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THE EXISTENCE OF WAVE OPERATORS FOR NONLINEAR EQUATIONS

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In this paper, conditions will be found on nonlinear perturbations of the evolution equation with skew adjoint generator sufficient to guarantee the existence of nontrivial mild global solutions of the perturbed equation which converge to solutions of the unperturbed equation as $t \to +\infty$ or $-\infty$. These results are used to prove the existence of wave operators for certain semi-linear hyperbolic equations.

Let H be a real separable Hilbert space and A a skew adjoint linear operator on H. Denote the one parameter unitary group generated by A by e^{tA}.

Let K be a map from $R \times H \to H$ denoted by $K_t(u)$ for $u \in H$, $t \in R$. Assume also that $K_t(0) = 0 \in H$ for all $t \in R$. The object of study in this paper will be functions $u : R \to H$ such that $K_t(u(t))$ is Bochner measurable and

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}K_s(u(s))ds \tag{1}$$

where u_0 is some element of H. Note that it is assumed that u is defined for all real t; i.e., it is a global solution of (1). An elementary sufficient condition that there exist such u is that K_t is globally Lipschitzian. That is, there exist k such that

$$\| K_t(u) - K_t(v) \| \leqq k \| u - v \|$$

for all $t \in R$ and $u, v \in H$.

If $u(t)$ is strongly differentiable, is in $\mathscr{D}_A$ for all t and satisfies (1) then

$$\frac{du}{dt} = Au + K_t(u) \qquad u(0) = u_0 \,. \tag{2}$$

For this reason solutions of (1) are called global mild solution of (2). These are the only type of solutions this paper will be interested in.

The main question of this paper is whether there are solutions of (1) which behave for large positive or negative t like $e^{tA}v$ for some $v \in H$. If $\| K_t(u) - K_t(v) \| \leqq \sigma(t) \| u - v \|$ for all $u, v \in H$ and σ bounded integrable then $u(t)$ exists for all $u_0 \in H$ and behaves in this manner (Prop. 3). If K does not depend on time, the results are not as straightforward. Propositions 4 and 5 give sufficient conditions on an element $v_0 \in H$ that there exist a solution of (1) behaving as $t \to -\infty$

or $+\infty$ like $e^{tA}v_0$. These results are then applied to the question of existence of wave operators certain semi-linear hyperbolic equations which may be written in the form (1) or (2).

2. Wave operators. It is convenient to define a number of terms which will be useful later.

DEFINITION. $\mathscr{W}_-(\mathscr{W}_+)$ is a map from a subset $\mathscr{D}_-(\mathscr{D}_+)$ of H to a subset $\mathscr{R}_-(\mathscr{R}_+)$ of H defined as follows. $u_0 \in \mathscr{D}_-$ if $\lim_{t\to-\infty} e^{-tA}u(t)$ exists where $u(t)$ is a solution of (1) with initial condition u_0.

$$\mathscr{W}_- u_0 = \lim_{t\to-\infty} e^{-tA}u(t) \ .$$

$\mathscr{W}_-(\mathscr{W}_+)$ is the backward (forward) wave operator.

REMARK. 1. The above objects depend on A and K_t. If necessary they will be written $\mathscr{W}_-^{A,K}$, $\mathscr{D}_-^{A,K}$, etc.

2. All proofs will be for $\mathscr{W}_-$, $\mathscr{D}_-$, $\mathscr{R}_-$. Similar proofs with the obvious changes will work for $\mathscr{W}_+$, $\mathscr{D}_+$, $\mathscr{R}_+$.

3. If $u_0 \in \mathscr{D}_-$ and $w_0 = \mathscr{W}_- u_0 \in R_-$ then $e^{tA}w_0$ satisfies (1) with $K_t \equiv 0$ and initial condition w_0. Moreover $\| e^{tA}w_0 - u(t) \| \to 0$ as $t \to -\infty$.

PROPOSITION 1. Let w_0 be in H. Suppose there exists a Bochner measurable function $v : R \to H$ such that

(a) $$\int_{-\infty}^{0} \| K_t(e^{tA}v(t)) \| \, dt < \infty$$

(b) $$v(t) = w_0 + \int_{-\infty}^{t} e^{-sA}K_s(e^{sA}v(s))ds \quad t \leqq 0 \ .$$

Then $w_0 \in \mathscr{R}_-$ and $v(0) \in \mathscr{D}_-$ with $\mathscr{W}_- v(0) = w_0$.

