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POINT-DETERMINING HOMOMORPHISMS ON MULTIPLICATIVE SEMI-GROUPS OF CONTINUOUS FUNCTIONS

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Let X and Y be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the algebras of real valued continuous functions on X and Y respectively with the usual sup norms. If T is an algebra homomorphism from $C(X)$ onto a dense subset of $C(Y)$ then by a theorem of Stone, T induces a homeomorphism μ from Y to X and it necessarily follows that $Tf(y) = 0$ if and only if $f(\mu(y)) = 0$.

In a more general setting, viewing $C(X)$ and $C(Y)$ as multiplicative semi-groups, let T be a semi-group homomorphism from $C(X)$ onto a dense point-separating set in $C(Y)$. No such map μ satisfying the above condition need exist. T is called point-determining in case for each y there is an x such that $Tf(y) = 0$ if and only if $f(x) = 0$. It is shown that such a homomorphism T induces a homeomorphism from Y into X in such a way that $Tf(y) = [\operatorname{sgn} f(x)]\,|f(x)|^{p(x)}$ for some continuous positive function p where x is related to y via the induced homeomorphism, that such a T is an algebra homomorphism followed by a semi-group automorphism, and that T is continuous.

Let X and Y be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the algebras of all continuous real-valued functions on X and Y respectively with the usual sup norm. Let T be an algebra homomorphism of $C(X)$ onto a dense set in $C(Y)$. For each $y \in Y$ consider the mapping γ_y of $C(X)$ into the reals defined by

$$\gamma_y(f) = Tf(y) .$$

γ_y maps $C(X)$ onto the reals for if $Tf(x) = 0$ for all $f \in C(X)$ then the image of T is not dense. The kernel is, by algebra, a maximal ideal in $C(X)$. By a theorem of Stone [3, 80] there is a point $x \in X$ so that the kernel of γ_y is the set of all $f \in C(X)$ such that $f(x) = 0$, this point being uniquely determined.

Consider the map μ of Y into X which assigns to each $y \in Y$ the x as described above. If e and e_1 are the unit functions in $C(X)$ and $C(Y)$ respectively it is easy to see that $Te = e_1$ and that μ is one-to-one. Now for each $f \in C(X)$ consider the function $Tf(y)e - f = g$ in $C(X)$. Then $Tg(y) = 0$ so that $g(\mu(y)) = 0$ and hence $Tf(y) = f(\mu(y))$. We especially note that

$$(*) \qquad Tf(y) = 0 \quad \text{if and only if} \quad f(\mu(y)) = 0 \,.$$

As we shall see, under a more general setting for T, this condition will imply that μ is bicontinuous (see Lemma 3.1 below).

In this paper we view $C(X)$ and $C(Y)$ as multiplicative semi-groups and let T be a semi-group homomorphism from $C(X)$ onto a dense set in $C(Y)$; the restriction on T being that for each $y \in Y$ there is an $x \in X$ such that $f(x) = 0$ if and only if $Tf(y) = 0$ (i.e. a condition such as (*) above is satisfied). For such a T we show that Y can be imbedded homeomorphically in X in such a way that $Tf(y) = [\operatorname{sgn} f(x)]\,|f(x)|^{p(x)}$ for some continuous positive function $p(x)$ where y is related to x via the induced homeomorphism. It is shown that each such homomorphism T is an algebra homomorphism followed by a semi-group automorphism and that T is continuous.

2. Definitions, Notation and Preliminaries.

We first note that in our more general setting no mapping μ satisfying (*) above need exist. To see this let $X = [0, 1] \cup \{2\}$, $Y = [0, 1]$ with the relative topology of the real line. For $t \in [0, 1]$ set

$$Tf(t) = f(t)f(2) \,.$$

T is a semi-group homomorphism of $C(X)$ onto $C(Y)$ but $Tf(t) = 0$ if and only if either $f(t) = 0$ or $f(2) = 0$.

