

# Pacific Journal of Mathematics

**ON THE SPECTRAL RADIUS OF HERMITIAN ELEMENTS IN  
GROUP ALGEBRAS**

ANDRZEJ HULANICKI

## ON THE SPECTRAL RADIUS OF HERMITIAN ELEMENTS IN GROUP ALGEBRAS

A. HULANICKI

Let  $G$  be a discrete group and  $\mathfrak{A}$  the  $L_1$  algebra over the field of complex numbers of  $G$ . The aim of the paper is to consider some combinatorial conditions on the group  $G$  which imply symmetry of the algebra  $\mathfrak{A}$ . One result is as follows:

If a group  $F$  contains a subgroup  $G$  of finite index such that any element of  $G$  has finitely many conjugates, then the group algebra  $\mathfrak{A}$  of  $F$  is symmetric.

A Banach  $*$ -algebra  $\mathfrak{A}$  is said to be symmetric if for every element  $x \in \mathfrak{A}$  the spectrum of the element  $x^*x$  is real and nonnegative. If the algebra  $\mathfrak{A}$  contains the unit element, this can be equivalently stated as

For every linear functional  $F$  on  $\mathfrak{A}$

$$(1) \quad F(y + yx^*x) = 0 \text{ for every } y \in \mathfrak{A} \text{ implies } F = 0.$$

It is well known and easy to prove that the group algebra of a finite or Abelian locally compact group is symmetric. If a discrete group is neither finite nor Abelian, all that is known about symmetry of its group algebra seems to be due to Bonic [2].

Bonic proved the following facts:

(i) The group algebra of the direct product of two groups one of which is Abelian, the other of which has a symmetric group algebra is symmetric.

(ii) The group algebra of a splitting extension of an Abelian group by the cyclic group of two elements is symmetric.

(iii) The group algebra of a free nonAbelian group is not symmetric.

(i) and (ii) together with the fact that the group algebras of finite and Abelian groups are symmetric establishes the scope of groups about which, using Bonic's results, one may assert symmetry. As a matter of fact, it is not more than the direct products of finitely many finite, Abelian and dihedral-type groups. The method applied by Bonic is based on (1), which in the case of group algebra of a discrete group is equivalent to the implication

$$(2) \quad \text{If } f \text{ is bounded and } f(t) + \sum_{s \in G} x^*x(s^{-1}t)f(s) \equiv 0, \text{ then } f \equiv 0.$$

It is not surprising that, if some of the elements of the group appear as complicated expressions of the generators, then to prove or disprove (2) may become difficult.

We propose another approach to this circle of problems.

Our starting point is a theorem of Raikov, cf. [9], [10]. Denote by  $\nu(x)$  the spectral radius of an element  $x$  in a Banach  $*$ -algebra  $\mathfrak{A}$ , that is, let

$$\nu(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}.$$

For any  $*$ -representation  $x \rightarrow T_x$  of  $\mathfrak{A}$  we have

$$\nu(x^*x) \geq \|T_{x^*x}\|,$$

cf. e.g. [11]. Raikov's theorem says:

A Banach  $*$ -algebra  $\mathfrak{A}$  is symmetric if, and only if,

$$(3) \quad \nu(x^*x) = \sup \|T_{x^*x}\|$$

where the least upper bound on the right hand side is taken over all  $*$ -representations of  $\mathfrak{A}$ .

