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## **CHARACTERIZATION OF THE SUBDIFFERENTIALS OF CONVEX FUNCTIONS**

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Each lower semi-continuous proper convex function  $f$  on a Banach space  $E$  defines a certain multivalued mapping  $\partial f$  from  $E$  to  $E^*$  called the subdifferential of  $f$ . It is shown here that the mappings arising this way are precisely the ones whose graphs are maximal "cyclically monotone" relations on  $E \times E^*$ , and that each of these is also a maximal monotone relation. Furthermore, it is proved that  $\partial f$  determines  $f$  uniquely up to an additive constant. These facts generally fail to hold when  $E$  is not a Banach space. The proofs depend on establishing a new result which relates the directional derivatives of  $f$  to the existence of approximate subgradients.

Let  $E$  be a topological vector space over the real numbers  $R$  with dual  $E^*$ . Let  $f$  be a *proper convex function* on  $E$ , i.e., an everywhere-defined function with values in  $(-\infty, +\infty]$ , not identically  $+\infty$ , such that

$$(1.1) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x$  and  $y$  in  $E$  when  $0 < \lambda < 1$ . A *subgradient* of  $f$  at  $x \in E$  is an  $x^* \in E^*$  such that

$$f(y) \geq f(x) + \langle y - x, x^* \rangle \quad \text{for all } y \in E.$$

(This says that  $f(x)$  is finite and that the graph of the affine function  $h(y) = f(x) + \langle y - x, x^* \rangle$  is a "nonvertical" supporting hyperplane at  $(x, f(x))$  to the epigraph of  $f$ , which is the convex subset of  $E \oplus R$  consisting of all the points lying above the graph of  $f$ .) For each  $x \in E$ , we denote by  $\partial f(x)$  the set of all subgradients of  $f$  at  $x$ , which is a weak\* closed convex set in  $E^*$ . If  $\partial f(x) \neq \emptyset$ ,  $f$  is said to be *subdifferentiable* at  $x$ . The *subdifferential* of  $f$  is the multivalued mapping (relation)  $\partial f$  which assigns the set  $\partial f(x)$  to each  $x$ .

The notion of subdifferentiability has been developed recently in [3], [6], [8], [11]. Much of the work has concerned the existence of subgradients. It is known, for example, that  $f$  is subdifferentiable wherever it is finite and continuous (see [6] or [8]). Results in [3] show among other things that, if  $E$  is a Banach space and  $f$  is lower semi-continuous (l.s.c.), then the set of points where  $f$  is subdiffer-

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entiable is dense in the effective domain of  $f$  (which is the convex set of all  $x$  such that  $f(x) < +\infty$ ). The present paper, however, will be concerned instead with general properties of the subdifferentials of l.s.c. proper convex functions, as relations on  $E \times E^*$ .

Some facts are already known about the global nature of subdifferentials. The most remarkable from a geometric point of view is the following.

**THEOREM (Moreau [11]).** *Let  $f$  be a l.s.c. proper convex function on a real Hilbert space  $H$ . Then the addition mapping  $(x, x^*) \rightarrow x + x^*$  from  $H \times H$  to  $H$  maps the graph of  $\partial f$  homeomorphically onto  $H$ .*

Among the many interesting consequences of this theorem is the arcwise connectedness of the set of points where  $f$  is subdifferentiable. Whether this arcwise connectedness is present when  $E$  is not a Hilbert space, is an open question. In the case where  $f$  is everywhere finite and Gateaux differentiable,  $\partial f$  reduces to the ordinary single-valued gradient mapping  $\nabla f$ . Then Moreau's theorem says that the equation

$$(1.2) \quad x + \nabla f(x) = u$$

has a unique solution  $x$  for each  $u \in H$ , and that this solution depends continuously on  $u$ .

Similar results have been arrived at independently through the study of monotone relations. A relation  $\rho$  on  $E \times E^*$  is said to be *monotone* if

$$\langle y - x, y^* - x^* \rangle \geq 0$$

holds whenever  $x^* \in \rho(x)$  and  $y^* \in \rho(y)$ . A *maximal* monotone relation is one whose graph is not properly contained in the graph of another monotone relation. The following theorem invites comparison with the one of Moreau.