DEFINITION 2.1. A semi-group homomorphism T will be called *point-determining* in case for each $y \in Y$ there is an $x \in X$ such that $f(x) = 0$ if and only if $Tf(y) = 0$.

The following result is immediate.

LEMMA 2.2. *If T is a point-determining semi-group homomorphism of $C(X)$ onto a dense set in $C(Y)$, e and e_1 the respective unit functions in $C(X)$ and $C(Y)$ then $Te = e_1$ and $TO = O$.*

DEFINITION 2.3. A subset $A \subseteqq C(Y)$ will be called *point-separating* in case for $y_1 \neq y_2$ in Y there is a $g \in A$ such that $g(y_1) = 0$ and $g(y_2) \neq 0$.

In the development that follows X and Y will be compact Hausdorff spaces, Y having no isolated points; $C(X)$ and $C(Y)$ will be viewed as multiplicative semi-groups and T will be a point-determining semi-group homomorphism of $C(X)$ onto a dense point-separating set in $C(Y)$. The hypothesis Y has no isolated points, however, is not used until Lemma 3.5. Multiplication is defined pointwise in $C(X)$ and $C(Y)$.

$\sim A$ will denote complement of A in any of the spaces considered.

$\varnothing$ will denote the empty set and the bar notation will denote closure.

We are indebted to a paper of Milgram [2] for suggesting the sequence of ideas and devices employed here.

3. Development of the main results. Notice that for each $y \in Y$, T determines a unique point $x \in X$. Thus T induces a well-defined single valued mapping $\mu: Y \to X$ defined by $\mu(y) = x$ in case $f(x) = 0$ if and only if $Tf(y) = 0$. In the material to follow notationally we let $\mu(Y) = X_0$. (μ turns out to be a special case of the multi-valued mappings studied in [1] although there we assumed T continuous.)

LEMMA 3.1. *μ is a homeomorphism of Y into X.*

Proof. μ is a one-to-one for say $\mu(y_1) = \mu(y_2)$. Then $Tf(y_1) = 0$ if and only if $Tf(y_2) = 0$. If $y_1 \neq y_2$ then since the range of T is point-separating there is an $h \in C(X)$ such that $Th(y_1) = 0$, $Th(y_2) \neq 0$ a contradiction.

To see μ is continuous we suppose contrarywise that μ is not continuous at some point $t_0 \in Y$. Then there is a net $\{t_\beta\}$ in Y, $t_\beta \to t_0$ and an open neighborhood U containing $\mu(t_0)$ such that $\mu(t_\beta) \notin U$ for any β. Now there is an $f \in C(X)$ such that $f(\mu(t_0)) = 1$ and $f(\sim U) = 0$ so that $f(\mu(t_\beta)) = 0$ for all β and hence $Tf(t_\beta) = 0$ for all β. But $Tf(t_0) \neq 0$ since $f(\mu(t_0)) \neq 0$ contradicting the fact that $Tf \in C(Y)$. Thus μ is continuous and it follows that μ is a homeomorphism.

If σ is a homeomorphism from Y into X, define

$$T: C(X) \to C(Y)$$

by

$$Tf(y) = f(\sigma(y)) , \quad f \in C(X) , \quad y \in Y .$$

T is onto (and continuous) so that we have the following.

THEOREM 3.2. *There is a point-determining semi-group homomorphism of $C(X)$ onto a dense point-separating set in $C(Y)$ if and only if Y is homeomorphic to a subset of X.*

We proceed now, to find the form of the general homomorphism in our theory.