Now let  $\mathfrak{A}$  be a group algebra  $\mathfrak{A} = \mathfrak{A}(G)$  of a discrete group  $G$  with the usual norm, multiplication, and involution

$$\|x\| = \sum_{s \in G} |x(s)|, \quad x*y(s) = \sum_{t \in G} x(t)y(t^{-1}s), \quad x^{\sim}(s) = \overline{x(s^{-1})},$$

respectively. Consider the left regular representation  $T$  of  $\mathfrak{A}(G)$   $x \rightarrow T_x$ , where the Hilbert space of  $T$  is  $L_2(G)$  equipped with norm

$$\|x\|_2 = \left(\sum_{s \in G} |x(s)|^2\right)^{1/2} \quad \text{and} \quad T_x f = x*f, \quad f \in L_2(G).$$

Denote

$$\lambda(x) = \|T_x\|.$$

It may happen (cf. [7]) that the left regular representation weakly contains (cf. [3]) all  $*$ -representation of  $\mathfrak{A}(G)$ , which is to say that  $\lambda(x^{\sim}*x) \geq \|T'_{x^{\sim}*x}\|$  for any  $*$ -representation  $T'$  of  $\mathfrak{A}(G)$ . Then (3) turns into the equality

$$(4) \quad \lambda(x^{\sim}*x) = \nu(x^{\sim}*x).$$

Our aim is to find sufficient conditions on a group  $G$  which imply (4) for any element  $x$  of its group algebra  $\mathfrak{A}(G)$ , and consequently the symmetry of  $\mathfrak{A}(G)$ . We formulate the conditions in the forthcoming § 1 and we devote § 2 to the proof that, in fact, they imply (4). Finally, in § 3, we discuss the classes of groups for which our conditions are satisfied.

The author is most indebted to Professor B. H. Neumann for a conversation about these classes of groups as well as to Dr. Neil W. Rickert who suggested several corrections and simplifications.

1. Let  $G$  be a discrete group,  $\mathfrak{A} = \mathfrak{A}(G)$  its group algebra. For any function  $x$  on  $G$  we denote by  $N(x)$  the support of the function  $x$ , i.e.,

$$N(x) = \{s: x(s) \neq 0\}.$$

Let  $A_1, \dots, A_n$  be a family of subset of  $G$ . By  $A_1 \dots A_n$  we denote the set of all products  $a_1 \dots a_n$ , where  $a_i \in A_i$ ,  $i = 1, \dots, n$ . We shall also use the abbreviated notation  $A^n$  for  $A \dots A$ . For a finite subset  $A$  of  $G$  we denote by  $|A|$  the number of the elements of  $A$ . Clearly,

$$(5) \quad |A_1 \dots A_n| \leq |A_1| \dots |A_n|.$$

For any two nonnegative integers  $m \leq n$  we denote by  $t(m, n)$  a sequence

$$t(m, n) = \langle t_1, \dots, t_n \rangle$$

of elements of  $G$  such that at most  $m$  of the  $t_i$ 's,  $i = 1, \dots, n$ , are different from the unity of  $G$ . We say that a group  $G$  satisfies condition (C) if

(C) There exists a constant  $k$  such that for any finite set  $A \subset G$  there exists a constant  $C = C(A, c)$  such that for any sequence  $\langle t_1, \dots, t_n \rangle = t(m, n)$

$$|At_1At_2 \dots At_n| \leq Ck^m c^n \quad \text{for any } c > 1.$$

For  $m = 0$  (or, which is the same, for bounded  $m$ ) condition (C) turns into the following one:

(A - S) For any finite subset  $A$  of  $G$

$$|A^n| = o(c^n) \quad \text{for any } c > 1.$$

This condition has been considered by G. M. Adel'son-Vel'skiĭ and Yu. A. Šreider in [1]. They have proved that (A - S) implies the existence of an invariant Banach mean value on  $G$ . Clearly, (A - S) implies the Følner conditions cf. [4].

2. First we formulate two simple lemmas.

LEMMA 1. *If  $G$  is a (discrete) group, then for any  $x \in \mathfrak{A}(G) \subset L_2(G)$ , we have  $\lambda(x) \geq \|x\|_2$ .*

In fact,

$$\lambda(x) = \sup_{\|y\|_2=1} \|x*y\|_2 \geq \|x*\delta\|_2 = \|x\|_2,$$

where  $\delta$ , the unit element of  $\mathfrak{A}(G)$ , is the function which takes value 1 at the unity of  $G$  and zero elsewhere.

