

Pacific Journal of Mathematics

SEMI-ALGEBRAS THAT ARE LOWER SEMI-LATTICES

EDWARD JOSEPH BARBEAU

SEMI-ALGEBRAS THAT ARE LOWER SEMI-LATTICES

E. J. BARBEAU

This paper is concerned with uniformly closed sets of continuous real-valued functions defined on a compact Hausdorff space that are at the same time semi-algebras (wedges closed under multiplication) and lower semi-lattices. The principal result is that any such set can be represented as an intersection of lower semi-lattice semi-algebras of three elementary types. This is an adaptation of a similar theorem of Choquet and Deny for lower semi-lattice wedges. A modified form of the theorem is also given for the case that the lower semi-lattice semi-algebra is in fact a lattice.

Throughout, E denotes a compact Hausdorff space and $C(E)$ the family of all continuous real-valued functions defined on E . For two functions f and g in $C(E)$, the functions $f \cap g$ and $f \cup g$, defined respectively for any point η of E by

$$(f \cap g)(\eta) \equiv \min\{f(\eta), g(\eta)\}; (f \cup g)(\eta) \equiv \max\{f(\eta), g(\eta)\},$$

are also in $C(E)$. A subset P of $C(E)$ is

- (a) a *lower semi-lattice* if and only if $f, g \in P \Rightarrow f \cap g \in P$,
- (b) an *upper semi-lattice* if and only if $f, g \in P \Rightarrow f \cup g \in P$,
- (c) a *lattice* if and only if P is both an upper and a lower semi-lattice,
- (d) a *wedge* if and only if $f, g \in P \Rightarrow \alpha f + \beta g \in P$, for any non-negative real numbers α, β ,
- (e) a *semi-algebra* if and only if P is a wedge and $f, g \in P \Rightarrow fg \in P$,
- (f) *closed under squaring* if and only if $f \in P \Rightarrow f^2 \in P$.

Choquet and Deny [4] determined those uniformly closed wedges contained in $C(E)$ which are semi-lattices in terms of certain classes of Radon measures which generate the dual wedge. The theorem for the lower semi-lattice case can be formulated as follows. Let σ be a positive Radon measure and ξ be a point of E . Define the sets

$$L_{\sigma, \xi} \equiv \{f: f \in C(E), \sigma(f) \leq f(\xi)\},$$

$$L_{\sigma} \equiv \{f: f \in C(E), \sigma(f) \leq 0\}.$$

Each of these is a uniformly closed wedge which is a lower semi-lattice. We use W' to denote the dual wedge of all those Radon measures which take nonnegative values on the wedge W , and δ_{ξ} to

denote the Radon measure with unit mass all concentrated at the point ξ of E .

THEOREM 1 (Choquet-Deny). *Let W be a uniformly closed wedge which is a lower semi-lattice contained in $C(E)$. Suppose that \mathcal{L}_1 is the family of all pairs (σ, ξ) , with ξ a point of E and σ a positive Radon measure satisfying $\sigma(\{\xi\}) = 0$, such that $\delta_\xi - \sigma \in W'$; suppose that \mathcal{L}_2 is the family of all positive Radon measures σ such that $-\sigma \in W'$. Then*

$$W = [\cap \{L_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{L}_1\}] \cap [\cap \{L_\sigma: \sigma \in \mathcal{L}_2\}].$$

For the proof, see [4]; the result is valid even if \mathcal{L}_1 is void or \mathcal{L}_2 consists of the zero measure alone. (The convention that a void intersection is the whole of the space is adopted.) An analogous theorem holds for upper semi-lattices. These results were originally given in a more general setting with the underlying space not necessarily compact, but with the function space given the topology of uniform convergence on compacta.

F. F. Bonsall, [1], [2], considered the relationship between lattice and algebraic properties of a function wedge. He showed that any uniformly closed semi-algebra A containing the function 1 and contained in $C^+(E)$ (the set of all nonnegative functions in $C(E)$) is a lattice if and only if it has the "type 1 property", i.e.,

$$f \in A \Rightarrow f/(1 + f) \in A.$$

In addition, he gave an interesting characterization of such semi-algebras as sets of functions monotone with respect to certain quasi-orderings on E . In [2], Bonsall gave intersection theorems for certain closed wedges and semi-algebras contained in $C^+(E)$ which were upper semi-lattices and permitted reduction by constants. (A subset K of $C(E)$ permits reduction by constants if and only if $f \in K, \lambda \geq 0 \Rightarrow (f - \lambda) \cup 0 \in K$.)