**THEOREM (Minty [7]).** *Let  $\rho$  be a maximal monotone relation on  $H \times H$ , where  $H$  is a real Hilbert space. Then the addition mapping  $(x, x^*) \rightarrow x + x^*$  from  $H \times H$  to  $H$  maps the graph of  $\rho$  homeomorphically onto  $H$ .*

We shall prove in § 5 that, if  $E$  is a Banach space, the subdifferential  $\partial f$  of each l.s.c. proper convex function  $f$  on  $E$  is a maximal monotone relation  $\rho$ . In particular, Moreau's theorem can be viewed as a special case of Minty's theorem. The maximal monotonicity of  $\partial f$  has previously been verified by Minty [6] in the case where  $f$  is also everywhere finite and continuous. Special connections between

convexity and monotonicity have also been noted by Kachurovskii [5]. We are indebted to the referee for bringing this latter paper to our attention.

Not every monotone relation arises from a convex function. For instance, every positive semi-definite linear mapping  $\rho$  on a real Hilbert space is a (single-valued) monotone relation, but such a mapping is the subdifferential  $\partial f$  of a proper convex function if and only if it is also self-adjoint. In general, one is led to ask for properties which characterize the relations which are subdifferentials.

In §2 we shall show that a given relation on  $E \times E^*$  is embedded in the subdifferential of some proper convex function on  $E$  if and only if it is “cyclically monotone” in a certain sense. When  $E$  is a Banach space, the subdifferentials of the l.s.c. proper convex functions on  $E$  turn out to be precisely the maximal cyclically monotone relations on  $E \times E^*$ . This will be proved in §4.

Our Banach space theorems depend heavily on the fundamental existence lemma for subgradients in [3]. They also make use of a new result in §3 which describes the directional derivatives of  $f$  in terms of the “approximate subgradients” introduced in [3].

**2. Embedding problem.** Let  $\rho$  be a relation on  $E \times E^*$ . When is  $\rho$  embedded in a subdifferential, i.e., when does there exist a proper convex function  $f$  such that  $\rho(x) \subseteq \partial f(x)$  for all  $x$ ? This can also be viewed as a kind of “integration” problem: given a set of pairs  $\{(x_i, x_i^*), i \in I\}$  in  $E \times E^*$  (namely, the graph of  $\rho$ ), one seeks a proper convex function  $f$  satisfying the “differential” conditions

$$f(y) \geq f(x_i) + \langle y - x_i, x_i^* \rangle, i \in I,$$

for all  $y \in E$ .

There is a simple necessary condition which  $\rho$  must satisfy if the embedding problem is to have a solution. Indeed, if  $f$  is a proper convex function on  $E$  and if

$$x_i^* \in \rho(x_i) \subseteq \partial f(x_i), \quad i = 0, 1, \dots, n,$$

then

$$\infty > f(x_j) \geq f(x_i) + \langle x_j - x_i, x_i^* \rangle$$

for all  $i$  and  $j$  and hence

$$(2.1) \quad 0 \geq \langle x_0 - x_n, x_n^* \rangle + \dots + \langle x_2 - x_1, x_1^* \rangle + \langle x_1 - x_0, x_0^* \rangle.$$

A relation  $\rho$  which satisfies (2.1) for every set of  $n + 1$  pairs in its graph will be called *monotone of degree  $n$* . A relation which is monotone of degrees  $n$  for all  $n > 0$  will simply be called *cyclically monotone*. The condition can also be stated as follows:  $\rho$  is cyclically

monotone if and only if

$$\sum \langle x_i, x_i^* \rangle \geq \sum \langle x_{\sigma(i)}, x_i^* \rangle$$

for every finite set of points in the graph of  $\rho$  and every permutation  $\sigma$ .

Monotonicity of degree  $n$  implies monotonicity of degree  $m$  for all  $m \leq n$ . Note that  $\rho$  is monotone of degree 1 if and only if it is a *monotone relation*. Thus every cyclically monotone relation is monotone. In the one-dimensional case, the converse is also true: every monotone relation is cyclically monotone. It is easy to see, however, that this is false when  $E = R^n$  with  $n > 1$ . The following conjecture does seem plausible, though: if  $E = R^n$ , then each relation on  $E \times E^*$  which is monotone of degree  $n$  is actually monotone of all degrees, i.e., is cyclically monotone. We have not seriously investigated this question.