LEMMA 2. *If  $x_1, \dots, x_n \in \mathfrak{A}(G)$ , then*

- (i)  $x_1 * \dots * x_n(s) = \sum_{t_1, \dots, t_n \in G} x_1(t_1) \dots x_n(t_n) \delta(t_n^{-1} \dots t_1^{-1}s)$
- (ii)  $N(x_1 * \dots * x_n) \subseteq N(x_1) \dots N(x_n)$
- (iii)  $N(x^\sim) = (N(x))^{-1} = \{t^{-1} : t \in N(x)\}$ .

The proof of (i) is obtained by an elementary induction. To verify (ii) we simply note that, by (i),  $s \in N(x_1 * \dots * x_n)$  implies

$$t_{n-1}^{-1} \dots t_1^{-1}s = t_n \in N(x_n) \quad \text{and} \quad t_i \in N(x_i), \quad i = 1, \dots, n.$$

Therefore  $s = t_1 \dots t_n \in N(x_1) \dots N(x_n)$ .

From now on we shall frequently use the abbreviated notation  $x^{*n}$  for  $x * \dots * x$ .

THEOREM 1. *If a group  $G$  satisfies (A – S), then for every  $x$  such that  $N(x)$  is finite equality (4) holds.*

*Proof.* Suppose  $N(x)$  is finite  $\neq \emptyset$ . Let  $A = (N(x))^{-1}N(x)$ . Then  $A$  is finite and, by Lemma 2 (ii)-(iii),  $N(x^\sim * x) \subset A$ . We have

$$\begin{aligned} \|(x^\sim * x)^{*n}\| &= \sum_{s \in G} |(x^\sim * x)^{*n}(s)| \\ &= \sum_{s \in A^n} |(x^\sim * x)^{*n}(s)| \leq |A^n|^{1/2} \left( \sum_{s \in A^n} |(x^\sim * x)^{*n}(s)|^2 \right)^{1/2} \\ &= |A^n|^{1/2} \|(x^\sim * x)^{*n}\|_2. \end{aligned}$$

Hence, by Lemma 1, since  $T_{x^\sim * x}$  is a hermitian operator,

$$\|(x^\sim * x)^{*n}\| \leq |A^n|^{1/2} \lambda((x^\sim * x)^{*n}) = |A^n|^{1/2} (\lambda(x^\sim * x))^{*n}.$$

Consequently,

$$\nu(x^\sim * x) = \lim_{n \rightarrow \infty} \|(x^\sim * x)^{*n}\|^{1/n} \leq \limsup_{n \rightarrow \infty} [|A^n|^{1/2n} \lambda(x^\sim * x)].$$

But, since  $G$  satisfies (A–S) and  $A \neq \emptyset$ ,

$$\limsup_{n \rightarrow \infty} |A^n|^{1/2n} = 1,$$

whence  $\nu(x^\sim * x) \leq \lambda(x^\sim * x)$ , which completes the proof of Theorem 1.

**COROLLARY 1.** *If a group  $G$  satisfies condition (A—S), then for any  $x \in \mathfrak{A}(G)$  such that  $N(x)$  is finite  $\text{Sp}_{\mathfrak{A}(G)}(x^*x) \geq 0$ .*

*Proof.* By a standard argument (cf. e.g. [9 pp. 300–301]) we reduce our task to proving that for each  $x \in \mathfrak{A}(G)$  with finite support  $(\delta + x^*x)^{-1}$  exists. Let  $\mathfrak{A}$  be the (commutative) algebra of hermitian elements generated (algebraically) by  $\delta$  and  $\delta + x^*x = y_0$ . Then, by Lemma 2 (ii)–(iii), for any  $y \in \mathfrak{A}$ ,  $N(y)$  is finite, whence, since  $G$  satisfies (A—S), by Theorem 1,  $\nu(y^*y) = \lambda(y^*y)$  and, consequently, since  $y^* = y$ ,  $\nu(y) = \lambda(y)$ .

