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AN INEQUALITY FOR OPERATORS IN A HILBERT SPACE

BERTRAM MOND

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Let A be a self-adjoint operator on a Hilbert space H satisfying $mI \leqq A \leqq MI$, $0 < m < M$. Set $q = M/m$. Let j and k be real numbers, $jk \neq 0$, $j < k$. Then

$$(A^k x, x)^{1/k}/(A^j x, x)^{1/j} \leqq \{j^{-1}(q^j - 1)\}^{-1/k}\{k^{-1}(q^k - 1)\}^{1/j}\{(k-j)^{-1}(q^k - q^j)(x,x)\}^{(1/k)-(1/j)}$$

for all $x \in H (x \neq 0)$. Letting $j = -1$ and $k = 1$, this inequality reduces to $(Ax, x)(A^{-1}x, x) \leqq [(M+m)^2/4mM](x,x)^2$, the well-known Kantorovich Inequality.

Preliminaries. We shall make use of the following four inequalities:

For $a > 0, b > 0$,

$$a^\alpha b^{1-\alpha} \leqq \alpha a + (1-\alpha)b \qquad \text{if } 0 < \alpha < 1 \tag{1}$$

$$a^\alpha b^{1-\alpha} \geqq \alpha a + (1-\alpha)b \qquad \text{if } \alpha < 0\,. \tag{2}$$

For $j < k, 1 \leqq y \leqq q$,

$$(q^k - 1)y^j - (q^j - 1)y^k - (q^k - q^j) \geqq 0 \qquad \text{if } jk > 0 \tag{3}$$

$$-(q^k - 1)y^j + (q^j - 1)y^k + (q^k - q^j) \geqq 0 \qquad \text{if } jk < 0\,. \tag{4}$$

(1) is the well-known inequality between the arithmetic and geometric means. Simple proofs of (2), (3) and (4) can be found in a recent paper by Goldman [3].

Let C be a self-adjoint operator on a Hilbert space H satisfying

$$I \leqq C \leqq qI \tag{5}$$

where I is the identity operator (and (5) is understood in the usual sense that $(x,x) \leqq (Cx,x) \leqq q(x,x)$ for all $x \in H$). To the real valued function $u(\lambda)$, defined and continuous on $[1, q]$, there is associated in a natural way a self-adjoint operator on H denoted by $u(C)$ (see e.g. [6] pp. 265–273).

We shall make use of the following [loc. cit.]:

LEMMA. *If $u(\lambda) \geqq 0$ for $1 \leqq \lambda \leqq q$, then $u(C) \geqq 0$, i.e., $u(C)$ is a positive operator.*

Results.

THEOREM 1. *Let C be a self-adjoint operator on a Hilbert space H satisfying $I \leqq C \leqq qI$. Let j and k be real numbers, $j < k, jk \neq 0$. The operator*

$$(q^k - 1)C^j - (q^j - 1)C^k - (q^k - q^j)I \tag{6}$$

is positive if $jk > 0$; while the operator

$$-(q^k - 1)C^j + (q^j - 1)C^k + (q^k - q^j)I \tag{7}$$

is positive if $jk < 0$.

Proof. The theorem follows directly from (3) and (4) by virtue of the Lemma.

Letting $j = -1$ and $k = 1$, Theorem 1 yields an inequality that is equivalent to one given by Diaz and Metcalf [2].

The following theorem, which is the main result of this paper, is a Hilbert space generalization of Cargo and Shisha [1] and Mond [5].

THEOREM 2. *Let A be self-adjoint operator on a Hilbert space H satisfying $mI \leqq A \leqq MI, 0 < m < M$. Set $q = M/m$. Let j and k be real numbers $jk \neq 0, j < k$. Then*

$$\begin{aligned}(A^k x, x)^{1/k}/(A^j x, x)^{1/j} \\ \leqq \{j^{-1}(q^j - 1)\}^{-1/k}\{k^{-1}(q^k - 1)\}^{1/j}\{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)}\end{aligned} \tag{8}$$

for all $x \in H (x \neq 0)$.

Proof. Set $C \equiv A/m$. It obviously suffices to prove

$$\begin{aligned}(C^k x, x)^{1/k}/(C^j x, x)^{1/j} \\ \leqq \{j^{-1}(q^j - 1)\}^{-1/k}\{k^{-1}(q^k - 1)\}^{1/j}\{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)} .\end{aligned} \tag{9}$$

Since C satisfies (5), by Theorem 1,

$$(q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \geqq (q^k - q^j)(x, x) \quad \text{if } jk > 0 \tag{10}$$

and

$$(q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \leqq (q^k - q^j)(x, x) \quad \text{if } jk < 0 . \tag{11}$$

Rewrite (10) as

$$\begin{aligned}\{-j(k - j)^{-1}\}\{j^{-1}(q^j - 1)(C^k x, x)\} + \{k(k - j)^{-1}\}\{k^{-1}(q^k - 1)(C^j x, x)\} \\ \geqq (k - j)^{-1}(q^k - q^j)(x, x)\end{aligned} \tag{12}$$

if $jk > 0$, and (11) as

$$\{-j(k-j)^{-1}\}\{j^{-1}(q^j-1)(C^kx,x)\}+\{k(k-j)^{-1}\}\{k^{-1}(q^k-1)(C^jx,x)\} \leqq (k-j)^{-1}(q^k-q^j)(x,x) \tag{13}$$

if $jk<0$.