LEMMA 3.3. *Let U be open in X. If $f \equiv 1$ on U then $Tf \equiv 1$ on $\mu^{-1}(U \cap X_0)$.*

Proof. $f \equiv 1$ on U implies that $fg = g$ for all $g \in C(X)$ such that $g(x) = 0$ on $\sim U$ and hence $TfTg = Tg$ for all such g. Let $y_0 \in \mu^{-1}(U \cap X_0)$ so that $\mu(y_0) = x_0 \in U \cap X_0 \subset U$ and note that $Tf(y_0) \neq 0$ since $f(x_0) = 1$. Now there is an $h \in C(X)$ such that $h(x_0) = 1$ and $h(\sim U) = 0$ so from the above $Tf(y_0)Th(y_0) = Th(y_0)$. But $h(x_0) = 1$ implies that $Th(y_0) \neq 0$ and therefore $Tf(y_0) = 1$ and since y_0 was arbitrary the result follows.

LEMMA 3.4. *Let U be open in X. If $f \equiv g$ on U then $Tf \equiv Tg$ on $\mu^{-1}(U \cap X_0)$.*

Proof. Let $y_0 \in \mu^{-1}(U \cap X_0)$ so that $\mu(y_0) = x_0 \in U \cap X_0 \subset U$. If $f(x_0) = 0 = g(x_0)$ then $Tf(y_0) = 0 = Tg(y_0)$. If $f(x_0) = g(x_0) \neq 0$ we may assume without loss of generality that $f(x_0) = g(x_0) = c > 0$. Then $W' = \{x \mid f(x) > c/2\}$ is open in X and $x_0 \in W'$. For $x \in X$ set $h'(x) = \max[f(x), c/2]$. Then h' and $h = 1/h'$, are in $C(X)$. Now $fh \equiv 1$ on W' and hence $fh \equiv 1 \equiv gh$ on $W = W' \cap U$. Thus by Lemma 3.3 $Tfh \equiv 1 \equiv Tgh$ on $\mu^{-1}(W \cap X_0)$ and so in particular $Tf(y_0)Th(y_0) = 1 = Tg(y_0)Th(y_0)$. Now $h(x_0) \neq 0$ so $Th(y_0) \neq 0$. Thus $Tf(y_0) = Tg(y_0)$ and the result follows.

LEMMA 3.5. *Let $x_0 = \mu(y_0)$. If $f(x_0) = 1$ then $Tf(y_0) = 1$.*

Proof. Suppose first that $f(x_0) = 1$ but that $Tf(y_0) > 1$. Then there is an open neighborhood W containing y_0 such that $Tf(y_0) \geqq c > 1$ for all $y \in W$. Now $\mu(W) = U \cap X_0$ for some open set U in X such that $x_0 \in U$. Let $V_n = \{x \in X \mid |f^n(x) - 1| < 1/n\}$ $n = 1, 2, 3, \cdots$ and set $U_n = V_n \cap U$. Note that $x_0 \in U_n$ an open set in X for each n and that there are points of $X_0 - \{x_0\}$ in U_n for every n since Y has no isolated points (and hence X_0 has none). We construct a sequence $\{x_n\}$ of distinct points such that $x_n \in U_n \cap X_0$ as follows:

Select $x_1 \in U_1 \cap X_0$ such that $x_1 \neq x_0$ and set $U_1 = W_0^{(0)}$. Select disjoint neighborhoods $W_0^{(1)}$ containing x_0 and $W_1^{(1)}$ containing x_1 such that $W_0^{(1)} \subset W_0^{(0)}$ and $W_1^{(1)} \subset W_0^{(0)}$.

In general select $x_n \in W_0^{(n-1)} \cap X_0 \cap U_n$ such that $x_n \neq x_0$ and disjoint neighborhoods $W_0^{(n)}$ containing x_0 and $W_1^{(n)}$ containing x_n such that $W_0^{(n)} \subset W_0^{(n-1)}$ and $W_1^{(n)} \subset W_0^{(n-1)}$. Note that $\{W_0^{(n)}\}$ is a decreasing sequence of neighborhoods containing x_0 where $x_i \in W_0^{(i-1)}$; $\{W_1^{(n)}\}$ is a collection of neighborhoods where $x_i \in W_1^{(i)}$ and $W_1^{(n)} \cap W_0^{(n)} = \varnothing$ $n = 1, 2, 3, \cdots$.