Proof. Let $u(t) = e^{tA}v(t)$. Then $u(0) = v(0)$. By (b),

$$\begin{aligned} u(t) &= e^{tA}w_0 + \int_{-\infty}^{t} e^{(t-s)A}K_s(u(s))ds \\ &= e^{tA}v(0) + \int_{0}^{t} e^{(t-s)A}K_s(u(s))ds \ . \end{aligned}$$

$u(t)$ is a solution of (1) and with initial data $v(0)$, and

$$\| e^{-tA}u(t) - w_0 \| = \| v(t) - w_0 \| \leqq \int_{-\infty}^{t} \| K_t(e^{tA}v(t)) \| \, dt \to 0 \text{ as } t \to -\infty .$$

COROLLARY 1. *Let u_0 be in H such there exists a solution $v(t)$ to*

(1) *with initial data* u_0. *Then if* $\int_{-\infty}^{0} \| K_s(u(s)) \| ds < \infty$, $u_0 \in \mathscr{D}_-$ *and*

$$\mathscr{W}_- u_0 = u_0 - \int_{-\infty}^{0} e^{-sA} K_s(u(s)) ds .$$

PROPOSITION 2. If K_t is Lipschitzian with constant k and u_0 and v_0 are $\mathscr{D}_-$ with $\mathscr{W}_- u_0 = \mathscr{W}_- v_0$ then $u_0 = v_0$ if

$$\int_{-\infty}^{s} \| u(t) - v(t) \| dt \leqq \gamma e^{cs} \qquad s \leqq 0,\ c > k$$

and $u(t)$ and $v(t)$ satisfy the hypothesis of Corollary 1.

Proof.

$$\begin{aligned} \| u(t) - v(t) \| &\leqq \int_{-\infty}^{t} \| K_s(u(s)) - K_s(v(s)) \| ds \\ &\leqq k \int_{-\infty}^{t} \| u(s) - v(s) \| ds \end{aligned}$$

Let $\sigma(t) = \| u(t) - v(t) \|$ and

$$\varphi(t) = \int_{-\infty}^{t} \| u(s) - v(s) \| ds \leqq \gamma e^{ct} .$$

$\varphi(t)$ is absolutely continuous and $\varphi' = \sigma$.

$$\begin{aligned} &\therefore \quad \varphi' \leqq k\varphi \\ &\therefore \quad \frac{d}{dt} (\varphi e^{-kt}) \leqq 0 \\ &\therefore \quad \int_{T}^{t} \frac{d}{dt} (\varphi e^{-kt}) \leqq 0 && t \geqq T \\ &\text{i.e.} \quad \varphi(t) e^{-kt} \leqq \varphi(T) e^{-kT} \leqq \gamma e^{(c-k)T} \longrightarrow 0 \quad \text{as} \quad T \longrightarrow -\infty . \\ &\therefore \quad \varphi(t) e^{-kt} \leqq 0. && t > -\infty \\ &\therefore \quad \varphi(0) = 0 \Rightarrow \sigma(0) \end{aligned}$$

The following classical result will be helpful.

LEMMA 1. *Let* $\rho, \sigma, \varepsilon$ *be positive real valued measurable functions,* ρ *and* ε *bounded and* σ *integrable such that*

$$\rho(t) \leqq \varepsilon(t) + \int_{-\infty}^{t} \sigma(s) \rho(s) ds .$$

Then

$$\rho(t) \leqq \varepsilon(t) + \exp\left(\int_{-\infty}^{t} \sigma(s) ds\right) \int_{-\infty}^{t} \varepsilon(s) \sigma(s) \exp\left(\int_{-\infty}^{s} \sigma(\alpha) d\alpha\right) ds .$$

The following is a result of Segal [2] and Strauss [6]. For $H = E^n$, this result is classical.

PROPOSITION 3. If $\| K_t(u) - K_t(v) \| \leqq \sigma(t) \| u - v \|$ where $\sigma(t)$ is bounded integrable then $\mathscr{D}_- = \mathscr{R}_- = H$, $\mathscr{W}_-$ is one-one, Lipschitz continuous with a Lipschitz continuous inverse.

