Pacific Journal of Mathematics

SUBSETS OF n-BOOKS IN E^3

C. A. Persinger

Vol. 18, No. 1 March 1966

PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 1, 1966

SUBSETS OF n-BOOKS IN E^3

C. A. PERSINGER

An n-book B^n in E^3 is defined to be the union of n closed disks in E^3 such that each pair of disks meets precisely on a single arc B on the boundary of each. The disks are called the leaves of B^n, and the arc B is its back.

The zero-dimensional subsets of a tame n-book in E^3 are shown to be limited by the fact that no wild Cantor set lies in such a book, even if the number of leaves is countable. However, wild arcs and disks abound in tame n-books. Each arc in a tame n-book is shown to lie in a tame 3-book in E^3, and the tame disks in tame n-books are shown to be characterized by the tameness of their boundaries.

One consequence of the Schoenflies Theorem for the plane follows.

THEOREM 1.1. *Each arc in a tame* 1-*book or a tame* 2-*book in* E^3 *is tame.*

The embedding of n-books in E^3 for one case is handled by the following.

THEOREM 1.2. *If each leaf of an n-book B^n in E^3 is tame, then B^n is tame* [3].

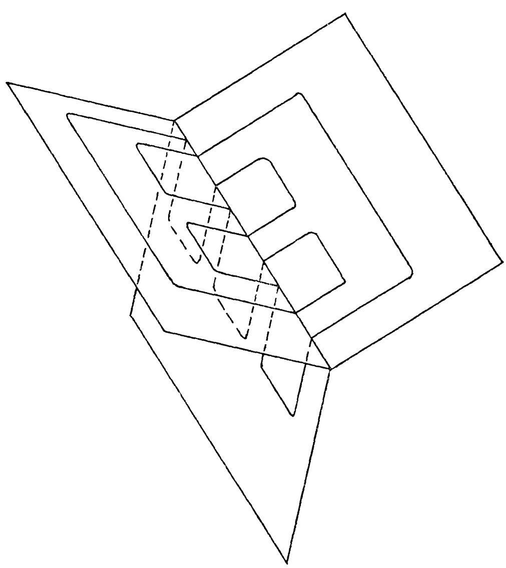

Figure 1.

It should not be surprising that if B^n is a tame n-book in E^3, $n \geqq 3$, then B^n contains a wild arc. Employing the method used by Posey [6] in the proof of Theorem 3.1 below, a trefoil can be embedded in a tame 3-book as shown in Figure 1. If a sequence of loops from a trefoil are put together as in Fox and Harrold [5], then a wild arc in a tame 3-book is obtained.

If D is a disk in E^3, Int D and Bd D will denote the interior and the boundary of D, respectively. The closure of a set N will be denoted by $\bar{N}$.

A topological polyhedron K in E^m is tamely embedded in E^m if there is a space homeomorphism that carries K onto a finite Euclidean polyhedron. Otherwise, K is wildly embedded. A set K in E^m is locally tame at a point p of K if there is a neighborhood N of p and a homeomorphism h of $\bar{N}$ such that $h(\bar{N} \cap K)$ is a finite Euclidean polyhedron. The definitions of tame and locally tame are due to Fox and Artin [4] and Bing [1], respectively.

2. Zero-dimensional subsets.

A Cantor set C in E^3 is tame if there is a homeomorphism of E^3 onto itself taking C into a tame arc. Otherwise, C is wild. It is clear that there are many zero-dimensional subsets of a tame n-book in E^3, but an obvious limitation is imposed by the following.

THEOREM 2.1. *No wild Cantor set lies in a tame n-book in E^3.*

Proof. Let B^n be a tame n-book, $D_1 D_2, \cdots, D_n$ the leaves of B^n, and B its back. Without loss of generality, we may assume that B^n is polyhedral. Let C be a Cantor set in B^n. We swell each leaf of B^n to obtain polyhedral 2-spheres $D_1', D_2', \cdots, D_n'$ which meet only on B and such that $D_i \subset D_i'$. Consider $C \cap D_i' = C \cap D_i$. This is a compact zero-dimensional subset of D_i' for each $i = 1, 2, \cdots, n$, and so we can pass a tame arc A_i through all the points of $C \cap D_i'$. Now A_i lies on D_i' and A_i contains a Cantor set C_i with $C \cap D_i' \subset C_i$. Since A_i is a tame arc, C_i is a tame Cantor set. Hence, by Theorem 6.1 of Bing [2], $\bigcup_{i=1}^n C_i$ is a tame Cantor set, and so $C \subset \bigcup_{i=1}^n C_i$ is tame.

Let $D_1, D_2, \cdots$ be a countable collection of tame closed disks in E^3 meeting on a single arc B on the boundary of each. Let the sequence $\{D_i\}$ converge to a tame disk D. We define a tame $\aleph_0$-book $B^{\aleph_0}$ to be $B^{\aleph_0} = (\bigcup_{i=1}^\infty D_i) \cup D$. Since Theorem 6.1 of [2] applies to a countable collection of tame Cantor sets, we may use the proof of Theorem 2.1 to obtain the following.

COROLLARY 2.2. *No wild Cantor set lies in a tame $\aleph_0$-book in E^3.*

3. **Arcs.** We have seen that there are wild arcs in tame n-books in E^3. The following Theorem has been proved by Posey [6].

