= z_4, v(z_2) = z_3, v(z_3) = z_2, v(z_4) = z_1 \end{aligned}$$

so that both  $Q/Z(Q)$  and  $Z(Q)$  are the direct sum of two irreducible  $V$ -modules. Form the subgroup  $Q_1 = (v, Q)$ . This subgroup is generated by the elements  $x_1x_2^{-1}, x_3x_4^{-1}, z_1z_4^{-1}, z_2z_3^{-1}$ , and  $(x_1x_2^{-1}, x_3) = z_1z_3^{-1}$ , and has order  $q^5$ . Then  $Q/Z(Q)$  is an extra special  $q$ -group of order  $q^3$ , generated (mod  $Q_1$ ) by the elements  $x_1x_2$  and  $x_3x_4$ , with center generated (mod  $Q_1$ ) by  $z_1z_2z_3z_4$ .  $Q/Q_1$  is fixed elementwise by  $v$ .

Writing  $R$  additively we may regard  $R$  as a  $QV$ -module. We take  $R$  to be the direct sum of  $p$  irreducible  $Q$ -modules,  $R_1, \dots, R_p$ , which are permuted by  $V$  according to the rules

$$w^{-1}R_jw = R_{j+1} \quad (\text{indices take mod } p)$$

$$v^{-1}R_1v = R_1, v^{-1}R_2v = R_p, v^{-1}R_3v = R_{p-1}, \dots, v^{-1}R_{(p-1)/2}v = R_{(p+1)/2}$$

so that the manner in which the  $R_i$  are permuted by  $V$  provides a faithful transitive permutation representation of  $V$  of degree  $p$ .  $v$  is assumed to act on  $R_1$  by scalar multiplication by  $-1$ .  $Q$  is represented irreducibly on  $R_1$  with kernel  $Q_1$ , that is  $R_1$  represents  $Q/Q_1$  faithfully. The matrices are

$$\rho(x_1) = \rho(x_2) = \text{diag} (1, c^{-1}, \dots, c^{-q+1})$$

$$\rho(x_3) = \rho(x_4) = \begin{pmatrix} 0 & 1 & \dots & 0 \\ & 1 & & \\ & & \ddots & \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

$$\rho(z_1) = \rho(z_2) = \rho(z_3) = \rho(z_4) = cI_q$$

where  $c$  is a primitive  $q$ th root modulo  $r$ , and  $I_q$  denotes the  $q$  by  $q$  identity matrix. By defining the representation of  $Q$  on  $R_i$  as the conjugate representation under  $w^{i-1}$ , the representation of  $QV$  on  $R$  is completely defined.

We next observe that  $V$  is fixed point free on  $G$ . First  $w^{-i}vw^i = vw^{2i}$  leaves  $R_{i+1} = w^{-i}R_1w^i$  invariant and for any element,  $e$ , in  $R_{i+1}$ , we have  $e = w^{-i}e', w^i$  for some  $e'$  in  $R_1$ . Then

$$\begin{aligned} (w^{-i}vw^i)^{-1}e(w^{-i}vw^i) &= (w^{-i}v^{-1}w^i)w^{-i}e'w^i(w^{-i}vw^i) \\ &= w^{-i}v^{-1}e'vw^i = w^{-i}(-e')w^i \\ &= -w^{-i}e'w^i = -e. \end{aligned}$$

Thus  $vw^{2i}$  acts on  $R_{i+1}$  by scalar multiplication by  $-1$ . Now suppose  $h$  is an element of  $R$  fixed by  $V$ . Then we may write  $h$  uniquely in the form  $h = h_1 + h_2 + \dots + h_p$ , where  $h_i \in R_i$ . Then  $w^{-i}vw^i$  sends each  $h_j \in R_j$  into some element of  $R_k$  where  $k \neq j$  unless  $j = i + 1$ . But since it fixes  $h$  it must fix  $h_{i+1}$ . On the other hand it acts on

$R_{i+1}$  by scalar multiplication by  $-1$ . Since  $r \neq 2$ , this implies that  $h_{i+1} = 0$ . Repeating this argument for each  $i = 1, 2, 3, \dots, p$ , we have that  $h = 0$ . Thus  $V$  is fixed point free on  $R$ . But  $V$  is also fixed point free on  $Q/Z(Q)$  and  $Z(Q)$  whence it must be fixed point free on all of  $G$ .