The cyclic monotonicity condition can be viewed heuristically as a discrete substitute for two classical conditions: that a smooth convex function has a positive semi-definite second differential, and that all circuit integrals of an integrable vector field must vanish.

**THEOREM 1.** *Let  $E$  be a topological vector space, and let  $\rho$  be a relation on  $E \times E^*$ . In order that there exist a proper convex function  $f$  on  $E$  such that  $\partial f \supseteq \rho$ , it is necessary and sufficient that  $\rho$  be cyclically monotone.*

*Proof.* The necessity of the condition was demonstrated in the preceding remarks. To prove the sufficiency, we suppose, therefore, that  $\rho$  is a cyclically monotone relation. There is no loss of generality if we also suppose  $\rho$  is nonempty and fix some  $x_0 \in E$  and  $x_0^* \in E^*$  with  $x_0^* \in \rho(x_0)$ . For each  $x \in E$ , let

$$f(x) = \sup \{ \langle x - x_n, x_n^* \rangle + \cdots + \langle x_1 - x_0, x_0^* \rangle \},$$

where  $x_i^* \in \rho(x_i)$  for  $i = 1, \dots, n$  and the supremum is taken over all possible finite sets of such pairs  $(x_i, x_i^*)$ . We shall show that  $f$  is a proper convex function with  $\partial f \supseteq \rho$ .

Note first that  $f$  is a supremum of a nonempty collection of affine functions, one for each choice of  $(x_1, x_1^*), \dots, (x_n, x_n^*)$ . Hence  $f(x) > -\infty$  for all  $x$  and the convexity condition (1.1) is satisfied. Furthermore,

$$f(x_0) = 0$$

because  $\rho$  is cyclically monotone. Hence  $f$  is a proper convex function.

Next, choose any  $\bar{x}$  and  $\bar{x}^*$  with  $\bar{x}^* \in \rho(\bar{x})$ . We shall show that  $\bar{x}^* \in \partial f(\bar{x})$ . It is enough to show that, for each  $\alpha < f(\bar{x})$ , we have

$$(2.2) \quad f(x) \geq \alpha + \langle x - \bar{x}, \bar{x}^* \rangle \quad \text{for all } x.$$

Given  $\alpha < f(\bar{x})$ , we can choose (by the definition of  $f$ ) pairs  $(x_i, x_i^*)$  such that  $x_i^* \in \rho(x_i)$  for  $i = 1, \dots, k$ , and

$$(2.3) \quad \alpha < \langle \bar{x} - x_k, x_k^* \rangle + \dots + \langle x_1 - x_0, x_0^* \rangle.$$

Let  $x_{k+1} = \bar{x}$  and  $x_{k+1}^* = \bar{x}^*$ . Then

$$f(x) \geq \langle x - x_{k+1}, x_{k+1}^* \rangle + \langle x_{k+1} - x_k, x_k^* \rangle + \dots + \langle x_1 - x_0, x_0^* \rangle$$

for all  $x$  by the definition of  $f$ . This implies (2.2) via (2.3). Therefore  $\partial f \supseteq \rho$  and the theorem has been proved.

**COROLLARY 1.** *If  $\rho$  is a maximal cyclically monotone relation, then  $\rho = \partial f$  for some proper convex function  $f$ .*

By a *maximal* cyclically monotone relation, we of course mean one whose graph is not properly contained in the graph of any other cyclically monotone relation. Thus Corollary 1 follows immediately from Theorem 1.

It is easy to prove, using Zorn's lemma, that every cyclically monotone relation is embedded in a maximal one. Since every subdifferential is cyclically monotone, we may, therefore, also state:

**COROLLARY 2.** *If  $f_1$  is any proper convex function on  $E$ , then there exists a proper convex function  $f_2$  on  $E$  such that  $\partial f_1 \subseteq \partial f_2$  and  $\partial f_2$  is a maximal cyclically monotone relation.*

Each maximal cyclically monotone relation is actually the subdifferential of some *lower semi-continuous* proper convex function. Indeed, it is easy to see that, if  $f$  is any proper convex function with a non-empty subdifferential, then the function  $g$  defined by

$$(2.4) \quad g(x) = \liminf_{y \rightarrow x} f(y) \quad \text{for all } x$$

is a l.s.c. proper convex function with

$$(2.5) \quad \partial g(x) \supseteq \partial f(x) \quad \text{for all } x.$$

If  $\partial f$  is maximal, we must, therefore, have  $\partial f = \partial g$ .