Therefore the completion  $\mathfrak{A}^\nu$  and  $\mathfrak{A}^\lambda$  of  $\mathfrak{A}$  in the norms  $\nu$  and  $\lambda$  respectively, are equal. But  $\mathfrak{A}^\lambda$  is isometrically isomorphic to a Banach \*-algebra with the unity of hermitian operators, so  $y_0^{-1}$  exists in  $\mathfrak{A}^\lambda$  and, consequently,  $y_0^{-1} \in \mathfrak{A}^\nu$ . This shows that in the Gelfand representation of  $\mathfrak{A}^{||\cdot||}$ , the completion of  $\mathfrak{A}$  in the norm  $||\cdot||$ , we have  $\hat{y}_0(M) \neq 0$  for any maximal ideal  $M$  of  $\mathfrak{A}^{||\cdot||}$ , which, by Gelfand's theorem, proves that  $y_0^{-1} \in \mathfrak{A}^{||\cdot||} \subset \mathfrak{A}(G)$ , as required.

By virtue of Theorem 1, if  $G$  satisfies (A—S), then equality (4) holds for a dense (in the norm  $||\cdot||$ ) set of hermitian elements  $x$  of  $\mathfrak{A}(G)$ . Clearly the function  $\lambda$  is continuous on  $\mathfrak{A}(G)$ , so if we knew that  $\nu$  were continuous, equality (4) would be an immediate consequence of Theorem 1. However, it is known that, in general, for a noncommutative Banach \*-algebra the function need not be continuous. The example of S. Kakutani, as presented in [11 p. 282], shows that  $\nu$  is not continuous on the algebra of the bounded operators of a Hilbert space. In this case, however,  $\nu$  is continuous on hermitian elements. We do not know whether in the case of a nonsymmetric group algebra the function  $\nu$  is continuous on hermitian elements.

**THEOREM 2.** *If a group  $G$  satisfies condition (C), then equality (4) holds for all elements  $x \in \mathfrak{A}(G)$ .*

*Proof.* Suppose  $G$  satisfies (C) and let  $z \in \mathfrak{A}(G)$ . For any positive  $\varepsilon$  we write

$$(6) \quad z^*z = y + x$$

where  $N(x) = A$  is finite and  $||y|| < \varepsilon$ . We are going to prove

$$(7) \quad \nu(z^*z) \leq k ||y|| + \lambda(x),$$

where  $k$  is the constant whose existence is postulated by condition (C). Since  $\lambda$  is continuous (with respect to the norm  $||\cdot||$ ), (7) implies

$$\nu(z \sim * z) \leq \lambda(z \sim * z),$$

as desired. To prove (7) we write

$$(z \sim * z)^{*n} = (y + x)^{*n} = \sum_{m=0}^n \sum_{\sigma, \tau} y^{*p_1} * x^{*q_1} * \dots * y^{*p_r} * x^{*q_r},$$

where the summation extends over the set  $C(m, n)$  of all sequences  $\sigma = \langle p_1, \dots, p_r \rangle$  and  $\tau = \langle q_1, \dots, q_r \rangle$  of nonnegative integers such that  $p_1 + \dots + p_r = m$ ,  $q_1 + \dots + q_r = m - n$ . Hence

$$(8) \quad \|(z \sim * z)^{*n}\| \leq \sum_{m=0}^n \sum_{\sigma, \tau} \|y^{*p_1} * x^{*q_1} * \dots * y^{*p_r} * x^{*q_r}\|.$$