For the sequence $\{x_n\}$ we have $\{x_{n+1}, x_{n+2}, \cdots\} \subset W_0^{(n)}$ and $W_1^{(n)}$ is a neighborhood containing x_n such that $\{x_{n+1}, x_{n+2}, \cdots\} \cap W_1^{(n)} = \varnothing$. Therefore $x_n \notin \overline{\{x_{n+1}, x_{n+2}, \cdots\}}$. Hence we can select a collection $\{O_n\}$ of open sets as follows:

Let O_1 be an open subset of U_1 such that $x_1 \in O_1 \subset \bar{O}_1 \subset U_1$ and $\bar{O}_1$ does not contain $x_2, x_3, \cdots$. In general let O_n be an open subset of U_n such that $x_n \in O_n \subset \bar{O}_n \subset U_n$ and $\bar{O}_n$ does not contain $x_{n+1}, x_{n+2}, \cdots$ and such that $\bar{O}_i \cap \bar{O}_n = \varnothing$ $i = 1, 2, \cdots, n-1$.

Now define a function g' on $\overline{\bigcup_{n=1}^{\infty} \bar{O}_n}$ by

$$g' = \begin{cases} f^n \text{ on } \bar{O}_n \\ 1 \text{ elsewhere on } \overline{\bigcup_{n=1}^{\infty} \bar{O}_n} . \end{cases}$$

Then g' is continuous on $\overline{\bigcup_{n=1}^{\infty} \bar{O}_n}$. To see this we need only examine $t \in \overline{\bigcup_{n=1}^{\infty} \bar{O}_n} - \bigcup_{n=1}^{\infty} \bar{O}_n$. At such a t, $g'(t) = 1$. Let Q be any open set in the reals containing 1 and choose $N > 0$ such that $(1 - 1/N, 1 + 1/N) \subset Q$. Now since $t \notin \bigcup_{i=1}^{N} \bar{O}_i$, a closed set, there is a neighborhood V containing t such that $V \cap \bigcup_{i=1}^{N} \bar{O}_i = \varnothing$. For $s \in V \cap \overline{\bigcup_{i=N+1}^{\infty} \bar{O}_i}$ either $g'(s) = 1$ or $s \in \bar{O}_k$ for some $k \in \{N+1, N+2, \cdots\}$ in which case $|\, g'(s) = 1 \,| = |\, f^k(s) - 1 \,| < 1/k < 1/N$. So in any case $|\, g'(s) - 1 \,| < 1/N$ i.e. $g'(s) \in Q$ and hence g' is continuous. By Tietze's extension theorem g' can be extended to a function $g \in C(X)$.

Now for $y \in \mu^{-1}(O_n \cap X_0) \subset W$ we have by Lemma 3.4

$$Tg(y) = Tf^n(y) = [Tf(y)]^n \geqq c^n$$

so Tg is not bounded and hence $Tg \notin C(Y)$ a contradiction. Thus if $f(x_0) = 1$ then $Tf(y_0) \leqq 1$.

Now suppose that $f(x_0) = 1$ but that $Tf(y_0) < 1$. Set $f_1 = f^2$. Then $f_1(x_0) = 1$ so $Tf_1(y_0) \neq 0$ and $Tf_1(y_0) = [Tf(y_0)]^2 > 0$. By the first part of the proof $Tf_1(y_0) \not> 1$. We rule out $Tf_1(y_0) < 1$ as follows.