When  $E$  is a Banach space, the converse is also true: the subdifferential of *every* l.s.c. proper convex function is a maximal cyclically monotone relation. This fact will be proved in § 4. The situation is not necessarily so simple when  $E$  is not a Banach space. There exists, for instance, a (reflexive) Frechet Montel space on which there is a l.s.c. proper convex function whose subdifferential is *empty* and hence certainly not maximal. (See [3].) The same counterexample shows

that, in general, two l.s.c. proper convex functions can have the same subdifferential and yet differ by more than just an additive constant. (Take a function with empty subdifferential, and compare it with its translates.) It will be demonstrated in § 4 that this difficulty never arises in Banach spaces.

One might define a proper convex function to be *maximal* if it is l.s.c. and its subdifferential is a maximal cyclically monotone relation. Although not every l.s.c. proper convex function on  $E$  need be maximal when  $E$  is not a Banach space, many maximal ones always do exist by Corollary 2 and the above remarks. It would be interesting to know whether the class of maximal functions in the non-Banach space case possesses any significant properties as a whole. For instance, do such functions satisfy conditions (A) and (B) in [3]?

**3. Approximate subgradients and directional derivatives.** In this section we shall establish a new result about the directional derivatives of a convex function. This result will be crucial in the proving of our Banach space theorems in § 4 and § 5.

We shall assume in the following that  $E$  is locally convex and Hausdorff, and that  $f$  is a lower semi-continuous proper convex function on  $E$ .

The *conjugate* of  $f$  is the function  $f^*$  on  $E^*$  defined by

$$(3.1) \quad f^*(x^*) = \sup \{ \langle x, x^* \rangle - f(x) \mid x \in E \}$$

for each  $x^* \in E^*$ . It is known that  $f^*$  is a proper convex function on  $E^*$ , l.s.c. in the weak\* as well as the strong topology. Furthermore, the conjugate  $f^{**}$  of  $f^*$  on  $E^{**}$  coincides with  $f$  on  $E$  (considered as a subspace of  $E^{**}$ ), i.e.

$$(3.2) \quad f(x) = \sup \{ \langle x, x^* \rangle - f^*(x^*) \mid x^* \in E^* \}$$

for all  $x \in E$ . This duality will be needed later. The reader interested in the theory of conjugate convex functions should consult [2], [4], [9], [11], [13], [14].

For each  $\varepsilon > 0$ , we define an "approximate subdifferential relation"  $\partial_\varepsilon f$  as in [3] by letting  $\partial_\varepsilon f(x)$  be the set of  $x^* \in E^*$  such that

$$(3.3) \quad f(y) \geq [f(x) - \varepsilon] + \langle y - x, x^* \rangle \quad \text{for all } y \in E.$$

If  $f(x)$  is finite,  $\partial_\varepsilon f(x)$  is a *nonempty* weak\* closed convex subset of  $E^*$  for each  $\varepsilon > 0$ , and  $\partial_\varepsilon f(x)$  decreases to  $\partial f(x)$  as  $\varepsilon \downarrow 0$ . The sense in which  $\partial_\varepsilon f$  approximates  $\partial f$  is explained by the following result, proved by A. Brøndsted and the author. We restate in here for convenience, since it will be applied both in § 4 and § 5.

LEMMA ([3]). Let  $E$  be a Banach space, and let  $f$  be a l.s.c. proper convex function on  $E$ . Let  $x \in E$ ,  $x^* \in E^*$ , and  $\varepsilon > 0$  be such that  $x^* \in \partial_\varepsilon f(x)$ . Select any  $\lambda$  with  $0 < \lambda < \infty$ . Then there exist  $\bar{x} \in E$  and  $\bar{x}^* \in E^*$  such that

$$\|\bar{x} - x\| \leq \lambda, \|\bar{x}^* - x^*\| \leq \varepsilon/\lambda, \bar{x}^* \in \partial f(\bar{x}).$$

Now let  $x$  be any point at which  $f$  is finite. It is a classical fact (e.g. see [1]) that *directional derivative*

$$(3.4) \quad f'(x; y) = \lim_{\lambda \downarrow 0} [f(x + \lambda y) - f(x)]/\lambda$$

exists for all  $y \in E$  (although it may be infinite) because the difference quotient decreases as  $\lambda \downarrow 0$ . Furthermore,  $f'(x; \cdot)$  is positively homogeneous, i.e.

$$f(x; \lambda y) = \lambda f(x; y) \quad \text{for all } \lambda > 0 \text{ and } y \in E,$$

and it is a proper convex function on  $E$  provided it is greater than  $-\infty$  for each  $y$ .