By Lemma 2 (i), we have

$$\begin{aligned} \|y^{*p_1} * x^{*q_1} * \dots * y^{*p_r} * x^{*q_r}\| &= \|y^{*p_1} * x^{*q_1} * \dots * y^{*p_r} * x^{*q_r} * \delta\| \\ &= \sum_{s \in G} \left| \sum_{\alpha, \beta} y(t_1) \dots y(t_m) x(s_1) \dots x(s_{n-m}) \delta(\alpha_r^{-1} b_r^{-1} \dots \alpha_1^{-1} b_1^{-1} s) \right|, \end{aligned}$$

where the summation is all over the set of sequences

$$\begin{aligned} \alpha &= \langle t_1, \dots, t_m \rangle, t_i \in G, & i &= 1, \dots, m; \\ \beta &= \langle s_1, \dots, s_{n-m} \rangle, s_i \in G, & i &= 1, \dots, n - m \\ \alpha_i &= s_{q_1 + \dots + q_{i-1} + 1} \dots s_{q_1 + \dots + q_i}, & i &= 1, \dots, r, q_0 = 0 \\ b_i &= t_{p_1 + \dots + p_{i-1} + 1} \dots t_{p_1 + \dots + p_i}, & i &= 1, \dots, r, p_0 = 0. \end{aligned}$$

Hence

$$\begin{aligned} (9) \quad \|y^{*p_1} * x^{*q_1} * \dots * y^{*p_r} * x^{*q_r}\| &\leq \sum_{s \in G} \sum_{\alpha} |y(t_1)| \dots |y(t_m)| \cdot \\ &\left| \sum_{\beta} x(b_1^{-1} s_1) x(s_2) \dots x(s_{q_1}) x(b_2^{-1} s_{q_1+1}) x(s_{q_1+2}) \dots x(s_{q_1+q_2}) \dots \right. \\ &\quad \left. x(b_r^{-1} s_{q_1 + \dots + q_{r-1} + 1}) x(s_{q_1 + \dots + q_{r-1} + 2}) \dots x(s_{n-m}) \delta(s_{n-m}^{-1} \dots s_1^{-1} s) \right| \\ &= \sum_{\alpha} |y(t_1)| \dots |y(t_m)| \|x_{b_1} * x^{*(q_1-1)} * \dots * x_{b_r} * x^{*(q_r-1)}\|, \end{aligned}$$

where  $x_{b_i}(s) = x(b_i^{-1} s)$ ,  $i = 1, \dots, r$ . But, since  $N(x) = A$ ,  $N(x_{b_i}) = b_i A$ ,  $i = 1, \dots, r$ . Hence, by Lemma 2(ii),

$$N(x_{b_1} * x^{*(q_1-1)} * \dots * x_{b_r} * x^{*(q_r-1)}) \subset b_1 A A^{(q_1-1)} \dots b_r A A^{(q_r-1)} = b_1 A^{q_1} \dots b_r A^{q_r} = P.$$

Therefore, by Lemma 1 and Schwarz inequality,

$$\begin{aligned} (10) \quad &\|x_{b_1} * x^{*(q_1-1)} * \dots * x_{b_r} * x_{b_r} * x^{*(q_r-1)}\| \\ &\leq |P|^{1/2} \|x_{b_1} * x^{*(q_1-1)} * \dots * x_{b_r} * x^{*(q_r-1)}\|_2 \\ &\leq |P|^{1/2} \lambda(x_{b_1})(\lambda(x))^{q_1-1} \dots \lambda(x_{b_r})(\lambda(x))^{q_r-1} \\ &= |P|^{1/2} (\lambda(x))^{n-m}, \end{aligned}$$

the last equality being true by virtue of the easily verified fact that  $\lambda(x_b) = \lambda(x)$ .

On the other hand,

$$|P| \leq \sup_{b_1, \dots, b_r \in G} |b_1 A^{q_1} \cdots b_r A^{q_r}| \leq \sup |t_1 A t_2 A \cdots t_{n-m} A|$$

where the least upper bound on the right hand side is taken over all sequences  $t(m, n - m) = \langle t_1, \dots, t_{n-m} \rangle$  with at most  $m$  of the  $t_i$ 's different from the unity of  $G$ . Hence, since  $G$  satisfies (C),

$$|P|^{1/2} \leq C(A, c) k^m c^n \text{ for any } c > 1.$$

Consequently, by (10) and (9),

$$\begin{aligned} (11) \quad & \| y^{*p_1} * x^{*q_1} * \cdots * y^{*p_r} * x^{*q_r} \| \\ & \leq \sum_{\alpha} |y(t_1)| \cdots |y(t_m)| k^m c^n C(\lambda(x))^{n-m} \\ & = c^n C k^m \|y\|^m (\lambda(x))^{n-m}. \end{aligned}$$