Proof. Let $u_0 \subset H$. Let $u(t)$ be the solution of (1) with data u_0. Thus

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}K_s(u(s))ds \qquad t \leqq 0 .$$

Thus

$$\| u(t) \| \leqq \| u_0 \| + \int_t^0 \sigma(s) \| u(s) \| ds .$$

Then by a classical result,

$$\| u(t) \| \leqq \| u_0 \| (1 + ce^c) \text{ where } c = \int_{-\infty}^0 \sigma(t)dt .$$

$$\therefore \quad \int_{-\infty}^0 \| K_s(u(s)) \| ds \leqq \int_{-\infty}^0 \sigma(s) \| u(s) \| ds < c(1 + ce^c) \| u_0 \| .$$

Thus $u_0 \in \mathscr{D}_-$ by Corollary 1. $\therefore$ $\mathscr{D}_- = H$. Let u_0 and v_0 be in H. Let $u(t)$ and $v(t)$ be the corresponding solutions. Then

$$\| u(t) - v(t) \| \leqq \| u_0 - v_0 \| + \int_t^0 \sigma(s) \| u(s) - v(s) \| ds \quad t \leqq 0 .$$

$$\therefore \quad \| u(t) - v(t) \| \leqq \| u_0 - v_0 \| (1 + ce^c) \qquad \forall t \leqq 0$$

$$\therefore \quad \| \mathscr{W}_- u_0 - \mathscr{W}_- v_0 \| \leqq \| u_0 - v_0 \| (1 + ce^c) .$$

Thus $\mathscr{W}_-$ is Lipschitz continuous. Now suppose $\mathscr{W}_- u_0 = \mathscr{W}_- v_0$.

$$e^{-tA}u(t) = u_0 + \int_0^t e^{-sA}K_s(u(s))ds .$$

$$e^{-tA}v(t) = v_0 + \int_0^t e^{-sA}K_s(v(s))ds .$$

Similarly,

$$e^{-tA}u(t) = e^{-TA}u(T) + \int_T^t e^{(t-s)A}K_s(u(s))ds$$

$$e^{-tA}v(t) = e^{-TA}v(T) + \int_T^t e^{(t-s)A}K_s(v(s))ds .$$

Therefore

$$\| e^{-tA}u(t) - e^{tA}v(t) \| = \| u(t) - v(t) \| \leqq \| u(T) - v(T) \| + \int_T^t \sigma(s) \| u(s) - v(s) \| ds \qquad 0 \geqq t \geqq T$$

$$\| u(t) - v(t) \| \leqq \| u(T) - v(T) \| + \int_{-\infty}^t \sigma(s) \| v(s) - v(s) \| ds \,.$$

By Lemma 1,

$$\| u(t) - v(t) \| \leqq \| u(T) - v(T) \| (1 + ce^c) \qquad 0 \geqq t \geqq T \,. \tag{3}$$

But $\| u(T) - v(T) \| \to 0$ as $T \to -\infty$.

$\therefore \quad \| u(t) - v(t) \| = 0 \quad \forall\, t \leqq 0.$ In particular for $t = 0$.

$\therefore \quad u_0 = v_0$.

Let $T \leqq 0$ be such that $\int_{-\infty}^T \sigma(s)ds = c' < 1$. Let $u_0 \in H$. Define recursively

$$u_n(t) = u_0 + \int_{-\infty}^t e^{-sA}K_s(e^{sA}u_{n-1}(s))ds \,. \qquad t \leqq T \,.$$

Claim (a) $\| u_n(t) - u_{n-1}(t) \| \leqq c'^n \| u_0 \| \qquad t \leqq T$

(b) $\| u_n(t) \| \leqq \| u_0 \| \sum_{k=0}^n c'^n \qquad t \leqq T$.

These are proved by induction. Some care must be taken since u_{n+1} is not defined until (b) is proved for u_n.

It follows that there exists a measurable $v(t)$ such that $u_n(t) \to v(t)$ uniformly on $(-\infty, T]$. Also, $\| v(t) \| \leqq \| u_0 \| /(1 - c')$ for $t \leqq T$.