Assume $k>0$. Set

$$a=j^{-1}(q^j-1)(C^kx,x),\ b=k^{-1}(q^k-1)(C^jx,x),\ \alpha=-j(k-j)^{-1}.$$

If $j>0$, applying (2) and combining with (12), we obtain

$$\{j^{-1}(q^j-1)(C^kx,x)\}^{-j/(k-j)}\{k^{-1}(q^k-1)(C^jx,x)\}^{k/(k-j)} \geqq (k-j)^{-1}(q^k-q^j)(x,x) \tag{14}$$

which when raised to the power $(k-j)/(-kj)$ yields

$$\{j^{-1}(q^j-1)(C^kx,x)\}^{1/k}\{k^{-1}(q^k-1)(C^jx,x)\}^{-1/j} \leqq \{(k-j)^{-1}(q^k-q^j)(x,x)\}^{(1/k)-(1/j)}. \tag{15}$$

If $j<0$ ($k>0$), applying (1) and combining with (13) yields the reverse of (14) which, when raised to the power $(k-j)/(-kj)$, yields (15).

Finally, if $j<k<0$, set

$$a=k^{-1}(q^k-1)(C^jx,x),\ b=j^{-1}(q^j-1)(C^kx,x),\ \alpha=k(k-j)^{-1}.$$

Applying (2) and combining with (12) yields (14) which, when raised to the power $(k-j)/(-kj)$ yields (15). In all cases, therefore, we have (15), a rearrangement of (9). (Compare the method of proof of Theorem 2 with Goldman [3].)

The well-known [4] Kantorovich inequality, $(Ax,x)(A^{-1}x,x)\leqq[(m+M)^2/4mM](x,x)^2$, is the special case of Theorem 2 with $j=-1$, $k=1$.

References

1. T. G. Cargo and O. Shisha, *Bounds on ratios of means*, J. Res. Nat. Bur. Standards **66B** (1962), 169-170.

2. J. B. Diaz and F. T. Metcalf, *Stronger forms of inequalities of Kantorovich and Strang for operators in a Hilbert space*, Notices Amer. Math. Soc. **11**, No. 1 (Jan. 1964), 92.

3. A. J. Goldman, *A generalization of Rennie's inequality*, J. Res. Nat. Bur. Standards **68B** (1964), 59-63.

4. L. V. Kantorovich, *Functional analysis and applied mathematics*, Uspekhi Math. Nauk, **3** (1948), Translated from the Russian by Curtis D. Benster, Nat. Bur. Standards, Report No. 1509., March 7, 1952.

5. B. Mond, *A matrix inequality including that of Kantorovich* to appear in J. Math. Analysis and Applications **13**, No. 1 (Jan. 1966), 49-52.

6. F. Riesz and B. Sz-Nagy, *Functional analysis*, Translated from the 2nd French edition by Leo F. Boron, Frederick Ungar Pub. Co., 1955.

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Edward Joseph Barbeau, *Semi-algebras that are lower semi-lattices* 1
Steven Fredrick Bauman, *The Klein group as an automorphism group without fixed point* 9
Homer Franklin Bechtell, Jr., *Frattini subgroups and Φ-central groups* 15
Edward Kenneth Blum, *A convergent gradient procedure in prehilbert spaces* 25
Edward Martin Bolger, *The sum of two independent exponential-type random variables* 31
David Wilson Bressler and A. P. Morse, *Images of measurable sets* 37
Dennison Robert Brown and J. G. LaTorre, *A characterization of uniquely divisible commutative semigroups* 57
Selwyn Ross Caradus, *Operators of Riesz type* 61
Jeffrey Davis and Isidore Isaac Hirschman, Jr., *Toeplitz forms and ultraspherical polynomials* 73
Lorraine L. Foster, *On the characteristic roots of the product of certain rational integral matrices of order two* 97
Alfred Gray and S. M. Shah, *Asymptotic values of a holomorphic function with respect to its maximum term* 111
Sidney (Denny) L. Gulick, *Commutativity and ideals in the biduals of topological algebras* 121
G. J. Kurowski, *Further results in the theory of monodiffric functions* 139
Lawrence S. Levy, *Commutative rings whose homomorphic images are self-injective* 149
Calvin T. Long, *On real numbers having normality of order k* 155
Bertram Mond, *An inequality for operators in a Hilbert space* 161
John William Neuberger, *The lack of self-adjointness in three-point boundary value problems* 165
C. A. Persinger, *Subsets of n-books in E^3* 169
Oscar S. Rothaus and John Griggs Thompson, *A combinatorial problem in the symmetric group* 175
Rodolfo DeSapio, *Unknotting spheres via Smale* 179
James E. Shockley, *On the functional equation $F(mn)F((m,n)) = F(m)F(n)f((m,n))$* 185
Kenneth Edward Whipple, *Cauchy sequences in Moore spaces* 191