THEOREM 3.1. *Let T, W, and E be subsets of an arc A such that*
(i) *A is locally tame at each point of T,*
(ii) *$W = A \backslash T$, and*
(iii) *E is a subset of W such that each neighborhood of each point in E meets T in a nonempty set. If E lies on a tame arc, then there is a space homeomorphism f such that f maps $E \cup T = J$ into the union of three half-planes which meet in their common edge.*

Posey's proof gives us a method for embedding some wild arcs in tame 3-books. We take a standard projection (all singular points are isolated double-points) of J onto a plane M such that the double-points along with E lie on a line segment L. For each double-point p on L, there is a "small" subarc of J which contains the "over point" and projects onto a segment perpendicular to L, and there is a subarc J_p of J which contains the "under point" and projects onto a subsegment L_p of L. If N is the half-plane lying "below" M, perpendicular to M, and meeting M in the line determined by L, we modify the projection so that each J_p (or a subarc of J_p) is projected onto a semicircle C_p lying in N.

COROLLARY 3.2. *If an arc A lies in a tame n-book B^n in E^3, then it is equivalent to an arc in a tame 3-book.*

Proof. Since B^n is tame, there is no loss of generality in assuming that the leaves of B^n are planar disks. Then the set of points at which A fails to be locally tame must lie on the back of the book. By the Theorem, there is a space homeomorphism that takes A into a tame 3-book.

The same proof applies to extend Corollary 3.2.

COROLLARY 3.3. *If an arc A lies in a tame $\aleph_0$-book in E^3, then it is equivalent to an arc in a tame 3-book.*

In view of these results, it is clear that we have the following.

THEOREM 3.4. *Let B^n be a tame n-book in E^3 such that each arc in B^n is tame. Then $n = 1$ or $n = 2$.*

We next combine Theorem 3.4 with Theorem 1.1.

THEOREM 3.5. *A tame n-book in E^3 contains only tame arcs if* *and only if* $n = 1$ *or* $n = 2$.

Thus we see that the consideration of wild arcs and simple closed curves in tame n-books in E^3 can be restricted to tame books which consist of three leaves. However, there are arcs in E^3 which lie in no tame n-book in E^3. An arc through the Antoine Necklace has this property.

In Fox and Artin [4], Example 1.1 is a noncellular arc, Example 1.2 is an arc which fails to have Property P, and Example 1.4 is an arc which fails to have Property Q. By Theorem 3.1, each of these arcs can be embedded in a tame 3-book. Fox and Harrold [5] define the Wilder arcs, and by Theorem 3.1, each of the uncountable number of distinct types of Wilder arcs can be embedded in a tame 3-book.

4. **Disks.** We will conclude with some results on disks in tame n-books in E^3.

THEOREM 4.1. *There are wild disks in tame n-books,* $n \geqq 3$.

Proof. Let Example 1.1 of Fox and Artin [4] be embedded in a tame 3-book B^3 according to the method outlined following Theorem 3.1. Since the leaves of B^3 may be taken to be planar disks, each subarc of the sequence in each leaf lies in a closed planar disk which meets the remainder of the arc only on the back of B^3 so that the union of these disks is a disk which lies in B^3 and has Example 1.1 as a spanning arc of its boundary. Hence, this disk is wild.

THEOREM 4.2. *Let D be a closed disk in a tame n-book* B^n *in* E^3, $n \geqq 3$. *Then D is tame if and only if* Bd D *is tame.*

Proof. Surely if D is tame, Bd D is also tame since D is a closed disk.

Conversely, suppose D is a disk in B^n and Bd D is tame. If D is not tame, there is a point p in Int D and D fails to be locally tame at p. Since the leaves of B^n may be taken as planar disks, p must lie on the back of B^n. Now p is in Int D and so there is a neighborhood U of p, $U \subset \operatorname{Int} D$ such that $\bar{U} \subset \operatorname{Int} D$, and $\bar{U}$ lies in precisely two leaves of B^n. Hence $\bar{U}$ is a sub-disk of a tame disk (the union of two tame leaves of B^n) and thus $\bar{U}$ is locally tame at p. Therefore $\bar{U}$ is tame and D is locally tame at all points of Int D. Hence, D is tame.

COROLLARY 4.3. *Let* B^n *be a topological n-book and let D be a* *wild disk in* B^n *such that Bd D is tame. Then some leaf of* B^n *is wild.*

REFERENCES

1. R. H. Bing, *Locally tame sets are tame*, Ann. of Math. **59** (1954), 145-158.
2. ———, *Tame Cantor sets in* E^3, Pacific J. Math. **11** (1961), 435-446.
3. P. H. Doyle, *On the embedding of complexes in 3-space*, Illinois J. Math. **8** (1964), 615-620.
4. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. **49** (1948), 979-990.
5. R. H. Fox and O. G. Harrold, Jr., *The Wilder arcs*, Topology of 3-Mainifolds and Related Topics, Prentice-Hall, Inc., (1962), 184-187.
6. E. E. Posey, *Proteus forms of wild and tame arcs*, Duke Math. J. **31** (1964), 63-72.

Received March 15, 1965.

VIRGINIA POLYTECHNIC INSTITUTE AND
AIR FORCE INSTITUTE OF TECHNOLOGY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California

R. M. BLUMENTHAL
University of Washington
Seattle, Washington 98105

*J. DUGUNDJI
University of Southern California
Los Angeles, California 90007

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH   B. H. NEUMANN   F. WOLF   K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics

Vol. 18, No. 1 March, 1966