It's trivial show there a unique $w(t)$: $R \to H$ satisfying

$$w(t) = e^{tA}v(T) + \int_T^t e^{(t-s)A}K_s(w(s))ds \,.$$

For $t < T$, $e^{-tA}w(t) = v(t)$. Let $\tilde{v}(t) = e^{-tA}w(t)$. $\tilde{v}(t)$ satisfies the hypothesis of Proposition 1. $\therefore\ u_0 \in \mathscr{R}_-$. $\therefore\ \mathscr{R}_- = H$. It follows from (3) that

$$\| u(t) - v(t) \| \leqq \| \mathscr{W}_- u_0 - \mathscr{W}_- v_0 \| (1 + ce^c) \qquad t \leqq 0$$

$$\therefore \quad \| u_0 - v_0 \| \leqq \| \mathscr{W}_- u_0 - \mathscr{W}_- v_0 \| (1 + ce^c) \text{ for } u_0 \ \&\ v_0 \in H \,.$$

$$\therefore \quad \| \mathscr{W}_-^{-1} u_0 - \mathscr{W}_-^{-1} v_0 \| \leqq \| u_0 - v_0 \| (1 + ce^c) \,.$$

COROLLARY 2. *$S = \mathscr{W}_+ \mathscr{W}_-^{-1}$ exists and is Lipschitz continuous.*

PROPOSITION 4. Let $\beta \in H$. Let K independent of t and Lipschitz cont. with Lipschitz constant k. If $\| K(e^{tA}\beta) \| \leqq de^{ct}$ for $t \leqq T$ (some fixed T) and $c > k$, then $\beta \in \mathscr{R}_-$ and in fact there is a solution $u(t)$ such that

$$(*) \qquad \| e^{-tA}u(t) - \beta \| \leqq \frac{d}{c-k} e^{ct} \qquad t \leqq T\,.$$

Moreover, $u(t)$ is unique among solutions satisfying $*$.

Proof. The proof is a Picard iteration. Because of the technical details, a proof is included. Let $v_0(t) = \beta$.

$$v_n(t) = \beta + \int_{-\infty}^{t} e^{-sA}K(e^{sA}v_{n-1}(s))ds \qquad t \leqq T\,.$$

Claim (a) $\| v_n(t) - v_{n-1}(t) \| \leqq \dfrac{dk^{n-1}}{c^n} e^{ct} \qquad t \leqq T$

(b) $\displaystyle\int_{-\infty}^{t} \| K(e^{sA}v_n(s)) \| \, ds \leqq \frac{d}{k} e^{ct} \sum_{j=1}^{n-1} \left(\frac{k}{c}\right)^j \qquad t \leqq T$

$$\| v_1(t) - \beta \| \leqq \int_{-\infty}^{t} \| K(e^{sA}\beta) \| \, ds \leqq \frac{d}{c} e^{ct}\,.$$

$\therefore$ (a) holds for $n = 1$. Moreover

$$\int_{-\infty}^{t} \| K(e^{sA}v_1(s) \| \, ds \leqq \int_{-\infty}^{t} \| K(e^{sA}v_1(s) - K(e^{sA}\beta) \| \, ds$$
$$+ \int_{-\infty}^{t} \| K(e^{sA}\beta) \| \, ds \leqq \frac{d}{k} e^{ct}\left(\frac{R}{c} + \frac{R^2}{c^2}\right) \qquad t \leqq T$$

$\therefore$ (b) holds for $n = 1$.
Assume $v_{n-1}(t)$ exists and satisfies (b). Thus $v_n(t)$ exists.

$$\int_{-\infty}^{t} \| K(e^{sA}v_n(s)) \| \, ds \leqq \int_{-\infty}^{t} \| v_n - \beta \| \, ds + \frac{d}{c} e^{ct} \qquad t \leqq T\,.$$