Therefore, by (8) and (11),

$$\| \tilde{z}^{\sim} * z^{\sim} \| \leq f_A(n, n) \sum_{m=0}^n \binom{n}{m} k^m \|y\|^m (\lambda(x))^{n-m} = c^n C (k \|y\| + \lambda(x))^n.$$

Hence we have

$$\begin{aligned} \nu(z^{\sim} * z) &= \lim_{n \rightarrow \infty} \| (z^{\sim} * z)^{\sim n} \|^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} c C^{1/n} (k \|y\| + \lambda(x)) = (k \|y\| + \lambda(x)) c. \end{aligned}$$

This completes the proof of Theorem 2.

3. In this section we present some propositions which exhibit some of the classes of groups for which conditions (A—S) and (C) are satisfied. First we establish some notations:

Let  $G$  be a group and let  $a, b \in G$ . We write

$$b^{-1}ab = a^b, a^{-1}b^{-1}ab = [a, b], [[a_1, \dots, a_{n-1}], a_n] = [a_1, \dots, a_n].$$

If  $A, B$  are subsets of  $G$ ,

$$A^B = \{a^b : a \in A, b \in B\}.$$

REMARK. If  $A$  is a subset of  $G$ , then for  $a_1, \dots, a_n \in A^G$  and  $b_1, \dots, b_n \in G$  we have

$$a_1 b_1 \cdots a_n b_n = b_1 \cdots b_n a'_1 \cdots a'_n, \text{ where } a'_1, \dots, a'_n \in A^G.$$

(This follows immediately from the fact that  $ab = ba^b$  for any  $a, b \in G$ ).



PROPOSITION 1. Let  $A$  be a finite subset of a group  $G$  such that  $A = A^G$ . Then  $|A^n| \leq n^{|A|}$ .

*Proof.* If  $a_1, \dots, a_n \in A$ , then

$$(12) \quad a_1 \cdots a_n = b_1^{n_1} \cdots b_k^{n_k}, \quad b_1, \dots, b_k \in A$$

and, moreover,  $n_1 + \dots + n_k \leq n$  and  $k \leq |A|$ . To prove (12) which is, in fact, a version of Dietzmann's Lemma (cf. [8 Vol. II, p. 154]), we suppose that  $k$  is minimal with respect to all representations of  $a_1 \cdots a_n$  of the form (12). Suppose  $k > |A|$ . Then there exists at least two elements  $b_i$  and  $b_j$ ,  $1 \leq i < j \leq k$  such that  $b_i = b_j$ . Then

$$b_1^{n_1} \cdots b_k^{n_k} = b_1^{n_1} \cdots b_{i-1}^{n_{i-1}} b_i^{n_i+n_j} (b_{i+1}^c)^{n_{i+1}} \cdots (b_{j-1}^c)^{n_{j-1}} b_j^{n_j-1} \cdots b_k^{n_k},$$

where  $c = b_j^j$ , which is a contradiction, since  $b_s^c \in A^G = A$ . By (12), we have

$$|A^n| \leq (\text{the number of the sequences } n_1, \dots, n_k \text{ with } \sum_i n_i \leq n \text{ and } k \leq |A|) \leq n^{|A|}.$$

DEFINITION. A group  $G$  is called a FC-group if for any  $a \in G$  the set  $\{a\}^G$  is finite. (cf. e.g. [8 Vol. II, pp. 154 and 269]).

PROPOSITION 2. If  $G$  is a FC-group, then it satisfies (C) with constant  $k = 1$ .