The main purpose of the present paper is to show that any uniformly closed lower semi-lattice semi-algebra contained in $C(E)$ is an intersection of ones of certain elementary types. The result obtained does not require the full force of the multiplication property of a semi-algebra, but only closure under squaring; its proof depends heavily on Theorem 1. In the final section, a similar intersection theorem for lattices is deduced. Unlike earlier results for semi-algebras, the theorems here are not restricted to nonnegative functions.

Because of the asymmetry introduced into the situation by the multiplication, one cannot trivially obtain a corresponding result for upper semi-lattice semi-algebras. It seems that the class of these

semi-algebras is much more extensive and varied than the class of lower semi-lattices, so that a complete determination is still in the future.

By abuse of notation, we use, for any Radon measure σ , the symbol σ to refer both to the continuous linear functional defined on $C(E)$ and to the corresponding regular measure defined on the Borel subsets of E , but no confusion will result from this. The support of σ in E is denoted by $S(\sigma)$.

2. **The Principal Result.** Let σ be a positive Radon measure with support $S(\sigma)$, ξ and ζ be two points of E and N a closed subset of E . Then it is clear that each of the sets

$$A_{\sigma, \xi} \equiv \{f: f \in C(E), \sigma(f) \leq f(\xi), 0 \leq f(\eta) \leq f(\xi) (\forall \eta \in S(\sigma))\}$$

$$B_{\xi, \zeta} \equiv \{f: f \in C(E), f(\xi) = f(\zeta)\}$$

$$C_N \equiv \{f: f \in C(E), f(\eta) = 0 (\forall \eta \in N)\}$$

is a uniformly closed semi-algebra which is a lower semi-lattice, and that any intersection of sets of these forms is such a semi-algebra. It will be shown that every uniformly closed lower semi-lattice semi-algebra is an intersection of sets of the forms $A_{\sigma, \xi}$, $B_{\xi, \zeta}$ and C_N .

LEMMA 1. *Let A be a closed subwedge of $C(E)$ closed under squaring, and suppose that $\delta_\xi - \sigma \in A'$ where σ is a positive Radon measure on E and ξ is a point of E . Then, for $f \in A$,*

$$|f(\eta)| \leq |f(\xi)|$$

whenever $\eta \in S(\sigma)$.

Proof. If $f \in A$, then $f^2 \in A$. Suppose $f(\xi) = 0$. Then $(-\sigma)(f^2) = 0$, so that f^2 vanishes almost everywhere (σ). Hence $f(\eta) = f^2(\eta) = 0$ whenever $\eta \in S(\sigma)$.

On the other hand, if $f \in A$ and $f(\xi) = \lambda \neq 0$, then $g \equiv \lambda^{-2}f^2 \in A \cap C^+(E)$ and $g(\xi) = 1$. Define

$$G \equiv \{\eta: \eta \in E, g(\eta) > 1\} = \{\eta: \eta \in E, |f(\eta)| > |f(\xi)|\}.$$

If G is void, then $|f(\eta)| \leq |f(\xi)|$ for η belonging to E , and, a fortiori, to $S(\sigma)$. If G is nonvoid, then G , being open, is σ -integrable. Let K be any compact subset of G , and let $\lambda_K \equiv \inf \{g(\eta): \eta \in K\}$. Since g attains its minimum on K , $\lambda_K > 1$. For m any power of 2, g^m belongs to A , and ϕ_K , the characteristic function of K , satisfies $\phi_K \leq \lambda_K^{-m}g^m$, so that

$$\sigma(K) \leq \lambda_K^{-m}\sigma(g^m) \leq \lambda_K^{-m}g^m(\xi) = \lambda_K^{-m}.$$

Hence $\sigma(K) = 0$. Since $\sigma(G) = \sup \{\sigma(K): K \text{ compact, } K \subseteq G\} = 0$, then $G \cap S(\sigma) = \phi$ and the result follows.