But

$$\| v_n - \beta \| \leqq \int_{-\infty}^{t} \| K(e^{sA}v_{n-1}(s)) \| \, ds \leqq \frac{d}{k} e^{ct} \sum_{j=1}^{n} \left(\frac{k}{c}\right)^j \qquad t \leqq T$$

$$\therefore \quad \int_{-\infty}^{t} \| K(e^{sA}v_n(s)) \| \, ds \leqq \frac{d}{k} e^{ct} + \frac{d}{c} e^{ct} \sum_{j=1}^{n} \left(\frac{k}{c}\right)^j$$
$$= \frac{d}{k} e^{ct} \sum_{j=1}^{n+1} \left(\frac{k}{c}\right)^j \qquad t \leqq T$$

$$\| v_n(t) - v_{n-1}(t) \| \leqq k \int_{-\infty}^{t} \| v_{n-1}(s) - v_{n-2}(s) \| \, ds$$
$$\leqq \frac{dk^{n-1}}{c^n} e^{ct} \qquad t \leqq T\,.$$

Thus $v_n(t)$ exist for all n and $t \leqq T$ and satisfy (a) and (b).
Thus $v_n(t) \to v(t)$ uniformly on $(-\infty, T)$. In fact,

$$\| v_n(t) - v(t) \| \leqq \left(\frac{k}{c}\right)^n \frac{dc^{ct}}{k-c}$$

$v(t)$ satisfies

$$v(t) = \beta - \int_{-\infty}^{t} e^{-sA} K(e^{sA} v(s)) ds \; .$$

If $T \geqq 0$ then $v(0)$ is already defined and by Theorem 1, $\beta \in R_-$. If $T < 0$, let $w(t)$ satisfy

$$w(t) = e^{tA} v(T) + \int_{T}^{t} e^{(t-s)A} K(w(s)) ds \; .$$

$w(t)$ exists for all t and $w(t) = e^{tA} v(t)$ for $t \leqq T$.

If $\tilde{v}(t) = e^{-tA} w(t)$ then $\tilde{v}(t)$ satisfies the hypothesis of Proposition 1. Thus $\beta \in \mathscr{R}_-$. $w(t)$ is our sought after $u(t)$. For $t \leqq T$,

$$\begin{aligned} \| e^{-tA} w(t) - \beta \| &= \| v(t) - \beta \| \leqq \| v(t) - v_n(t) \| + \| v_n(t) - \beta \| \\ &\leqq \left(\frac{k}{c}\right)^n \frac{de^{ct}}{c-k} + \frac{de^{ct}}{k} \sum_{j=1}^{n} \left(\frac{k}{c}\right)^j = \frac{de^{ct}}{c-k} \; . \end{aligned}$$

Suppose $\mathscr{W}_- z(0) = \beta \; \| z(t) - e^{tA}\beta \| \leqq \gamma' e^{c't} \; c' > k$ for $t \leqq T'$.
Then

$$\| z(t) - w(t) \| \leqq \gamma e^{c't} + \frac{d}{c-k} e^{ct} \qquad t < \min(T, T') \; .$$

Then $z(0) = w(0)$ by Proposition 2.

PROPOSITION 5. Let

$$k(s) = \sup_{\|u\|, \|v\| \leqq r} \frac{\| K(u) - K(v) \|}{\| u - v \|} \; .$$

Assume $k(s)$ is bounded on compact subsets of R^+. Assume also that for any $u_0 \in H$, there is a solution of (1) with data u_0. Let $\beta \in H$ and assume $\| K(e^{tA}\beta) \| \leqq \gamma e^{ct}$ for $t \leqq T''$ and $c > k(2 \| \beta \|)$. Then $\beta \in \mathscr{R}_-$.

Proof. Let T' be such that if $t \leqq T', \gamma e^{ct} \leqq (c-k) \| \beta \|$, $k = k(2 \| \beta \|)$. Let $T = \min(T', T'')$. Let $v_0(t) = \beta$. Let

$$v_1(t) = \beta + \int_{-\infty}^{t} e^{-sA} K(e^{sA}\beta) ds \qquad t \leqq T$$

(i) $\| v_1(t) - \beta \| \leqq (\gamma/c) e^{ct} \qquad t \leqq T$

(ii) $\| v_1(t) \| \leqq \| \beta \| + (\gamma/c) e^{ct} \leqq (2 - k/c) \| \beta \| \qquad t \leqq T$.

It follows that $v_2(t)$ is defined as is Proposition 4.