In fact, let  $A$  be a finite subset of  $G$ . Since  $G$  is FC-group, the set  $B = A^G$  is finite. Hence

$$|t_1 A t_2 A \cdots t_n A| \leq |A^{t_1 t_2 \cdots t_n} A^{t_3 \cdots t_n} \cdots A^{t_n}| \leq |B^n| \leq n^{|B|} = o(c^n)$$

for any  $c > 1$ .

PROPOSITION 3. If  $F$  is a group which contains a subgroup  $G$  of finite index in  $F$  such that  $G$  is a FC-group, then  $F$  satisfies (C).

*Proof.* We show first that if  $M$  is a finite subset of  $G$ , then the set  $M^F$  is finite. Let  $M_0 = M^G$ . Since  $G$  is FC-group,  $M_0$  is finite. Denote by  $\phi$  the function which selects one element  $\phi(f)$  out of each of the cosets  $Gf, f \in F$ , and let  $\phi(G) = 1$  — the unit element of  $F$ .

Let

$$\phi(F) = H = \{h_1, \dots, h_k\}.$$

We have

$$M^F = M_1 = \bigcup_{f \in F} M_0^{\phi(f)}.$$

In fact, if  $s, f \in F$ , there exists a  $g \in G$  such that

$$\phi(f)s = g\phi(\phi(f)s) = g\phi(fs).$$

Hence

$$M_1^s = \bigcup_{f \in F} M_0^{\phi(f)s} = \bigcup_{f \in F} M_0^{g\phi(fs)} = \bigcup_{f \in F} M_0^{\phi(fs)} = M_1.$$

Thus, since  $\phi(F)$  is finite,  $M^F$  is finite.

Now, let

$$Q = \{h_i h_j (\phi(h_i h_j))^{-1} : i, j = 1, \dots, k\}.$$

Clearly,  $Q$  is finite and  $Q \subset G$ . By virtue of what we have proved above,  $Q^F = Q_1$  is finite. We now prove that for any sequence  $u_1, \dots, u_n$  with  $u_i \in \{1, \dots, k\}$ ,  $i = 1, \dots, n$ , we have

$$(13) \quad h_{u_1} \dots h_{u_n} \in Q^n H \subset Q_1^n H = HQ_1^n.$$

Let  $f_0 = 1, f_i = h_{u_1} \dots h_{u_i}, i = 1, \dots, n$ , and let  $a_i = \phi(f_i)$ . Then

$$f_n a_n^{-1} = a_0 h_{u_1} a_1^{-1} a_1 h_{u_2} a_2^{-1} a_2 \dots a_{n-1}^{-1} a_{n-1} h_{u_n} a_n^{-1}.$$

But for any  $i = 1, \dots, n$ ,

$$a_i = \phi(f_i) = \phi(f_{i-1} h_{u_i}) = \phi(\phi(f_{i-1}) h_{u_i}) = \phi(a_{i-1} h_{u_i}).$$

Hence

$$f_n a_n^{-1} = a_0 h_{u_1} (\phi(a_0 h_{u_1}))^{-1} \dots a_{n-1} h_{u_n} (\phi(a_{n-1} h_{u_n}))^{-1} \in Q^n.$$

and, consequently,

$$h_{u_1} \dots h_{u_n} = f_n \in Q^n a_n \subset Q^n H,$$

which completes the proof of (13).

Now let  $A$  be a finite subset of  $F$ . There exists a finite set  $M \subset G$  such that  $A \subset HM$ . Consider the finite set

$$B = HM^F.$$

Clearly,  $A \subset B$ . For every two nonnegative integers  $m \leq n$ , let  $t(m, n)$  be as in Section 1. We have

$$(14) \quad |t_1 B \dots t_n B| = |(B \dots B)^{s_1} (B \dots B)^{s_2} \dots (B \dots B)^{s_{m+1}}|,$$

where  $\langle t_1, \dots, t_n \rangle = t(m, n)$ ,  $t_{j_1}, \dots, t_{j_m}$  are the  $t_j$ 's different from the unity of  $F$ , for every  $i = 1, \dots, m$   $s_i = t_{j_i} \dots t_{j_m}$  and  $s_{m+1} = 1$ .