Set $W = \{x \mid f_1(x) > 1/2\}$. W is open, $x_0 \in W$ and if we set $f_2(x) = \max[1/2, f_1(x)]$ then f_2 is nowhere zero, $f_2 \in C(X)$ and f_2 agrees with f_1 on W. Hence by Lemma 3.4 $Tf_1 \equiv Tf_2$ on $\mu^{-1}(W \cap X_0)$ and so $0 < Tf_2(y_0) < 1$. Now $f_3 = 1/f^2 \in C(X)$, $f_3(x_0) = 1$ and $Tf_3(y_0) = 1/Tf_2(y_0) > 1$ a contradiction by the first part of the proof. Hence $Tf_1(y_0) = 1$ so that $Tf(y_0) = \pm 1$. But by assumption $Tf(y_0) < 1$ so $Tf(y_0) = -1$.

As done above let g be a strictly positive function in $C(X)$ agreeing with f on some neighborhood U containing x_0. Then Tf and Tg agree on $\mu^{-1}(U \cap X_0)$. But $g > 0$ everywhere on X implies that $Tg \geqq 0$ everywhere on Y and hence $Tg(y_0) \neq -1$ so $Tf(y_0) \neq -1$, a contradiction. Thus $Tf(y_0) = +1$ and the proof is finished.

LEMMA 3.6. *If $x_0 = \mu(y_0)$ and if $f(x_0) = g(x_0)$ then $Tf(y_0) = Tg(y_0)$.*

Proof. We need only consider $f(x_0) = c = g(x_0) \neq 0$. Let $h(x) = 1/c$ for all $x \in X$ so that $h \in C(X)$ and $hf(x_0) = hg(x_0) = 1$. By Lemma 3.5 $Thf(y_0) = 1 = Thg(y_0)$ i.e. $Th(y_0)Tf(y_0) = Th(y_0)Tg(y_0) = 1$. But $Th(y_0) \neq 0$

so the result follows.

Notice that Lemma 3.6 implies us that functions in $C(X)$ which agree on $X_0 = \mu(Y)$ have the same images in $C(Y)$. We will show that T is actually restriction to X_0 followed by a semi-group automorphism.

Suppose we regard the real numbers, R, as a multiplicative semi-group. We have the following.

LEMMA 3.7. *Let α be a semi-group homomorphism from R onto a dense subset of R. Then α is either unbounded in every neighborhood of zero or α is order preserving.*

Proof. Since the range of α is dense in R it follows that $\alpha(0) = 0$ and $\alpha(1) = 1$. If we show that $\alpha(-t) = -\alpha(t)$ for all t then only positive numbers need be considered in verifying the lemma. To this end note $\alpha(1) = [\alpha(-1)]^2$ so $\alpha(-1) = \pm 1$. We rule out $\alpha(-1) = +1$ for suppose $\alpha(-1) = +1$. Then $\alpha(\pm t) = \alpha(t)$ for all t. Let $\{t_n\}$ be a sequence in R such that $\alpha(t_n) \rightarrow -1$. Then $\alpha(-t_n) = \alpha(t_n) \rightarrow -1$ so that $\alpha(|t_n|) \rightarrow -1$. But $|t_n| = s_n^2$ for some $s_n \in R$ and $\alpha(s_n^2) = [\alpha(s_n)]^2 \rightarrow -1$ a contradiction. Hence $\alpha(-t) = -\alpha(t)$.

Now let $a, b \in R$ such that $0 < a < b$. Suppose $\alpha(a) > \alpha(b)$. Then since $\alpha(a/b) = \alpha(a)/\alpha(b)$ we have $\alpha[(a/b)^n] = [\alpha(a)/\alpha(b)]^n \rightarrow \infty$ while $(a/b) \rightarrow 0$ i.e. α is unbounded in every neighborhood of zero. Now suppose $\alpha(a) = \alpha(b)$ and that α is bounded in some neighborhood of zero. Then $r \in [a, b]$ implies that $\alpha(r) = \alpha(a)$ since otherwise either $\alpha(r) < \alpha(a)$ or $\alpha(r) > \alpha(b)$ and in both cases by the above α would be unbounded in every neighborhood of zero contradicting our assumption. Hence for $r, r' \in [a, b]$ $\alpha(r/r') = \alpha(r)/\alpha(r') = 1$. Now let z be any positive real number. There is an n such that $a/b \leqq z^{1/n} \leqq b/a$ i.e. there is an n such that $z^{1/n} = r/r'$, where $r, r' \in [a, b]$. Then $1 = \alpha(z^{1/n})$ and so $\alpha(z) = [\alpha(z^{1/n})]^n = 1$. So $z > 0$ implies that $\alpha(z) = 1$, $z < 0$ implies that $\alpha(z) = -1$ and $\alpha(0) = 0$, a contradiction since the image of α is dense in R. Thus $\alpha(a) = \alpha(b)$ implies that α is unbounded in every neighborhood of zero.