Claim $\| v_n(t) \| \leqq \left(2 - \left(\frac{k}{c}\right)^n\right) \| \beta \|$. $t \leqq T$ True to $n = 1$.

$$\| v_n(t) - v_{n-1}(t) \| \leqq \int_{-\infty}^{t} \| K(e^{sA} v_{n-1}(s)) - K(e^{sA} v_{n-2}(s)) \| \, ds \quad t \leqq T .$$

By induction assumption, $\| v_{n-1}(s) \|$ and $\| v_{n-2}(s) \|$ are both less than $2 \| \beta \|$ for $s \leqq T$.

$$\therefore \quad \| v_n(t) - v_{n-1}(t) \| \leqq k \int_{-\infty}^{t} \| v_{n-1}(s) - v_{n-2}(s)) \| \, ds .$$

But $\| v_0 \|, \| v_1 \| \cdots \| v_{n-1} \| \leqq 2 \| \beta \|$.
$\therefore$ for $j < n$,

$$\| v_j(t) - v_{j-1}(t) \| \leqq k \int_{-\infty}^{t} \| v_{j-1}(s) - v_{j-2}(s) \| \, ds \leqq \frac{\gamma k^{j-1}}{c^j} e^{ct}$$

by induction.

$$\therefore \quad \| v_n(t) - v_{n-1}(t) \| \leqq \frac{\gamma k^{n-1}}{c^n} e^{ct}$$

$$\therefore \quad \| v_n(t) \| \leqq \| v_{n-1}(t) \| + \frac{\gamma k^{n-1}}{c^n} e^{ct} \leqq \left(2 - \left(\frac{k}{c}\right)^n\right) \| \beta \| .$$

Now proceed as in Proposition 4.

3. Nonlinear wave equation. Consider the partial differential equations

$$(4) \qquad \Box u = 0 \qquad u(0) = r(\vec{x}) \;\; u_t(0) = s(\vec{x})$$

$$(5) \qquad \Box u = qF(u) \qquad u(0) = r(\vec{x}) \;\; u_t(0) = s(\vec{x})$$

where $\Box = \sum_{j=1}^{3} (\partial^2/\partial x_j^2) - (\partial^2/\partial t^2)$, q is a function on R^3, and F is a real valued function of a real variable. Let $H = \mathscr{D}_{[\sqrt{-\Delta}]} \oplus L_2$ where $\mathscr{D}_{[\sqrt{-\Delta}]}$ is the completion of the domain of $\sqrt{-\Delta}$ in L_2 with respect to the norm $\|\nabla u\|_2 = \left(\int_{R^3} |\nabla u|^2\right)^{1/2}$. The Sobolev inequality shows that we may treat the elements of $\mathscr{D}_{[\sqrt{-\Delta}]}$ as functions. ($\mathscr{D}_{[\sqrt{-\Delta}} \subset L_6$). Let

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

on H, i.e., $A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix}$ for such $\begin{pmatrix} u \\ v \end{pmatrix}$ where it is defined. A is skew-adjoint with respect to $\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 = \| \nabla u \|_2^2 + \| v \|_2^2$ where $\| \; \|_2$ is the L_2 norm. Let $K : H \to H$ be defined by $K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ qF(u) \end{pmatrix}$. If

$r \in C_0^\infty$ and $s \in C_0^\infty$ then $u \in C_0^\infty$ for each t. Then $\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} \in H$ and

$$(6') \qquad \frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = A\begin{pmatrix} u \\ u_t \end{pmatrix} + K\begin{pmatrix} u \\ u_t \end{pmatrix}$$

where u satisfies equation (5). In general true solutions to

$$(6) \qquad \frac{d\alpha}{dt} = A\alpha + K\alpha$$

are not strict solutions to (5) but are solutions in a weak sense. In fact however this paper will be interested in solutions to (6); i.e., mild solutions. For a discussion of the above see [2, 3, 4, 5].

PROPOSITION 6. Assume $F(0) = 0$, $|F(x) - F(y)| < k|x - y|$ and $|q(x)| \leqq \gamma e^{-c|x|}$, $|x| = (x_1^2 + x_2^1 + x_3^2)^{1/2}$ where

$$C > \left(\frac{2k\gamma\pi^{1/3}d}{3}\right)^{1/2}.$$

$d = \sup_{u \in C_0^\infty}(\|u\|_6/\|\nabla u\|_2)$. ($d$ is finite by the Sobolev lemma). Then if $\varphi_0 \in C_0^\infty \oplus C_0^\infty \subset H$, $\varphi_0 \in \mathscr{R}_-$ (and $\mathscr{R}_-$).