LEMMA 2. *Let \mathcal{L}_1 and \mathcal{L}_2 be the families defined as in Theorem 1 with respect to the uniformly closed wedge A , and suppose that A is closed under squaring. Let*

$N \equiv \{\eta: \eta \in E, f(\eta) = 0 (\forall f \in A)\}$. Then:

- (a) $\sigma \in \mathcal{L}_2$ if and only if $S(\sigma) \subseteq N$;
- (b) if $(\sigma, \xi) \in \mathcal{L}_1$ and every function in A takes a nonnegative value at the point ξ , then $A \subseteq A_{\sigma, \xi}$;
- (c) if $(\sigma, \xi) \in \mathcal{L}_1$ and some function in A takes a negative value at the point ξ , then there exists closed disjoint subsets M_0 and M_1 of $S(\sigma)$ (either possibly void) such that

- (i) $M_0 \cup M_1 = S(\sigma)$,
- (ii) $M_0 \subseteq N$,
- (iii) $\eta \in M_1 \Rightarrow A \subseteq B_{\xi, \eta}$,
- (iv) $\sigma(M_1) = 1$.

Proof. (a) If $A \subseteq L_\sigma$ and $f \in A$, then $(-\sigma)(f^2) = 0$, from which f^2 , and hence f , vanishes on $S(\sigma)$. This part is now clear.

(b) It must be shown that whenever $\eta \in S(\sigma)$ and $f \in A$, then $0 \leq f(\eta) \leq f(\xi)$. By Lemma 1, we know that $|f(\eta)| \leq f(\xi)$. Suppose, if possible, that, for some point ζ in $S(\sigma)$, some positive ε and some $f \in A$, $f(\zeta) = -\varepsilon$. Choose a positive integer m such that $f(\xi) < m\varepsilon$, and let $h = f \cap mf$. Then $h \in A$ and $|h(\zeta)| = -h(\zeta) = m\varepsilon > f(\xi) = h(\xi)$, so that Lemma 1 is contradicted.

(c) Let $f, g \in A$ and suppose that $f(\xi) = -1$, $g(\xi) = +1$. Then $f + g \in A$ and $(f + g)(\xi) = 0$, so that, by Lemma 1, $(f + g)(\eta) = 0$ for every point η in $S(\sigma)$. In particular, $(f + f^2)(\eta) = 0$ for $\eta \in S(\sigma)$, with the consequence that f takes only the values 0 and -1 on $S(\sigma)$. Define $M_0 \equiv S(\sigma) \cap \{\eta: f(\eta) = 0\}$ and $M_1 \equiv S(\sigma) \cap \{\eta: f(\eta) = -1\}$. Evidently M_0 and M_1 are closed, disjoint sets satisfying (i).

Let $h \in A$. If $h(\xi) = 0$, then, by Lemma 1, h vanishes everywhere on $S(\sigma)$. If $h(\xi) > 0$, then the argument of the last paragraph with g replaced by $(h(\xi))^{-1}h$ yields $f(\eta) + (h(\xi))^{-1}h(\eta) = 0$ for each point η in $S(\sigma)$. If $h(\xi) < 0$, then $(-h(\xi))^{-1}h \in A$, and the argument of the last paragraph applied to $(-h(\xi))^{-1}h$ and f^2 yields $(-h(\xi))^{-1}h(\eta) + f^2(\eta) = 0$ for each point η in $S(\sigma)$. In any case

$$h(\eta) = \begin{cases} 0 & (\eta \in M_0) \\ h(\xi) & (\eta \in M_1) \end{cases},$$

so that (ii) and (iii) are true. Part (iv) may be seen by noting that, for $h \in A$, $h(\xi)\sigma(M_1) = \sigma(h) \leq h(\xi)$ and both positive and negative

values are possible for h at ξ .

THEOREM 2. *Let A be a uniformly closed subwedge of $C(E)$ such that (i) A is a lower semi-lattice,*

(ii) A is closed under squaring.

Let \mathcal{F}_1 be the family of all pairs (σ, ξ) , with ξ a point of E and σ a positive Radon measure satisfying $\sigma(\{\xi\}) = 0$, such that $A \subseteq A_{\sigma, \xi}$; let \mathcal{F}_2 be the family of all pairs (ξ, ζ) of distinct points of E such that $A \subseteq B_{\xi, \zeta}$; let $N \equiv \{\eta: \eta \in E, f(\eta) = 0 (\forall f \in A)\}$.