Each element of the set on the right hand side of (14) is of the form

$$f = (h_1 g_1 \dots h_{r_1} g_{r_1})^{s_1} \dots (h_{r_{m+1}} g_{r_{m+1}} \dots h_{r_{m+1}} g_{r_{m+1}})^{s_{m+1}},$$

where  $g_i \in M^F, i = 1, \dots, n$ . Then

$$f = (h_1 \cdots h_{r_1})^{s_1} \cdots (h_{r_m+1} \cdots h_{r_{m+1}})^{s_{m+1}} g'_1 \cdots g'_n,$$

where  $g'_i \in M^F, i = 1, \dots, n$ . But, by (13), for any  $i = 1, \dots, m + 1$  we have

$$h_{r_{i-1}} \cdots h_{r_i} = h'_i q_i, \text{ where } h'_i \in H \text{ and } q_i \in Q^{(r_i - r_{i-1})}.$$

Consequently, since  $Q_1^F = Q_1$ ,

$$f = h_1^{s_1} \cdots h_{m+1}^{s_{m+1}} q'_1 \cdots q'_{m+1} g'_1 \cdots g'_n$$

and so, since

$$\sum_{i=1}^{m+1} (r_i - r_{i-1}) = n,$$

$$f \in H^{s_1} \cdots H^{s_{m+1}} Q_1^n M_1^n.$$

Therefore, since  $|H^s| = |H|$  for any  $s \in F$ , and  $|H| = k$ ,

$$|t_1 B \cdots t_n B| \leq |H^{s_1}| \cdots |H^{s_{m+1}}| |Q_1^n| |M_1^n| \leq |H|^m |H| n^{(|Q_1| + |M_1|)},$$

whence, since  $|H| n^{(|Q_1| + |M_1|)} = o(c^n)$ , Proposition 3 follows.

**COROLLARY 2.** *If a group  $G$  contains a FC-group as a subgroup of finite index, then equality (4) holds for any element  $x$  of the group algebra  $A(G)$  of  $G$  and, consequently,  $A(G)$  is symmetric.*

**PROPOSITION 4.** The direct product of finitely many groups which satisfy (C) satisfies (C).

**PROPOSITION 5.** If  $G$  is a locally nilpotent group, then  $G$  satisfies (A—S).

This follows from the fact that if  $A = \{a_1, \dots, a_k\}$  is a finite subset of  $G$ ,  $H$  is the subgroup generated by  $A$ , which, by assumption, is nilpotent of class, say,  $s$ , then any element of  $A^n \subset H$  is of the form

$$\prod_{i=1}^k a_i^{n(i)} \prod_{i_1, i_2=1}^k [a_{i_1}, a_{i_2}]^{n(i_1, i_2)} \cdots \prod_{i_1, \dots, i_s=1}^k [a_{i_1}, \dots, a_{i_s}]^{n(i_1, \dots, i_s)}$$

where  $n(i_1, \dots, i_s)$  are nonnegative integers  $\leq f_j(n), j = 1, \dots, s$ , and  $f_j(n)$  is an integral valued polynomial. (The exact form of  $f_j(n)$  can be easily established by the use of the collecting process as described in [6] and [5] but it is irrelevant to us here.) Thus

$$|A^n| \leq (f_1(n))^{k^1} \cdots (f_s(n))^{k^s} = o(c^n) \text{ for any } c > 1.$$

**PROPOSITION 6.** Condition (C) is not satisfied by nilpotent groups, in general.

PROPOSITION 7. Condition (A—S) is not satisfied by soluble groups, in general.

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Received January 10, 1965.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$ 8.00; single issues, \$ 3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$ 4.00 per volume; single issues \$ 1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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