Proof.

$$\|K(\rho) - K(\psi)\| \leqq \eta\|\varphi - \psi\| \quad \text{where} \quad \eta = \frac{2k\gamma\pi^{1/3}d}{3c}.$$

Thus there exist unique solutions of (6) for any initial data. By Huygen's principle, $e^{tA}\varphi_0$ is detached; i.e., $\exists t_0$ such that if

$$\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = e^{tA}\varphi_0,$$

$u(t)$ and $u_t(t)$ vanishes in a cone $\{|x| < |t| + t_0\}$. Thus

$$\begin{aligned} \|K(e^{tA}\varphi_0)\|^2 &= \int q^2F^2(u(t))d_3x \leqq k^2\int q^2u^2 = k^2\int_{|x| \geqq |t|+t_0} q^2u^2 \\ &\leqq k^2d^2\left(\int_{|x|\geqq|t|+t_0} q^3\right)^{2/3}\|\varphi_0\|^2 \\ &\leqq k^2d^2\|\varphi_0\|^2\gamma^2\frac{(4\pi)^{2/3}e^{-2ct_0}}{9c^2}e^{-2(c-\varepsilon)|t|} \end{aligned}$$

for any $\varepsilon > 0$ and $|t| > T(\varepsilon)$. Our result then follows from Proposition 4.

REMARK. Note that only the fact that $u(t)$ is detached is used.

PROPOSITION 7. Assume $F(u) = u^3$, $|q(x)| \leqq \gamma e^{-c|x|^2}$. Then if $\varphi_0 \in C_0^\infty \oplus C_0^\infty$, $\varphi_0 \in \mathscr{R}_+$ and $\mathscr{R}_-$.

Proof. It is proved in Segal [2] that for this case, solutions exist for any initial data. $\|K(\alpha) - K(\beta)\| \leqq 3\gamma d^3 r^2 \|\alpha - \beta\|$ where $\|\alpha\|$, $\|\beta\| \leqq r$, i.e, $k(s)$, is the notation of Proposition 5, is $3\gamma d^3 r$ (use Sobolev inequality). Suppose $\varphi_0 \in C_0^\infty \oplus C_0^\infty$. It will be shown that for $|t|$ sufficiently large

$$\|K(e^{tA}\varphi_0)\| \leqq \gamma_1 e^{-c'|t|} \quad \text{where} \quad c' > k(2\|\varphi_0\|)\ .$$

The result then follows from Proposition 5. Let $\varphi_0(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix}$.

$$\|K(e^{tA}\varphi_0)\|^2 = \int q^2 u^6\ .$$

But $u(t)$ is detached so

$$\begin{aligned} \|K(e^{tA}\varphi_0)\|^2 &= \int_{|x| \geqq |t| + t_0} q^2 u^6 \leqq e^{-c(|t|+t_0)^2} \int_{|x| \geqq |t| + t_0} u^6 \\ &\leqq e^{-c(|t|+t_0)^2} d^6 \|\varphi_0\|^6\ . \\ \therefore \quad \|K(e^{tA}\varphi_0)\| &\leqq d^3 \|\varphi_0\|^3 e^{-c(|t|+t_0)^2}\ . \end{aligned}$$

For $|t|$ sufficiently large

$$e^{-c(|t|+t_0)^2} \leqq \gamma_1^2 e^{-c'|t|}\ .$$

for arbitrary c'.

By Propositions 7 and 5 there exists a $\psi_t(t)$ satisfying (2) mildly such that

$$\|\psi_\pm(t) - e^{tA}\varphi_0\| \leqq \gamma e^{-c|t|} \qquad |t| \leqq T(c)$$

where φ and K satisfies the hypothesis of Proposition 7 where c is arbitrary. Fix c. Then for $|t|$ sufficiently large, $\|\psi_\pm\| \leqq 2\|\varphi_0\|$. Thus

$$\begin{aligned} \|K(\psi_\pm(t))\| &\leqq \|K(\psi_\pm(t)) - K(e^{tA}\varphi_0)\| + \|K(e^{tA}\varphi_0)\| \\ &\leqq k(2\|\varphi_0\|)\gamma e^{-c|t|} + \gamma' e^{-c|t|} = \gamma e^{-c|t|} \end{aligned}$$

i.e.