There is an elementary relationship between subgradients and directional derivatives. Namely, if  $f(x)$  is finite one evidently has

$$(3.5) \quad x^* \in \partial f(x) \Leftrightarrow f'(x; y) \geq \langle y, x^* \rangle \quad \text{for all } y.$$

Given any nonempty weak\* closed convex set  $C^*$  in  $E^*$ , let us denote the *support function* of  $C^*$  on  $E$  by  $\sigma(C^*; \cdot)$ . Thus

$$\sigma(C^*; y) = \sup \{\langle y, x^* \rangle \mid x^* \in C^*\}$$

for each  $y \in E$ , and  $\sigma(C^*; \cdot)$  is a positively homogeneous l.s.c. proper convex function on  $E$ . Formula (3.5) says that, if  $\partial f(x) \neq \emptyset$ ,

$$f'(x; y) \geq \sigma(\partial f(x); y) \quad \text{for all } y.$$

But equality need not always hold, not even in the finite dimensional case, although it may be shown from (3.5) using the theory of conjugate convex functions that

$$\sigma(\partial f(x); y) = \liminf_{z \rightarrow y} f'(x; z)$$

for all  $y$  when  $\partial f(x)$  is nonempty.

The following theorem says that, just as  $\partial f(x)$  is the intersection of the  $\partial_\varepsilon f(x)$  for  $\varepsilon > 0$ , so is  $f'(x; \cdot)$  the infimum of the (l.s.c.) support functions of the  $\partial_\varepsilon f(x)$  for  $\varepsilon > 0$ , even when  $\partial f(x)$  is empty. This throws new light on the discontinuities of the directional derivative function. In connection with the lemma stated above, it will also provide us with a powerful means translating facts about the directional derivatives of  $f$  into facts about its subdifferential.



**THEOREM 2.** *Let  $E$  be a locally convex Hausdorff topological vector space, and let  $f$  be a l.s.c. proper convex function on  $E$ . Let  $x$  be a point at which  $f$  is finite. Then, for all  $y \in E$ ,*

$$(3.6) \quad \sigma(\partial_\varepsilon f(x); y) \downarrow f'(x; y) \text{ as } \varepsilon \downarrow 0.$$

*Proof.* Suppose we are given a nonempty weak\* closed convex set  $C^*$  in  $E^*$  of the form

$$(3.7) \quad C^* = \{x^* \mid f^*(x^*) - \langle x, x^* \rangle \leq \beta\},$$

where  $f^*$  is the conjugate of  $f$  and

$$(3.8) \quad \infty > \beta > \inf \{f^*(x^*) - \langle x, x^* \rangle \mid x^* \in E^*\} > -\infty.$$

The author showed in [13] that the support function of  $C^*$  could then be calculated by the formula.

$$\sigma(C^* \mid y) = \inf_{\lambda > 0} [f(x + \lambda y) + \beta]/\lambda.$$

By the definition of  $\partial_\varepsilon f$  and  $f^*$ ,  $x^* \in \partial_\varepsilon f(x)$  if and only if

$$f(x) - \varepsilon - \langle x, x^* \rangle \leq \inf_y \{f(y) - \langle y, x^* \rangle\} = -f^*(x^*).$$

Thus  $\partial_\varepsilon f(x) = C^*$  in (3.7), where

$$\beta = \varepsilon - f(x)$$

satisfies (3.8) by (3.2). Thus

$$(3.9) \quad \sigma(\partial_\varepsilon f(x); y) = \inf_{\lambda > 0} [f(x + \lambda y) - f(x) + \varepsilon]/\lambda$$

for each  $\varepsilon > 0$ . But

$$(3.10) \quad f'(x; y) = \inf_{\lambda > 0} [f(x + \lambda y) - f(x)]/\lambda,$$

since the difference quotient in (3.4) decreases as  $\lambda \downarrow 0$ . Formula (3.6) is obvious from (3.9) and (3.10).