Then:

$$(I) \quad A = [\cap \{A_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{F}_1\}] \cap [\cap \{B_{\xi, \zeta}: (\xi, \zeta) \in \mathcal{F}_2\}] \cap C_N.$$

Proof. By Theorem 1, with \mathcal{L}_1 and \mathcal{L}_2 defined with reference to A , we have that A is the intersection of all sets of the form $L_{\sigma, \xi}$ with $(\sigma, \xi) \in \mathcal{L}_1$ and of the form L_σ with $\sigma \in \mathcal{L}_2$. Denote by F the set on the right hand side of (I). Clearly, $A \subseteq F$. On the other hand, if $f \in F$, then, by Lemma 2(a), $f \in L_\sigma$ for each $\sigma \in \mathcal{L}_2$. Let $(\sigma, \xi) \in \mathcal{L}_1$. If every function in A is nonnegative at ξ , then, by Lemma 2(b), $A \subseteq A_{\sigma, \xi}$, so that $(\sigma, \xi) \in \mathcal{F}_1$ and $F \subseteq A_{\sigma, \xi} \subseteq L_{\sigma, \xi}$. If some function in A is negative at ξ , then there is a decomposition $\{M_0, M_1\}$ of $S(\sigma)$ satisfying the conditions of Lemma 2(c). Let $f \in F$. Then f belongs to C_N and so vanishes on $M_0 \subseteq N$. Also, if $\eta \in M_1$, then $A \subseteq B_{\xi, \eta}$, so that $(\xi, \eta) \in \mathcal{F}_2$ and $f \in B_{\xi, \eta}$, i.e., $f(\eta) = f(\xi)$. Since $\sigma(M_1) = 1$, this yields $f(\xi) = f(\xi) \sigma(M_1) = \sigma(f)$, so that $f \in L_{\sigma, \xi}$. In either case, $F \subseteq L_{\sigma, \xi}$. Hence

$$A \subseteq F \subseteq [\cap \{L_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{L}_1\}] \cap [\cap \{L_\sigma: \sigma \in \mathcal{L}_2\}] = A.$$

REMARK. The result is valid if any of \mathcal{F}_1 , \mathcal{F}_2 and N are void. If \mathcal{F}_1 is void, A is a lattice, so that the property of being a lower semi-lattice but not a lattice forces all the functions in A to be non-negative at least on a nonvoid subset of E .

CONSEQUENCES OF THEOREM 2. (a) Since all sets of the forms $A_{\sigma, \xi}$, $B_{\xi, \zeta}$ and C_N are semi-algebras, the wedge A satisfying the conditions of Theorem 2 is automatically a semi-algebra.

(b) Theorem 2 holds if the condition (ii) is strengthened to “ A is a semi-algebra”.

(c) Any wedge A contained in $C^+(E)$ which satisfies the conditions of Theorem 2 is an ideal of some semi-algebra T which has the type 1 property. For $(\sigma, \xi) \in \mathcal{F}_1$, let

$$T_{\sigma, \xi} \equiv \{f: f \in C^+(E), f(\eta) \leq f(\xi) (\forall \eta \in S(\sigma))\}.$$

Then $A_{\sigma, \xi} \cap C^+(E)$ is an ideal of $T_{\sigma, \xi}$, so that T may be taken to be

$$T \equiv [\cap \{T_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{F}_1\}] \cap [\cap \{B_{\xi, \zeta}: (\xi, \zeta) \in \mathcal{F}_2\}] \cap C_N .$$

(d) Any wedge A contained in $C^+(E)$ which satisfies the conditions of Theorem 2 and in addition contains the function 1 has the type 1 property, and hence is a lattice.

3. Semi-algebras which are lattices. In this section, let A be a uniformly closed semi-algebra contained in E which is a lattice. Since A is in particular a lower semi-lattice, the representation (I) given in Theorem 2 is valid, and, in fact, when \mathcal{F}_1 is void, expresses A as an intersection of lattices. However if \mathcal{F}_1 is nonvoid, then (I) is unsatisfactory since semi-algebras of the form $A_{\sigma, \xi}$ are not lattices unless σ is either the zero measure or has all of its mass concentrated at one point. This section will be concerned with modifying the family \mathcal{F}_1 so that A is given as the intersection of certain elementary lattices.