LEMMA 3.8. *If α is order preserving then α is actually onto and in this case $\alpha(t) = (\operatorname{sgn} t)\,|t|^p$ for some positive number p.*

Proof. Let $r_0 \in R$ and $\{r_n\}$ a sequence in R such that $\{r_n\} \downarrow r_0$. Then $\alpha(r_n) \rightarrow \alpha(r_0)$ since $\alpha(r_n) > \alpha(r_0)$ and if $\alpha(r_n) \geqq m > \alpha(r_0)$ there is an $s \in R$, $m > s > \alpha(r_0)$ and a $q \in R$ such that $\alpha(q) = s$. But $\alpha(r_0) < \alpha(q) < \alpha(r_n)$ for all n so $r_0 < q < r_n$, a contradiction since $r_n \rightarrow r_0$.

To see α is onto say r_0 is such that $\alpha(r) \neq r_0$ for any $r \in R$. We can choose a sequence of distinct points $\{\alpha(r_n)\}$ such that $\alpha(r_n) \downarrow r_0$. This implies $\{r_n\}$ is a bounded decreasing sequence so there is an r' such that $r_n \downarrow r'$ and hence from the above $\alpha(r_n) \rightarrow \alpha(r')$, a contradiction since $\alpha(r') \neq r_0$. Thus α is onto. Milgram [2, 4.3] has shown that in this case there is a $p > 0$ such that $\alpha(t) = (\operatorname{sgn} t) \mid t \mid^p$ which completes the proof.

In view of Lemma 3.6 for each $y \in Y$, $\alpha_{\mu(y)}: R \rightarrow R$ defined for arbitrary $f \in C(X)$ by $\alpha_{\mu(y)}(f(\mu(y))) = Tf(y)$ is well-defined. The image of $\alpha_{\mu(y)}$ is a dense subset in R, for fix $y \in Y$ and let $r \in R$. There is a function $g \in C(Y)$ such that $g(y) = r$ and a sequence $\{f_n\} \subset C(X)$ such that $Tf_n(y) \rightarrow g(y) = r$ i.e. $\alpha_{\mu(y)}(f_n(\mu(y))) \rightarrow r$.

Note that from Lemmas 3.7 and 3.8 we can say that $\alpha_{\mu(y)}$ is unbounded in every neighborhood of zero or $\alpha_{\mu(y)}$ is continuous.

LEMMA 3.9. *The mappings $\{\alpha_{\mu(y)}\}$ are discontinuous for at most a finite number of points.*

Proof. Suppose otherwise at $\{\mu(y'_n)\}$ where the y'_n are all distinct $n = 1, 2, 3, \cdots$. We can choose a subsequence $\{\mu(y_n)\}$ of distinct points such that no $\mu(y_n)$ is a limit point of the others as follows:

If no point in $\{\mu(y'_n)\}$ is a limit point of the other we are finished. If y'_{n_0} is a limit point of a subset of $\{\mu(y'_n)\}$ where $y'_{n_0} \in \{\mu(y'_n)\}$, by a process similar to that used in selecting the sequence $\{x_n\}$ in the proof of Lemma 3.5 with y'_{n_0} in the role of x_0 we obtain a sequence $\{\mu(y_n)\}$ such that $\mu(y_n) \notin \overline{\{\mu(y_{n+1}), \mu(y_{n+2}), \cdots\}}$, $n = 1, 2, 3, \cdots$. Hence for any $\mu(y_n)$ there is an open set V containing $\mu(y_n)$ such that $V \cap \{\mu(y_n)\} - \mu(y_n) = \emptyset$ so that $\{\mu(y_n)\}$ is the desired collection.