In each case  $Z(Q)$  is represented on  $R_i$  by scalar multiplication by  $c$ ; thus  $(Z(Q), R) = R \subseteq G'$  and  $Z(Q) = Q' \subseteq G'$ , whence  $G'$  contains the subgroup  $RZ(Q)$ , which is not nilpotent.

**COROLLARY 3.1.** *Let  $V$  be a solvable group containing a non-trivial subgroup  $W$  such that  $W$  is normal in  $V$  and  $V/W$  is the symmetric group of degree three. Then there exists a group  $G$  having order prime to  $|V|$ , admitting  $V$  as a fixed point free group of operators, such that  $G'$  is not nilpotent.*

*Proof.* This case is not directly subsumed under those cases listed in Theorem 3, but the required example is easily provided by that theorem. Let  $V^*$  be the unique subgroup of index 2 in  $V$  containing  $W$ . Then  $V^*$  is a solvable group containing a nontrivial normal subgroup, namely  $W$ , of index 3, which is an odd prime. Thus  $V^*$  satisfies (a) of Theorem 3. Accordingly, there exists a group  $G_1$  having order prime to  $V^*$  which admits  $V^*$  as a fixed point free group of operators and which has a commutator subgroup which is not nilpotent. Let the element  $v$  generate  $V$  module  $V^*$ , so that  $v^2$  is an element of  $V^*$ . Also, let  $G_2$  be an isomorphic copy of  $G_1$  and let  $f$  be the isomorphism  $f: G_1 \rightarrow G_2$ . Then if  $H = G_1 \times G_2$ , the action of  $V$  on  $H$  can be defined as follows:  $G_1$  already admits  $V^*$ . Let  $v(g) = f(g)$  for every  $g \in G_1$ , and  $v(g) = v^2(k)$ , where  $g = f(k)$ , for every  $g \in G_2$ . The latter is well defined since  $f$  is onto and one to one (making  $k$  unique) and  $v^2$  is an element of  $V^*$  whose action on  $G_1$  is already known. For any  $u \in V^*$ , and  $g \in G_2$  we define  $u(g)$  to be  $f[(v^{-1}uv)(k)]$  where  $f(k) = g$ ; thus, writing  $v$  for  $f$  when the domain of  $v$  is in  $G_1$ , this becomes  $v[v^{-1}uv(v^{-1}(g))] = u(g)$  so that  $V^*$  can in this way be regarded as a group of operators applying to  $H$ . Clearly  $V^*$  acts in fixed point free manner on the subgroup  $G_2$  since, if  $f(k) = g$  and  $V^*$  fixes  $g$ ,  $v^{-1}V^*v = V^*$  must fix  $k$ , which is impossible unless  $k$  (and hence  $g$ ) is the identity. Thus  $V$  acts in fixed point free manner on  $H = G_1 \times G_2$ . Now by hypothesis  $G_1'$  is not nilpotent, and hence its isomorphic copy,  $G_2'$  is also not nilpotent. But it is obvious that both of these subgroups lie in  $H'$ , whence  $H'$  is not nilpotent.

**THEOREM 4.** *Let  $V$  be a solvable group with the property that*



if  $G$  is any group admitting  $V$  as a fixed point free group of operators and  $G$  has order prime to  $|V|$ , then  $G$  is always nilpotent. Then  $V$  is one of the following groups:

- (i)  $V$  is cyclic of prime order
- (ii)  $V$  is one of the groups of order 4
- (iii)  $V$  is the symmetric group of degree three.

*Proof.* Case I.  $V$  has a factor group of odd prime order.

If  $W \triangleleft V$  and  $V/W$  has odd prime order, then by Theorem 3, since  $V$  satisfies (a) if  $W \neq E$ , we must suppose that  $V$  is cyclic of prime order.

Case II.  $V$  has no factor groups of odd prime order.