**REMARK.** It is easy to deduce from the symmetric formulas (3.1) and (3.2) that

$$(3.11) \quad x^* \in \partial f(x) \iff x \in \partial f^*(x^*),$$

$$(3.12) \quad x^* \in \partial_\varepsilon f(x) \iff x \in \partial_\varepsilon f^*(x^*).$$

In fact the conditions in (3.11) and (3.12) are equivalent respectively to

$$(3.11') \quad f(x) + f^*(x^*) = \langle x, x^* \rangle,$$

$$(3.12') \quad f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon .$$

The dual version of the above lemma, in which unstarred elements are everywhere interchanged with their starred counterparts, is therefore valid. This is true despite the fact that  $E$  may not be reflexive. The dual of Theorem 2 must likewise be valid, by (3.12) and the fact that the support function formula quoted in the proof of Theorem 2 was established in a symmetric context in [13]. The duals of the Lemma and of Theorem 2 will both be needed below in the proof of Theorem 3.

**4. Characterization problem.** We shall now show that, in the Banach space case, the subdifferentials of the l.s.c. functions are completely characterized as the maximal cyclically monotone ones. Counter-examples described in § 2 show that this can fail outside of Banach spaces. Note the interesting resemblance between our result and the fundamental theorem of calculus.

**THEOREM 3.** *Let  $E$  be a Banach space and let  $\rho$  be a relation on  $E \times E^*$ . In order that there exist a l.s.c. proper convex function  $f$  on  $E$  such that*

$$\partial f = \rho ,$$

*it is both necessary and sufficient that  $\rho$  be a maximal cyclically monotone relation. Moreover, the solution  $f$  is then unique up to an arbitrary additive constant.*

*Proof.* The sufficiency of the condition was demonstrated in general in §2 (see Corollary 1 and the remark following Corollary 2). To prove its necessity, assume that  $f$  is l.s.c. proper convex. By Corollary 2 in §2 (and the remark following it), there exists a l.s.c. proper convex function  $g$  such that

$$(4.1) \quad \partial g \supseteq \partial f$$

and  $\partial g$  is a *maximal* cyclically monotone relation. We shall show that (4.1) implies

$$g = f + \text{const.}$$

This will complete the proof of the theorem.

Fix any  $z \in E$  such that  $\partial f(z) \neq \emptyset$ . This is possible, since the lemma in §3 trivially implies  $\partial f$  is not empty. By (4.1) we also have  $\partial g(z) \neq \emptyset$ , so both  $f(z)$  and  $g(z)$  are finite. It will be shown first that

$$(4.2) \quad f(x) - f(z) \leq g(x) - g(z) \text{ for all } x .$$

Fix any  $x \in E$  and any  $\alpha < f(x) - f(z)$ . Let

$$Q(\lambda) = f(z + \lambda(x - z)) \text{ for all } \lambda.$$

Then  $Q$  is a l.s.c. proper convex function on the real line, with  $Q(0)$  finite and

$$Q(1) - Q(0) = f(x) - f(z) > \alpha.$$

The set of  $\lambda$  in  $[0, 1]$ , for which  $Q(\lambda) < \infty$ , is an interval containing 0, and the right derivative  $Q'_+(\lambda)$  is well-defined and nondecreasing on this interval. Indeed, by the classical one-dimensional theory of convex functions, there must exist  $\lambda_i$  such that

$$(4.3) \quad \begin{aligned} 0 &\leq \lambda_0 < \lambda_1 < \cdots < \lambda_n < 1 \\ Q(\lambda_i) &< \infty, \quad Q'_+(\lambda_i) > -\infty, \\ (\lambda_1 - \lambda_0)Q'_+(\lambda_0) &+ \cdots + (\lambda_{n+1} - \lambda_n)Q'_+(\lambda_n) > \alpha, \end{aligned}$$

where  $\lambda_{n+1} = 1$  and  $\lambda_0$  can be chosen arbitrarily small. Let

$$(4.4) \quad x_i = z + \lambda_i(x - z), \quad i = 0, 1, \dots, n.$$

In terms of the directional derivatives of  $f$ , (4.3) says

$$(4.5) \quad \begin{aligned} f(x_i) &< \infty, f'(x_i; x_{i+1} - x_i) > -\infty, \\ f'(x_0; x_1 - x_0) &+ \cdots + f'(x_n; x_{n+1} - x_n) > \alpha, \end{aligned}$$