Now the $\alpha_{\mu(y_n)}$ are unbounded in each neighborhood of the origin so that if $\{t'_m\}$ is a sequence of distinct points decreasing to zero we have $\alpha_{\mu(y_n)}(t'_m) \rightarrow \infty$ for all n as $m \rightarrow \infty$. We select a subsequence $\{t_n\} \downarrow 0$ such that $\alpha_{\mu(y_n)}(t_n) \rightarrow \infty$ as follows:

There is a $t \in \{t'_m\}$ such that $\alpha_{\mu(y_1)}(t) > 1$. Set $t = t_1$. In general there is a $t < t_{n-1} < \cdots < t_1$, $t \in \{t'_m\}$ such that $\alpha_{\mu(y_n)}(t) > n$. Set $t = t_n$ to yield the desired sequence.

Define a function f' on $\overline{\{\mu(y_n)\}}$ by $f'(\mu(y_n)) = t_n$ and $f' = 0$ on $\overline{\{\mu(y_n)\}} - \{\mu(y_n)\}$. f' is continuous on $\overline{\{\mu(y_n)\}}$ since for $y_0 \in \overline{\{\mu(y_n)\}} - \{\mu(y_n)\}$ we have $f'(y_0) = 0$ and letting $\{\mu(y_m)\}$ be any subsequence converging to y_0, $f'(\mu(y_m)) = t_m \rightarrow 0 = f'(y_0)$.

Now we can extend f' to a continuous function f on all of X. But then $Tf(y_n) = \alpha_{\mu(y_n)}(f(\mu(y_n))) = \alpha_{\mu(y_n)}(t_n) \rightarrow \infty$ contradicting the fact that $Tf \in C(Y)$ and the lemma is proved.

We have via Lemma 3.8, that except for at most a finite number of points y,

$$\alpha_{\mu(y)}f(\mu(y)) = [\operatorname{sgn} f(\mu(y))]\,|f(\mu(y))|^{p(\mu(y))} \quad \text{where} \quad p(\mu(y))$$

is a positive function. We note that p is continuous where it is defined, i.e. on the set $\{\mu(y) \mid \alpha_{\mu(y)}$ is continuous$\}$, since for the constant function 2 we have $T2(y) = \alpha_{\mu(y)}(2) = [\operatorname{sgn} 2]\,|2|^{p(\mu(y))} = 2^{p(\mu(y))}$ and since $T2$ is continuous the result follows.

Using the fact that Y has no isolated points we show a stronger result.

LEMMA 3.10. *There is a positive continuous function p on X_0 such that*

$$\alpha_{\mu(y)}(f(\mu(y))) = [\operatorname{sgn} f(\mu(y))]\,|f(\mu(y))|^{p(\mu(y))}\,.$$

Proof. In view of the preceeding remarks we need only show that $\alpha_{\mu(y)}$ is continuous for all y. To this end suppose that $\alpha_{\mu(y)}$ is discontinuous at y_0. Set $A = \{\mu(y) \mid \alpha_{\mu(y)}$ is continuous$\}$. By Lemma 3.9 all but a finite number of the $\mu(y)$ are in A and hence since Y has no isolated points every open neighborhood about $\mu(y_0)$ contains points of A.