Since  $V$  is solvable, it contains a normal subgroup  $V_1$  of prime index, and because of the case division the prime must be 2. If  $V_1 = E$ ,  $V$  is cyclic of order 2, and so  $V$  enjoys (i). Thus we may take  $V_1 \neq E$ . Select  $V_2$  maximal with respect to being a proper subgroup of  $V_1$  and normal in  $V$ . Then, since  $V$  is solvable,  $V_1/V_2$  is elementary abelian, and is irreducible as a  $V/V_1$ -module. If  $[V_1:V_2]$  is a power of 2 it is in fact equal to 2, so  $V/V_2$  is a group of order 4. Then if  $V_2 \neq E$ ,  $V$  satisfies (b) and so by Theorem 3, we would be confronted with a counter example to the hypothesis of this theorem. Thus we must suppose, in this case, that  $V_2 = E$ , whence (ii) holds. If  $[V_1:V_2]$  is not a power of 2, it is an odd prime,  $p$ . If  $p \geq 5$ , by Theorem 3, the hypothesis would be denied. Thus  $p = 3$ . Then if  $V_2 \neq E$ , by Corollary 3.1, the hypothesis is once more denied. Consequently,  $V_2 = E$  and  $V$  is the symmetric group of degree three.

**COROLLARY 4.1.** *The condition that  $V$  is solvable can be dropped in Theorem 4.*

*Proof.* Let  $T$  be a proper subgroup of  $V$  and suppose that  $T$  is not cyclic of prime order, does not have order 4 and is not isomorphic to  $S_3$ , the symmetric group of degree 3. Let  $1 = x_1, x_2, \dots, x_k$  be a full set of right coset representatives of  $T$  in  $V$ . By induction on the order of  $T$ , there exists a group,  $G_1$ , fixed point free under  $T$  such that  $G_1'$  is not nilpotent. Let  $G$  be the formal set of  $k$ -tuples  $(g_1, x_2^{-1}g_2x_2, \dots, x_k^{-1}g_kx_k)$ , where the  $g_i$  lie in  $G_1$ . Under the rule that  $x_i x_i^{-1} = 1$ ,  $G$  becomes a group (under component-wise multiplication) isomorphic to a direct product of  $k$  copies of  $C_1$ . By defining  $t^{-1}gt = g^t$  for all  $t \in T$  and  $g \in G_1$  (the exponential notation indicating that  $t$

acts as an operator in the manner given by the induction hypothesis), the action of  $V$  on  $G$  is defined by componentwise conjugation. Then it is easy to verify that  $V$  is fixed point free on  $G$ , and that  $G'$  is not nilpotent. We may thus suppose that any proper subgroup of  $V$  is either cyclic of prime order, has order 4, or is  $S_3$ .

By Theorem 4, it now suffices to show that  $V$  is solvable. Assume  $V$  is not solvable. Then a 2-Sylow subgroup,  $S_2$ , of  $V$ , being proper, has order 2 or 4. If  $N_V(S_2) = V$ ,  $V/S_2$  is metacyclic and hence  $V$  is solvable. If  $N_V(S_2) < V$ , it is clear that since  $A_4$  is not a proper subgroup of  $V$ ,  $S_2$  lies in the center of its normalizer and so  $V$  has a normal 2-complement,  $K$ , which is also metacyclic and hence is solvable. Thus  $V$  is solvable, contrary to our assumption.

#### REFERENCES

1. S. Bauman, *The Klein group as a fixed point free automorphism group*, Ph. D. Thesis, University of Illinois, 1962.
2. A. H. Clifford, *Representations induced in an invariance subgroup*, Ann. of Math. **38** (1937), 533-550.
3. G. Glauberman, *Fixed point in groups with operator groups*, Math. Zeit. **84** (1964), 120-125.
4. D. Gorenstein and I. N. Herstein, *Finite groups admitting a fixed point free automorphism of order 4*. Amer. J. Math. **83** (1961), 71-78.
5. F. Hoffman, *Nilpotent height of finite groups admitting fixed-point free automorphisms*, Math. Zeit. **85** (1964), 260-267.
6. E. Shult, *On groups admitting fixed point free abelian operator groups*, Ill. J. Math. **9** (1965), 701-720.
7. J. G. Thompson, *Finite groups with fixed point free automorphisms of prime order*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578-581.
8. ———, *Automorphisms of solvable groups*, J. of Alg. **1** (1964), 259-267.

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