where  $x_{n+1} = x$  and  $x_0$  can be chosen arbitrarily close to  $z$ . We may now choose  $\alpha_i \in \mathbb{R}$  and  $\delta > 0$  such that

$$(4.6) \quad \begin{aligned} f'(x_i; x_{i+1} - x_i) &> \alpha_i && \text{for } i = 0, 1, \dots, n, \\ \alpha_0 + \alpha_1 + \cdots + \alpha_n &> \alpha + (n + 1)\delta. \end{aligned}$$

By Theorem 2, for any  $\varepsilon > 0$  we can find  $x_i^* \in E^*$  for  $i = 0, 1, \dots, n$ , such that

$$x_i^* \in \partial_\varepsilon f(x_i) \quad \text{and} \quad \langle x_{i+1} - x_i, x_i^* \rangle > \alpha_i.$$

Applying the Lemma in §3 with  $\lambda^2 = \varepsilon$ , we then get  $\bar{x}_i$  and  $\bar{x}_i^*$  for  $i = 0, 1, \dots, n$  such that

$$\|\bar{x}_i - x_i\| \leq \varepsilon^{1/2}, \quad \|\bar{x}_i^* - x_i^*\| \leq \varepsilon^{1/2}, \quad \bar{x}_i^* \in \partial f(\bar{x}_i).$$

By choosing  $\varepsilon$  sufficiently small, we can ensure that also

$$(4.7) \quad \langle \bar{x}_{i+1} - \bar{x}_i, \bar{x}_i^* \rangle > \alpha_i - \delta \quad \text{for } i = 0, 1, \dots, n,$$

where we have set  $\bar{x}_{n+1} = x_{n+1} = x$ . At the same time, by also choosing  $x_0$  sufficiently near to  $z$ , we can ensure that  $\bar{x}_0$  is as near as we please to  $z$ . Now (4.7) implies by the definition of  $\alpha_i$  and  $\delta$  in (4.6) that

$$(4.8) \quad \langle x - \bar{x}_n, \bar{x}_n^* \rangle + \cdots + \langle \bar{x}_1 - \bar{x}_0, \bar{x}_0^* \rangle > \alpha .$$

But

$$\bar{x}_i^* \in \partial f(\bar{x}_i) \subseteq \partial g(\bar{x}_i)$$

by (4.1), so (4.8) implies

$$\alpha < [g(x) - g(\bar{x}_n)] + \cdots + [g(\bar{x}_1) - g(\bar{x}_0)] = g(x) - g(\bar{x}_0) .$$

Furthermore, since each neighborhood of  $z$  contains some  $\bar{x}_0$  for which this last inequality is true, we have

$$g(x) - \alpha \geq \liminf_{y \rightarrow z} g(y) = g(z)$$

because  $g$  is l.s.c. We have shown this for an arbitrary  $\alpha < f(x) - f(z)$ , so we may conclude (4.2) holds as desired.

To prove the inequality complementary to (4.2), we invoke duality to see that

$$(4.9) \quad f^*(x^*) - f^*(z^*) \leq g^*(x^*) - g^*(z^*) \text{ for all } x^* ,$$

where  $f^*$  and  $g^*$  are the conjugates of  $f$  and  $g$ , and  $z^*$  is any fixed element of  $E^*$  with

$$(4.10) \quad z^* \in \partial f(z) \subseteq \partial g(z) ,$$

$z$  as before. The proof of (4.9) is completely parallel to that of (4.2). It is valid, even though  $E$  might not be reflexive, because of the duality explained following the proof of Theorem 2. (In particular,  $\partial g^* \subseteq \partial f^*$  by (4.1) and (3.11).) Now (4.10) implies

$$\begin{aligned} f(z) + f^*(z^*) &= \langle z, z^* \rangle , \\ g(z) + g^*(z^*) &= \langle z, z^* \rangle , \end{aligned}$$

by the definition of the subgradients and conjugate functions, as already in (3.11'). Substituting in (4.9), we get

$$f^*(x^*) + f(z) \leq g^*(x^*) + g(z) \text{ for all } x^* .$$

Therefore, for all  $x \in E$ ,

$$f(z) + \inf_{x^*} \{f^*(x^*) - \langle x, x^* \rangle\} \leq g(z) + \inf_{x^*} \{g^*(x^*) - \langle x, x^* \rangle\} .$$

But this is the same as

$$(4.11) \quad f(z) - f(x) \leq g(z) - g(x) \text{ for all } x$$

by formula (3.2) for the conjugate of the conjugate function in § 3. In view of (4.2), we must actually have equality in (4.11). Thus  $g$  differs from  $f$  by at most a constant, as we wanted to prove.