Now for $0 < s < 1$ define $S \in C(X)$ by $S(x) = s$. Since $\alpha_{\mu(y_0)}$ is unbounded in every neighborhood of zero we can find an $s_0 \in (0, 1)$ such that $\alpha_{\mu(y_0)}(s_0) > 2$. Let U be any neighborhood containing $\mu(y_0)$ and take $\mu(y) \in U \cap A$. Then $TS_0(y) = \alpha_{\mu(y)}(s_0) = [\operatorname{sgn} s_0]\,|s_0|^{p(\mu(y))} < 1$ but $TS_0(y_0) = \alpha_{\mu(y_0)}(s_0) > 2$ which contradicts the continuity of TS_0.

LEMMA 3.11. *The semi-group homomorphism T is an algebra homomorphism followed by a semi-group automorphism. Moreover T is continuous.*

Proof. From Lemma 3.10 we have

$$Tf(y) = [\operatorname{sgn} f(\mu(y))]\,|f(\mu(y))|^{p(\mu(y))}\,.$$

Identify Y as the subset X_0 of X and define $T_1\colon C(X) \to C(Y)$ by $T_1 f = f \mid Y$ (i.e. f restricted to Y) and note that T_1 is an onto algebra homomorphism. Define $T_2\colon C(Y) \to C(Y)$ by $T_2 g(y) = [\operatorname{sgn} g(y)]\,|g(y)|^{p(y)}$ where $p(y)$ is the continuous positive function arising in the previous lemma. T_2 is a semi-group automorphism. To see that T_2 is one-to-one suppose $f_1, f_2 \in C(Y)$ where $f_1 \neq f_2$. Then there is a $y \in Y$ such that $f_1(y) \neq f_2(y)$. Now if $|f_1(y)| \neq |f_2(y)|$ then $T_2 f_1(y) \neq T_2 f_2(y)$ and if $|f_1(y)| = |f_2(y)|$ then $\operatorname{sgn} f_1(y) \neq \operatorname{sgn} f_2(y)$ so that $T_2 f_1(y) \neq T_2 f_2(y)$. Thus T_2 is one-to-one. Clearly $T = T_2 T_1$.

To see that T is continuous it suffices to show that T_2 is continuous (T_1 is clearly continuous). A standard argument shows this to be the case.

Combining some of the previous results we have the following.

THEOREM 3.12. *Let X and Y be compact Hausdorff spaces, Y having no isolated points. Let $C(X)$ and $C(Y)$ be the multiplicative semi-groups of all continuous real valued function on X and Y respectively. If T is a point-determining semi-group homomorphism of $C(X)$ onto a dense point-separating set in $C(Y)$ then Y can be imbedded homeomorphically in X in such a way that*

$$Tf(y) = [\operatorname{sgn} f(x)]\,|\,f(x)\,|^{p(x)}$$

for some continuous positive function p where x is the unique point related to y by the induced homeomorphism. Such a homomorphism is continuous and is an algebra homomorphism followed by a semi-group automorphism.

COROLLARY 3.13. *Let X and Y be compact Hausdorff spaces, Y having no isolated points. Let T be a semi-group homomorphism of $C(X)$ onto a dense point-separating set of $C(Y)$. Then*

(i) *T is an algebra homomorphism of $C(X)$ into $C(Y)$ if and only if T is point-determining and $Tc = c$ for each constant function c.*

(ii) *If T is point-determining then $T(-f) = -Tf$.*

Proof. (i) If T is an algebra homomorphism of $C(X)$ we have already seen that T is point-determining and in fact that $Tf(y) = f(\mu(y))$ where μ is the induced homeomorphism. Hence $Tc = c$ for all constant functions c.

If T is point-determining and $Tc = c$ for all constant functions c then by the above theorem, for all y,

$$2 = T_2(y) = [\operatorname{sgn} 2]2^{p(\mu(y))} = 2^{p(\mu(y))}$$

and hence $p(\mu(y)) = 1$ for all y. Thus for $f \in C(X)$, $Tf(y) = f(\mu(y))$ so T is an algebra homomorphism.

The proof of (ii) is obvious by the form of the homomorphism shown in the above theorem.

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