Suppose \mathcal{F}_1 contains the pair (σ, ξ) with $S(\sigma)$ containing at least two points. For $\eta \in S(\sigma)$, define the function $p \equiv p_{\sigma, \xi}$ by

$$p(\eta) \equiv \sup \{f(\eta): f \in A, f(\xi) = 1\} .$$

(There is no loss of generality in supposing that the supremum is taken over a nonvoid set, for otherwise $S(\sigma) \cup \{\xi\}$ would be a subset of N , defined as in Theorem 2.) Note that $0 \leq p(\eta) \leq 1$ for each point η of $S(\sigma)$ and that $p(\eta) = 0$ if and only if $\eta \in N$. The set

$$\begin{aligned} P_{\sigma, \xi} &\equiv \{f: f \in C(E), p(\eta)f(\xi) \geq f(\eta) \geq 0 (\forall \eta \in S(\sigma))\} \\ &= C_{N \cap S(\sigma)} \cap [\cap \{A_{\rho, \xi}: \rho = p(\eta)^{-1}\delta_\eta, \eta \in S(\sigma) \setminus N\}] \end{aligned}$$

is a uniformly closed lattice semi-algebra which contains A . We show that $P_{\sigma, \xi} \subseteq A_{\sigma, \xi}$, so that

$$\begin{aligned} A &\subseteq [\cap \{P_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{F}_1\}] \cap [\cap \{B_{\xi, \zeta}: (\xi, \zeta) \in \mathcal{F}_2\}] \cap C_N \\ &\subseteq [\cap \{A_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{F}_1\}] \cap [\cap \{B_{\xi, \zeta}: (\xi, \zeta) \in \mathcal{F}_2\}] \cap C_N = A . \end{aligned}$$

Let $u \in P_{\sigma, \xi}$. If $u(\xi) = 0$, then $u(\eta) = 0$ for each η belonging to $S(\sigma)$ so that $\sigma(u) = 0 = u(\xi)$ and $u \in A_{\sigma, \xi}$. If $u(\xi) \neq 0$, suppose, with no loss of generality, that $u(\xi) = 1$. Since $u(\eta) \leq p(\eta)$ for $\eta \in S(\sigma)$ and since u is continuous, for given positive ε and given point $\zeta \in S(\sigma)$, there exists a function $f_\zeta \in A$ and an open subset V_ζ of E such that $\zeta \in V_\zeta$, $f_\zeta(\xi) = 1$ and $f_\zeta(\eta) > u(\eta) - \varepsilon$ for each point η of $V_\zeta \cap S(\sigma)$. Because $S(\sigma)$ is compact, there exists a finite set $\zeta_1, \zeta_2, \dots, \zeta_k$ of points of $S(\sigma)$ such that

$$S(\sigma) \subseteq \cup \{V_{\zeta_i}: i = 1, 2, \dots, k\} .$$

The function $f \equiv f_{\zeta_1} \cup f_{\zeta_2} \cup \dots \cup f_{\zeta_k}$ belongs to A and $f(\xi) = 1$, $f(\eta) > u(\eta) - \varepsilon$ for each point η of $S(\sigma)$. Hence

$$\begin{aligned}\sigma(u) &\leq \sigma(f + \varepsilon) = \sigma(f) + \sigma(\varepsilon) \\ &\leq f(\xi) + \sigma(\varepsilon) = 1 + \sigma(\varepsilon) .\end{aligned}$$

Since $\sigma(u) \leq 1 + \sigma(\varepsilon)$ for each positive ε , $\sigma(u) \leq 1$. It is deduced that for any function u in $P_{\sigma, \xi}$, u belongs to $A_{\sigma, \xi}$.

We can now obtain the following result.