**5. Maximal monotone relations.** Every cyclically monotone relation is monotone, as pointed out in §2. It does not follow from this, however, that every *maximal* cyclically monotone relation is a *maximal* monotone relation. We do not know whether or not this is true in general, but the following theorem implies (via Theorem 3) that it is true in Banach spaces.

**THEOREM 4.** *Let  $E$  be a Banach space, and let  $f$  be a l.s.c. proper convex function on  $E$ . Then  $\partial f$  is a maximal monotone relation on  $E \times E^*$ .*

*Proof.* Suppose that  $z \in E$  and  $z^* \in E^*$  have the property that

$$(5.1) \quad \langle x - z, x^* - z^* \rangle \geq 0 \quad \text{whenever} \quad x^* \in \partial f(x).$$

We must show that then

$$(5.2) \quad z^* \in \partial f(z).$$

Actually, replacing  $f$  by

$$h(x) = f(z + x) - \langle x, z^* \rangle$$

if necessary, we may assume that  $z = 0$  and  $z^* = 0$ . Thus it is enough to prove the following fact: if

$$(5.3) \quad 0 \notin \partial f(0)$$

then there exists some  $\bar{x}$  and  $\bar{x}^*$  such that

$$(5.4) \quad \bar{x}^* \in \partial f(\bar{x}) \quad \text{and} \quad \langle \bar{x}, \bar{x}^* \rangle < 0.$$

Now (5.3) implies by definition that  $f(0)$  is not the minimum of  $f$  on  $E$ . Thus there exists some  $x_0$  with

$$f(0) > f(x_0).$$

Let  $Q(\lambda) = f(\lambda x_0)$  for all  $\lambda \in R$ . Then  $Q$  is a l.s.c. proper convex function on the real line and  $Q(0) > Q(1)$ . Hence, by the well-known theory of one-dimensional convex functions, there exists some  $\lambda_0$  such that

$$(5.5) \quad 0 < \lambda_0 \leq 1, \quad Q(\lambda_0) < \infty, \quad Q'_-(\lambda_0) < 0,$$

where  $Q'_-$  is the left derivative of  $Q$ . In terms of  $f$  and its directional derivatives, (5.5) says

$$f(\lambda_0 x_0) < \infty, \quad -f'(\lambda_0 x_0; -x_0) < 0.$$

Thus for  $x = \lambda_0 x_0$  we have

$$(5.6) \quad f(x) < \infty, f'(x; -x) > 0.$$

Choose any  $\varepsilon > 0$ . Then by (5.6) and Theorem 2 there exists some  $x^*$  with

$$x^* \in \partial_\varepsilon f(x) \quad \text{and} \quad \langle -x, x^* \rangle > 0.$$

Applying the lemma in §3 with  $\lambda^2 = \varepsilon$ , we can get an  $\bar{x}$  and  $\bar{x}^*$  with

$$\|\bar{x} - x\| \leq \varepsilon^{1/2}, \|\bar{x}^* - x^*\| \leq \varepsilon^{1/2}, \bar{x}^* \in \partial f(\bar{x}).$$

Since  $\langle x, x^* \rangle < 0$ , we will also have  $\langle \bar{x}, \bar{x}^* \rangle < 0$  when  $\varepsilon$  is sufficiently small. This shows that (5.4) can be satisfied, thereby completing the proof of the theorem.

REMARK. The preceding proof closely resembles Minty's proof of his Theorem 2 in [6], except towards the end. The difference is that in the special case treated by Minty  $f$  was necessarily subdifferentiable at every point, whereas here an approximation argument based on the difficult results in §3 was required before any conclusion could be reached about the subgradients of  $f$ . A much simpler proof valid for Hilbert spaces has been given by Moreau [11].

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