$$\|K(\psi_\pm(t))\| < \gamma_+ e^{-ct} \quad \text{for} \quad t \geqq T_+(c) \qquad (*)$$

$$\|K(\psi_-(t))\| < \gamma_- e^{ct} \quad \text{for} \quad t \leqq T_-(c) \qquad (**)$$

for arbitrary positive e. Thus:

COROLLARY. *If K is as in Proposition 7, there exist solutions to $(d\varphi/dt) = A\varphi + K(\varphi)$, satisfying (*) and (**).*

REMARK 1. It is not clear whether there exist φ satisfying (*)

and (**) simultaneously.

REMARK 2. Strauss [4, 5] by using energy inequalities has shown for the case $F(u) = qu^3$ that there exist solutions $u(x, t)$ such that $\| K(u(t)) \|$ are integrable which implies they are in the domain of $\mathscr{W}_-$ by Proposition 1.

4. Nonlinear relativistic wave equation. Consider the partial differential equation

$$\Box\, u = m^2 u + qF(u) \qquad u(0) = r \qquad u_t(0) = s \tag{7}$$

where $m > 0$. By a result of R. W. Goodman [1], has no finite energy detached solutions. As in § 3, consider mild solutions to

$$\frac{d\varphi}{dt} = A\varphi + K(\varphi) \qquad \varphi(0) = \varphi_0$$

where in this case $H = \mathscr{D}_{[\sqrt{m^2-\Delta}]} \oplus L_2$ and

$$A = \begin{pmatrix} 0 & I \\ -(m^2 - \Delta) & 0 \end{pmatrix}$$

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 = \| \sqrt{m^2 - \Delta}\, u \|_2^2 + \| v \|_2^2 .$$

The following will be proved elsewhere (in a somewhat stronger form).

LEMMA. *Let $r(x)$ and $s(x)$ be in $L_2(R^3)$. Let $\hat{r}$ and $\hat{s}$ be their Fourier transforms. Assume $(m^2 + |x|^2)^2\hat{r}$ and $(m^2 + |x|^2)^2\hat{s}$ are in Γ where Γ is the image under Fourier transform of $L_1(R^3)$. Then $\| u(x, t) \|_\infty = 0(1/|t|)$ where $u(x, t)$ satisfies (4) with $F \equiv 0$.*

PROPOSITION 8. Suppose q is bounded, F Lipschitz and $F \equiv 0$ in a neighborhood of 0. Then if the initial data satisfy the requirements of the above lemma $\| K(e^{tA}\varphi_0) \| \to 0$ exponentially. In fact $\| K(e^{tA}\varphi_0) \| = 0$ for $|t|$ sufficiently large. Thus $\mathscr{W}_\pm$ have nonempty range by Proposition 4.

Proof. As before, if

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in H,\ K(\varphi) = \begin{pmatrix} 0 \\ -qF(u) \end{pmatrix} .$$

Let $\varphi_0 = \begin{pmatrix} r \\ s \end{pmatrix}$ where r and s satisfy the above lemma. Then

$$e^{tA}\varphi_0 = \begin{pmatrix} u(x, t) \\ u_t(x, t) \end{pmatrix}.$$

Thus

$$K(e^{tA}\varphi_0) = \begin{pmatrix} 0 \\ -qF(u(x, t)) \end{pmatrix}.$$

But there exists $\varepsilon > 0$ such that if $|y| < \varepsilon$, $F(y) = 0$ by hypothesis. By the above lemma, there exists a T such that if $|t| > T$, $\|u(x, t)\|_\infty < \varepsilon$. Thus for $|t| > T$, $F(u(x, t)) = 0$ for all $x \in R^3$. If F is Lipschitz continuous, so then is K.

I. E. Segal has recently obtained by similar methods stronger results along the above lines.

Added in proof. The hypothesis of Proposition 4 and 5 may be weakened to $\int_{-\infty}^{0} e^{-ct} \| K(e^{tA}\beta) \| dt < \infty$ for some $c > k$.

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