THEOREM 3. *Let A be a uniformly closed subwedge of $C(E)$ which is a lattice and closed under squaring. Let \mathcal{F}_1 be the family of all pairs (σ, ξ) , with ξ a point of E and σ a positive Radon measure which either is the zero measure or has total mass at least unity all concentrated at a point distinct from ξ , such that $A \subseteq A_{\sigma, \xi}$; let \mathcal{F}_2 and N be defined as in Theorem 2. Then the equation*

$$A = [\cap \{A_{\sigma, \xi}: (\sigma, \xi) \in \mathcal{F}_1\}] \cap [\cap \{B_{\xi, \zeta}: (\xi, \zeta) \in \mathcal{F}_2\}] \cap C_N$$

expresses A as an intersection of uniformly closed lattice semi-algebras.

REMARK. If the wedge A is contained in $C^+(E)$, then a simpler representation for A is possible. Define for $0 \leq \alpha \leq 1$ and points ξ, η of E the set

$$Q_{\alpha, \xi, \eta} \equiv \{f: f \in C^+(E), \alpha f(\xi) \geq f(\eta)\} .$$

Then A can be expressed as an intersection of semi-algebras of the form $Q_{\alpha, \xi, \eta}$. (Observe that $C^+(E) \subseteq A_{0, \xi}$, that $B_{\xi, \zeta} \cap C^+(E) = Q_{1, \xi, \zeta} \cap Q_{1, \zeta, \xi}$ and that $C_N \cap C^+(E) = \cap \{Q_{0, \xi, \eta}: \xi \in E, \eta \in N\}$.)

BIBLIOGRAPHY

1. F. F. Bonsall, *Semi-algebras of continuous functions*, Proc. Lon. Math. Soc. (3) **10** (1960), 122-140.
2. ———, *Algebraic properties of some convex cones of functions*, Quart. J. Math. (Oxford) (2) **14** (1963), 225-230.
3. N. Bourbaki, *Intégration-Chapitres I-IV*, Hermann (ASI 1175), 1952.
4. G. Choquet and J. Deny, *Ensembles semi-réticulés et ensembles réticulés de fonctions continues*, Jour. de Math. Pures et App. **36** (1957), 179-189.

Received February 26, 1965.

THE UNIVERSITY OF WESTERN ONTARIO
LONDON, CANADA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

*J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Pacific Journal of Mathematics

Vol. 18, No. 1

March, 1966

Edward Joseph Barbeau, <i>Semi-algebras that are lower semi-lattices</i>	1
Steven Fredrick Bauman, <i>The Klein group as an automorphism group without fixed point</i>	9
Homer Franklin Bechtell, Jr., <i>Frattini subgroups and Φ-central groups</i>	15
Edward Kenneth Blum, <i>A convergent gradient procedure in prehilbert spaces</i>	25
Edward Martin Bolger, <i>The sum of two independent exponential-type random variables</i>	31
David Wilson Bressler and A. P. Morse, <i>Images of measurable sets</i>	37
Dennison Robert Brown and J. G. LaTorre, <i>A characterization of uniquely divisible commutative semigroups</i>	57
Selwyn Ross Caradus, <i>Operators of Riesz type</i>	61
Jeffrey Davis and Isidore Isaac Hirschman, Jr., <i>Toeplitz forms and ultraspherical polynomials</i>	73
Lorraine L. Foster, <i>On the characteristic roots of the product of certain rational integral matrices of order two</i>	97
Alfred Gray and S. M. Shah, <i>Asymptotic values of a holomorphic function with respect to its maximum term</i>	111
Sidney (Denny) L. Gulick, <i>Commutativity and ideals in the biduals of topological algebras</i>	121
G. J. Kurowski, <i>Further results in the theory of monodiffrie functions</i>	139
Lawrence S. Levy, <i>Commutative rings whose homomorphic images are self-injective</i>	149
Calvin T. Long, <i>On real numbers having normality of order k</i>	155
Bertram Mond, <i>An inequality for operators in a Hilbert space</i>	161
John William Neuberger, <i>The lack of self-adjointness in three-point boundary value problems</i>	165
C. A. Persinger, <i>Subsets of n-books in E^3</i>	169
Oscar S. Rothaus and John Griggs Thompson, <i>A combinatorial problem in the symmetric group</i>	175
Rodolfo DeSapio, <i>Unknotting spheres via Smale</i>	179
James E. Shockley, <i>On the functional equation $F(mn)F((m, n)) = F(m)F(n)f((m, n))$</i>	185
Kenneth Edward Whipple, <i>Cauchy sequences in Moore spaces</i>	191